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Frequency Aanalysis of Annular Plates Having a Small Core and Guided Edges at Both Inner and Outer Boundaries

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ABSTRACT

This paper deals with frequency analysis of annular plates having a small core and guided edges at both inner and outer boundaries. Using classical plate theory the governing differential equation of motion for the annular plate having a small core is derived and solved for the case of plate being guided at inner and outer edge boundaries. The fundamental frequencies for the first six modes of annular plate vibrations are computed for different materials and varying values of the radius parameter. The fundamental frequencies thus obtained may be classified into to axisymmetric and/or non-axisymmetric modes of vibration. The exact values of fundamental frequencies presented in this paper clearly show that no mode switching takes place for the case of annular plates with guided edges. The results presented in this paper will be of use in design and also serve as benchmark values to enable the researchers to validate their results obtained using numerical methods such as differential quadrature or finite element methods.

Keywords : Annular plate; Vibrations; Guided edge; Mode switching.

1 INTRODUCTION

VIBRATION characteristics of annular plates are of interest in design since annular plates are commonly used structural components. There exists a considerable amount of literature on frequencies of annular plates with classical boundary conditions [1-7]. Except for an annular plate being free at both inner and outer edges, the plate fundamental axisymmetric frequency generally corresponds to the mode with no nodal diameter. Moreover, as the core radius for the annular plate with clamped or simply supported inner periphery and outer periphery being free is continuously decreased, the plate fundamental frequency may switch from no nodal diameter to one nodal diameter [8-10].

An unconstrained completely free thin circular or annular plate can vibrate transversely in nodes which may be characterized by any number of nodal circles crossed by uniformly spaced nodal diameters. Southwell [11] studied the vibration behavior of the annular plate with clamped-free boundary conditions by using the asymptotic expansions method and the special case of inner radius *bR* shrinking towards zero. Through this study, he found that the frequency varies singularly from zero for smaller values of *bR*. Kim and Dickinson [12] studied the problem of the effect of elastic edge restraints on the natural frequencies of isotropic and polar orthotropic annular and circular

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plates. Recently the present authors [13] presented results for buckling of annular plates involving elastic restraints and guided ends. In another study [14], the present authors analyzed the problem of vibrations of circular plates resting on elastic foundation and involving guided edge conditions. Wang and Wang [15] studied and presented the results for the fundamental frequencies of annular plates having values of *bR* less than 0.1 *R*. However, their study deals with annular plates having classical boundary at both interior and exterior periphery.

The present study aims at determination of fundamental frequency of the annular plates having small core and both edges being guided edges at both interior and exterior periphery. The annular plates having small core problems are very significant in various applications like vibration control by nailing or bolting the interior of a plate. Also, importantly the condition such as guided edge needs to be modeled to simulate the dynamics of moving parts such as piston heads**.** In addition, the following issues will be addressed in this paper: (i). Is there any mode switch observed? (ii). Is the fundamental frequency governed by symmetric or asymmetric mode? (iii). Is the fundamental frequency zero or finite as $b \to 0$? (iv). Where is the location of the mode switch? (v). How much does the Poisson's ratio influences the transition core radius parameter? (vi). How does the boundary condition affect the core radius parameter? (vii). How does the fundamental frequency vary with the boundary conditions? (viii) How does the fundamental frequency vary with the Poisson's ratio? The present paper therefore aims at presenting results obtained from using an exact method of solution for the problem and further addressing the above listed questions for the case of annular plate having guided edges at the inner and outer plate boundaries.

2 MATHEMATICAL FORMULATIONS

Consider a thin circular annular plate of outer radius *R*, inner radius *bR*, uniform thickness *h*, Poisson's ratio *v*, Young's modulus *E*, and flexural rigidity *D*. The material of the annular plate is assumed to be isotropic, linearly elastic and homogeneous. Let the subscripts I and II denote respectively the outer region $b \le r \le 1$ and the inner region $0 \le r \le b$. All lengths are normalized with respect to *R*, i.e., the radius of outer region is 1 and that of inner region is *bR*.

According to the classical plate theory [1], the following fourth order differential equation describes the free flexural vibrations of a thin circular uniform plate:

$$D\nabla^4 w + \rho h \frac{\delta^2 w}{\delta t^2} = 0 \tag{1}$$

where. Here, lateral displacement of the vibration of thin plate can be expressed as $w = u(r)\cos(n\theta)e^{i\Omega t}$, where *w* is the transverse displacement, *n* is the integer and Ω is the frequency. The displacement $u(r)$ in this case turns out to be a linear combination of the Bessel functions such as $J_n(kr), Y_n(kr), I_n(kr) \,\&\, K_n(kr)$, where $k = R\left(\frac{\rho h \Omega^2}{D}\right)^{1/4}$ is the square root of the non-dimensional frequency [1].

The general solution is therefore given by

$$u(r) = C_1 J_n(kr) + C_2 Y_n(kr) + C_3 I_n(kr) + C_4 K_n(kr) \tag{2}$$

The boundary conditions considered in this study are guided (G) and guided (G) edges (i.e., G-G annular plate as shown in Fig. 1.

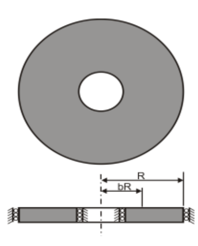

Fig.1
Annular plate with small core (G-G annular plate).

Case (i) : Outer edge of an annular plate is guided.

Boundary conditions at outer region of annular plate is

$$\frac{\partial w_I(r,\theta)}{\partial r}=0 \tag{3}$$

$$V_I(r,\theta)=0 \tag{4}$$

At $r=1$, the Eqs. (2), (3) and (4) can be written as:

$$u'(1)=0 \tag{5}$$

$$u'''(1)+u''(1)-\left[1+n^2(2-\nu)\right]u'(1)+n^2(3-\nu)u'(1))=0 \tag{6}$$

Case (ii): Inner edge of an annular plate is guided.
Boundary conditions at inner region of annular plate is

$$\frac{\partial w_{II}(r,\theta)}{\partial r}=0 \tag{7}$$

$$V_{II}(r,\theta)=0 \tag{8}$$

At $r=b$, the Eqs. (2), (7) and (8) can be written as:

$$u'(b)=0 \tag{9}$$

$$u'''(b)+u''(b)-\left[1+n^2(2-\nu)\right]u'(b)+n^2(3-\nu)u'(b))=0 \tag{10}$$

Eqs. (2), (5), (6), (9) and (10) yields the following

$$[T_1]C_1+[U_1]C_2+[V_1]C_3-[W_1]C_4=0 \tag{11}$$

$$\begin{aligned}&\left[\frac{k^3}{8}T_3+\frac{k^2}{4}T_2-\frac{k}{2}\left(\frac{3}{4}k^2+n^2(2-\nu)+1\right)T_1+\left(n^2(3-\nu)-\frac{k^2}{2}\right)J_n(k)\right]C_1+\\&\left[\frac{k^3}{8}U_3+\frac{k^2}{4}U_2-\frac{k}{2}\left(\frac{3}{4}k^2+n^2(2-\nu)+1\right)U_1+\left(n^2(3-\nu)-\frac{k^2}{2}\right)Y_n(k)\right]C_2+\\&\left[\frac{k^3}{8}V_3+\frac{k^2}{4}V_2+\frac{k}{2}\left(\frac{3}{4}k^2+n^2(-2+\nu)-1\right)V_1+\left(n^2(3-\nu)+\frac{k^2}{2}\right)I_n(k)\right]C_3+\\&\left[-\frac{k^3}{8}W_3+\frac{k^2}{4}W_2+\frac{k}{2}\left(-\frac{3}{4}k^2+n^2(2-\nu)+1\right)W_1+\left(n^2(3-\nu)+\frac{k^2}{2}\right)K_n(k)\right]C_4=0\end{aligned} \tag{12}$$

$$\left[-T_1'\right]C_1-\left[U_1'\right]C_2-\left[V_1'\right]C_3+\left[W_1'\right]C_4=0 \tag{13}$$

$$\begin{aligned}&\left[\frac{k^3}{8}T_3'+\frac{k^2}{4b}T_2'-\frac{k}{2}\left(\frac{3}{4}k^2+n^2(2-\nu)+1\right)T_1'+\left(\frac{n^2(3-\nu)}{b^3}-\frac{k^2}{2b}\right)J_n(kb)\right]C_1+\\&\left[\frac{k^3}{8}U_3'+\frac{k^2}{4b}U_2'-\frac{k}{2}\left(\frac{3}{4}k^2+n^2(2-\nu)+1\right)U_1'+\left(\frac{n^2(3-\nu)}{b^3}-\frac{k^2}{2b}\right)Y_n(kb)\right]C_2+\\&\left[\frac{k^3}{8}V_3'+\frac{k^2}{4b}V_2'+\frac{k}{2}\left(\frac{3}{4}k^2+n^2(-2+\nu)-1\right)V_1'+\left(\frac{n^2(3-\nu)}{b^3}+\frac{k^2}{2b}\right)I_n(kb)\right]C_3+\\&\left[-\frac{k^3}{8}W_3'+\frac{k^2}{4b}W_2'+\frac{\kappa}{2}\left(-\frac{3}{4}k^2+n^2(2-\nu)+1\right)W_1'+\left(\frac{n^2(3-\nu)}{b^3}+\frac{k^2}{2b}\right)K_n(kb)\right]C_4=0\end{aligned} \tag{14}$$

where

$$T_1 = J_{n-1}(k) - J_{n+1}(k); T_2 = J_{n-2}(k) + J_{n+2}(k); T_3 = J_{n-3}(k) - J_{n+3}(k);$$

$$U_1 = Y_{n-1}(k) - Y_{n+1}(k); U_2 = Y_{n-2}(k) + Y_{n+2}(k); U_3 = Y_{n-3}(k) - Y_{n+3}(k);$$

$$V_1 = I_{n-1}(k) + I_{n+1}(k); V_2 = I_{n-2}(k) + I_{n+2}(k); V_3 = I_{n-3}(k) + I_{n+3}(k);$$

$$W_1 = K_{n-1}(k) + K_{n+1}(k); W_2 = K_{n-2}(k) + K_{n+2}(k); W_3 = K_{n-3}(k) + K_{n+3}(k);$$

$$T_1^{'} = J_{n-1}(kb) - J_{n+1}(kb); T_2^{'} = J_{n-2}(kb) + J_{n+2}(kb); T_3^{'} = J_{n-3}(kb) - J_{n+3}(kb);$$

$$U_1^{'} = Y_{n-1}(kb) - Y_{n+1}(kb); U_2^{'} = Y_{n-2}(kb) + Y_{n+2}(kb); U_3^{'} = Y_{n-3}(kb) - Y_{n+3}(kb);$$

$$V_1^{'} = I_{n-1}(kb) + I_{n+1}(kb); V_2^{'} = I_{n-2}(kb) + I_{n+2}(kb); V_3^{'} = I_{n-3}(kb) + I_{n+3}(kb);$$

$$W_1^{'} = K_{n-1}(kb) + K_{n+1}(kb); W_2^{'} = K_{n-2}(kb) + K_{n+2}(kb); W_3^{'} = K_{n-3}(kb) + K_{n+3}(kb);$$

3 SOLUTION

For a set values $n, \nu, \& b$, Eqs. (11)-(14) given above results in an exact characteristic equation for non-trivial solutions involving the constant coefficients such as $C_1, C_2, C_3 \& C_4$. appearing in these equations. To obtain non-trivial solutions, the determinant of $[C]_{4x4}$ must be factored out solved accordingly. Frequency parameter k is determined for a chosen accuracy from the characteristic equation by applying a simple root search method such as bisection method. Using, Mathematica software with symbolic capabilities is effectively utilized in obtaining solutions for this problem.

4 RESULTS AND DISCUSSIONS

Poisson ratio ranges from 0.1 for Beryllium to 0.2 for marble to 0.3 for metals to 0.4 for polymers to 0.5 for rubber. The values of frequency parameters k for $n \le 5$ modes, for various values of Poisson's ratio ($\nu = 0.1, 0.2, 0.3, 0.4 \& 0.5$) and core radius parameter ($b = 0.1$ to $b = 0.5$) are tabulated (Table.1). Frequency parameter k, is presented for first six modes of vibration ($n = 0$ to $n = 5$ mode) in Figs. 2-6 corresponding to $\nu = 0.1, 0.2, 0.3, 0.4 \& 0.5$.

Table 1
First frequency parameter, k ($n = 0$ to $n = 5$ mode) for small core radius parameter b, and $\nu = 0.1$.

b	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
0	3.83171	1.8365	2.9364	4.08419	5.20275	6.3033
0.02	3.83639	2.00674	2.93715	4.08419	5.20275	6.3033
0.04	3.85018	2.0523	2.93939	4.0842	5.20275	6.3033
0.06	3.87254	2.08418	2.94305	4.08425	5.20275	6.3033
0.08	3.90295	2.10796	2.948	4.08437	5.20275	6.3033
0.1	3.94094	2.12536	2.9541	4.08462	5.20276	6.3033
0.2	4.23575	2.13259	2.99316	4.08924	5.20315	6.30333
0.3	4.70578	2.03814	3.01459	4.09696	5.20449	6.3035
0.4	5.39118	1.90073	2.97549	4.08274	5.19734	6.30014
0.5	6.39316	1.76276	2.86962	4.00574	5.14427	6.26697

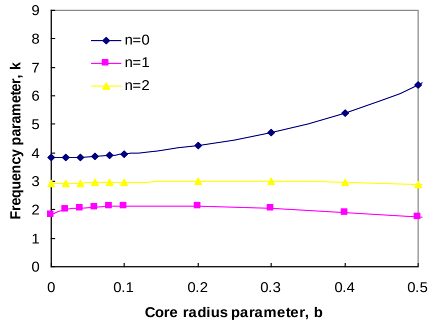

Fig.2
Frequencies for small core radius parameter b, and $\nu = 0.1$ (G-G annular plate).

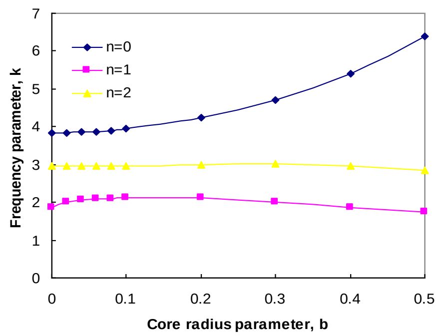

Fig.3
Frequencies for small core radius parameter b, and $\nu = 0.2$ (G-G annular plate).

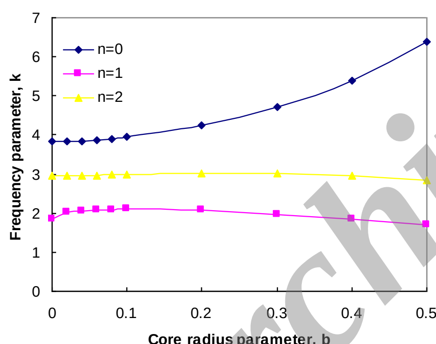

Fig.4
Frequencies for small core radius parameter b, and $\nu = 0.3$ (G-G annular plate).

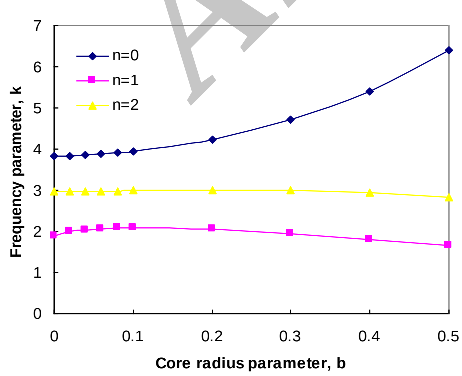

Fig.5
Frequencies for small core radius parameter b, and $\nu = 0.4$ (G-G annular plate).

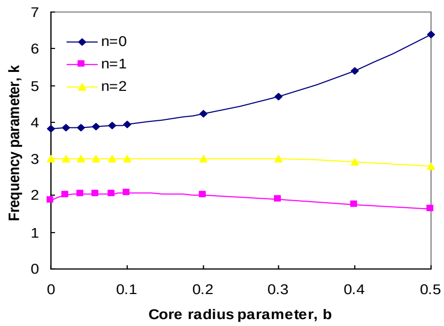

Fig.6
Frequencies for small core radius parameter *b*, and $\nu = 0.5$ (G-G annular plate).

Adopting poisson ratio $\nu = 0.1$, Fig. 2 represents frequency parameter k for $n = 1$ mode and it gives the fundamental frequency for lower values of *b*. This fundamental frequency is completely governed by asymmetric mode ($n = 1$ mode). It is noticed from Figs. 3-6, that the fundamental frequency is always governed by asymmetric ($n = 1$) mode irrespective of its core sizes. There is no mode switch observed for G-G annular plate. For $b = 0.1$ the fundamental frequency parameter is 2.12536, 2.11395, 2.0998, 2.08225, 2.06035 for $\nu = 0.1, 0.2, 0.3, 0.4 \,\&\, 0.5$ respectively. The frequency parameter decreases with increase in Poisson ratio. However, the change is just considerable. The percentage decrease is just about 3% (3.059%). However, when $b = 0$, the frequency parameter is 3.83171, that is independent of Poisson's ratio and obtained from $n = 0$ mode.

5 CONCLUSIONS

Fundamental frequency for annular plates with small core sizes are determined exactly and are presented in this paper and a closed form solution for vibration of annular plate with small core and guided edges is given. Results are presented for the first six modes of vibrations. The fundamental frequency is related to asymmetric ($n = 1$) mode only. There is no transition or mode switching observed for the case considered in this paper. The results presented in this paper are expected to be of use in design of such annular plates with small core and guided edge at both peripheries the annular plate.

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