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Rayleigh Wave in an Incompressible Fibre-Reinforced Elastic Solid Half-Space

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ABSTRACT

In this paper, the equation of motion for an incompressible transversely isotropic fibre-reinforced elastic solid is derived in terms of a scalar function. The general solution of the equation of motion is obtained, which satisfies the required radiation condition. The appropriate traction free boundary conditions are also satisfied by the solution to obtain the required secular equation for the Rayleigh wave speed. Iteration method is used to compute the numerical values of non-dimensional speed of Rayleigh wave. The dependence of the non-dimensional wave speed on non-dimensional material parameter is illustrated graphically. Effect of transverse isotropy is observed on the Rayleigh wave speed.

Keywords : Rayleigh wave; Fibre-reinforced; Incompressibility; Transverse isotropy.

1 INTRODUCTION

BELFIELD et al. [2] gave the idea of introducing a continuous self-reinforcement at every point of an elastic solid. Fibre-reinforced composite concrete structures are significant due to their low weight and high strength. A reinforced composite has characteristic property where its components act together as a single anisotropic unit till they remain in the elastic condition. During an earthquake, the artificial structures on the surface of the earth are excited which gives rise to violent vibrations in some cases. The material structures which resist the oscillatory vibration are of much interest for engineers and architects. Hashin and Rosen [11] investigated the elastic moduli for fibre-reinforced materials.

Bose and Mal [3] studied the propagation of time-harmonic elastic waves in a fibre- reinforced composite. Scott and Hayes [23] discussed the small vibrations of a fibre reinforced composite. Scott [24-25] studied the waves in a fibre-reinforced elastic material. Sengupta and Nath [26] considered the surface waves in fibre-reinforced anisotropic elastic media. They expressed the plane strain displacement components in terms of two scalar potentials to decouple the plane motion into qP and qSV waves. Singh [28] showed that, for wave propagation in fibre-reinforced anisotropic media, this decoupling cannot be achieved by the introduction of the displacement potentials. The reflection of qP and qSV waves at the free surface of a fibre-reinforced anisotropic elastic half-space is studied by Singh and Singh [29] by using a direct method without the introduction of potentials. Singh [30] obtained the reflection coefficients from free surface of an incompressible transversely isotropic fibre-reinforced elastic half-space for the case when outer slowness section is re- entrant. Singh and Yadav [31] studied the reflection of plane waves from a free surface of a rotating fibre-reinforced elastic solid half-space with magnetic field. Surface waves are very important in the study of earthquake, geophysics and

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geodynamics. Rayleigh waves cause destruction to the structure due to its slower attenuation of the energy than that of the body waves. Surface waves in elastic solids were first studied by Lord Rayleigh [18] for an isotropic elastic solid. The extension of surface wave analysis and other wave propagation problems to anisotropic elastic materials has been the subject of many studies; see, for example, Musgrave [15]; Anderson [1]; Stoneley [32]; Crampin and Taylor [6]; Chadwick and Smith [4]; Royer and Dieulesaint [22]; Dowaikh and Ogden [10]; Mozhaev [14]; Nkemzi [17]; Nair and Sotiropoulos [16]; Malischewsky [12]; Destrade [7-8]; Ting [33]; Destrade [9]; Ogden and Vinh [19]; Ogden and Singh [20]; Vinh and Linh [34]; Shams and Ogden [27]; Ogden and Singh [21].

The aim of this paper is to obtain a secular equation for the Rayleigh wave in an incompressible transversely isotropic fibre-reinforced elastic medium. Using iteration method, non-dimensional wave speed of Rayleigh wave is plotted against non-dimensional material parameter to show the effect of transverse isotropy.

2 EQUATIONS OF MOTION

Let us consider an incompressible transversely isotropic fibre-reinforced elastic medium. The constitutive equation explaining the stress-strain response to small deformation of such a material is given by Chadwick [5] as:

$$\sigma = -pI + 2\mu_T \varepsilon + 2(\mu_L - \mu_T)\{e \otimes (\varepsilon e) + (\varepsilon e) \otimes e\} + 4(\mu_E - \mu_L)\{e(\varepsilon e)\} e \otimes e \tag{1}$$

where ε, σ and I denote the infinitesimal strain and stress tensors and the 3×3 identity tensors, respectively, and e is a unit vector defining the axis of transversely isotropy. In Eq. (1), μ_L and μ_T are longitudinal and transverse shear moduli and μ_E is a weighted shear modulus, given by $\mu_E = \frac{E_L}{E_T}\mu_T$ where E_L and E_T are longitudinal and transverse Young's moduli. p is the hydrostatic pressure required to maintain the incompressibility constraint.

$$tr\ \varepsilon = 0 \tag{2}$$

Consider a Cartesian coordinate system $Ox_1x_2x_3$, such that Ox_1 is parallel to the direction of transversely isotropy, the constitutive relation (1) is written in component form as:

$$\sigma_{ij} = -p\delta_{ij} + 2\mu_T \varepsilon_{ij} + 2(\mu_L - \mu_T)(\delta_{i1}\varepsilon_{j1} + \varepsilon_{i1}\delta_{j1}) + 4(\mu_E - \mu_L)\varepsilon_{11}\delta_{i1}\delta_{j1} \tag{3}$$

The Eq. (3) may be written in matrix form (stiffness matrix) as:

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{pmatrix} -p + (4\frac{E_L}{E_T} - 1)\mu_T & 0 & 0 & 0 & 0 & 0 \\ 0 & -p + 2\mu_T & 0 & 0 & 0 & 0 \\ 0 & 0 & -p + 2\mu_T & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\mu_L & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\mu_L & 2\mu_L \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{13} \\ \varepsilon_{12} \end{bmatrix} \tag{4}$$

Let $(x_1 x_2 x_3)$ be Cartesian coordinates and consider a transversely isotropic fibre-reinforced elastic material occupying the half-space $x_2 \prec 0$, with traction-free boundary $x_2 = 0$. We consider a plane motion in the (x_1, x_2) plane with displacement components (u_1, u_2, u_3) such that

$$u_1 = u_1(x_1, x_2, t), \quad u_2 = u_2(x_1, x_2, t), \quad u_3 = 0 \tag{5}$$

where t is time. For an incompressible material, we have

$$u_{1,1} + u_{2,2} = 0 \tag{6}$$

From which we deduce the existence of a scalar function, denoted $\psi(x_1, x_2, t)$, such that

$$u_1 = \psi_{,2}, \quad u_2 = -\psi_{,1} \tag{7}$$

Using the strain-displacement relation $2\varepsilon_{ij} = u_{i,j} + u_{j,i}$ in Eq.(4), the stress components are written in terms of displacement components and pressure as:

$$\sigma_{11} = (c_1^2 - c_2^2)u_{1,1} - p \tag{8}$$

$$\sigma_{12} = c_3^2(u_{1,2} + u_{2,1}) \tag{9}$$

$$\sigma_{22} = 2c_2^2 u_{2,2} - p \tag{10}$$

where $p = p(x_1, x_2, t)$ is the hydrostatic pressure associated with the incompressibility constraint, and $c_1^2 = 4\mu_E - \mu_T$, $c_2^2 = \mu_T$, $c_3^2 = \mu_L$. The requirements of a physically reasonable response ensure that c_1^2, c_2^2 and c_3^2 are all positive. In absence of body forces, the equations of motion are

$$\sigma_{ij,j} = \rho \ddot{u}_i \tag{11}$$

which upon using Eqs.(8) to (10) become

$$(c_1^2 - c_2^2)u_{1,11} + c_3^2(u_{1,22} + u_{2,12}) - p_{,1} = \rho \ddot{u}_1 \tag{12}$$

$$c_3^2(u_{1,12} + u_{2,11}) + 2c_2^2 u_{2,22} - p_{,2} = \rho \ddot{u}_2 \tag{13}$$

with the help of Eq. (7), the Eqs. (12) and (13) become

$$(c_1^2 - c_2^2)\psi_{,211} + c_3^2(\psi_{,222} - \psi_{,112}) - p_{,1} = \rho \ddot{\psi}_{,2} \tag{14}$$

$$c_3^2(\psi_{,212} - \psi_{,111}) - 2c_2^2\psi_{,122} - p_{,2} = -\rho \ddot{\psi}_{,1} \tag{15}$$

Elimination of p by cross differentiation leads to the following equation of motion

$$c_3^2\psi_{,1111} + (c_1^2 + c_2^2 - 2c_3^2)\psi_{,1122} + c_3^2\psi_{,2222} = \rho(\ddot{\psi}_{,11} + \ddot{\psi}_{,22}) \tag{16}$$

3 RAYLEIGH WAVES

We now consider a half-space occupying the region $x_2 \langle 0$ in the reference configuration with boundary $x_2 = 0$ and Rayleigh surface waves propagating along the direction x_1 and we write ψ in the form

$$\psi(x_1+x_2,t)=\varphi(y)\exp\left[ik(x_1-ct)\right] \tag{17}$$

where $y=kx_2$ and k is wave number. Using Eq. (17) into Eq. (16), we obtain

$$c_3^2\varphi''''(y)-(c_1^2+c_2^2-2c_3^2-\rho c^2)\varphi''(y)+(c_3^2-\rho c^2)\varphi(y)=0 \tag{18}$$

The boundary conditions on $x_2=0$ are

$$\sigma_{21}=0, \qquad \sigma_{22}=0 \tag{19}$$

which are written as:

$$c_3^2(u_{1,2}+u_{2,1})=0 \tag{20}$$

$$2c_2^2u_{2,2}-p=0 \tag{21}$$

with the help of Eq.(7), the Eq. (20) is written as:

$$c_3^2(\psi_{22}-\psi_{11})=0 \tag{22}$$

Differentiating Eq.(21) with respect to x_1 and using Eq. (7) and Eq.(14), we have

$$c_3^2(\psi_{222}-\psi_{112})+(c_1^2+c_2^2)\psi_{112}-\rho\ddot{\psi}_2=0 \tag{23}$$

In addition to conditions (22) and (23), we need also the following condition on ψ

$$\psi(x_1,x_2,t)\to 0, \quad as \quad x_2\to-\infty \tag{24}$$

In terms of φ, the conditions (22) to (24) become

$$\varphi''(0)+\varphi(0)=0 \tag{25}$$

$$c_3^2\varphi'''(0)+(c_3^2-c_1^2-c_2^2+\rho c^2)\varphi'(0)=0 \tag{26}$$

$$\varphi(x_2)\to 0, \quad as \quad x_2\to-\infty \tag{27}$$

The general solution $\varphi(y)$ of Eq. (18) that satisfies radiation condition (27) is:

$$\varphi(y)=A\exp(s_1y)+B\exp(s_2y) \tag{28}$$

where A, B are constants and s_1,s_2 are solutions of following quadratic equation in s^2

$$c_3^2s^4-(c_1^2+c_2^2-2c_3^2-\rho c^2)s^2+(c_3^2-\rho c^2)=0 \tag{29}$$

If the roots s_1^2 and s_2^2 of the quadratic Eq. (29) are real, the they must be positive to ensure that s_1 and s_2 can have a positive real part. If they are complex then they are conjugate. In either case, the product $s_1^2s_2^2$ must be positive and hence a real (surface) wave speed satisfies the inequalities

$$0\langle \rho c^2 \langle c_3^2 \tag{30}$$

Using solutions (28) into boundary conditions (25) and (26), we have

$$(1+s_1^2)A + (1+s_2^2)B = 0 \tag{31}$$

$$\left[c_3^2 s_1^3 + (c_3^2 - c_1^2 - c_2^2 + \rho c^2)s_1\right]A + \left[c_3^2 s_2^3 + (c_3^2 - c_1^2 - c_2^2 + \rho c^2)s_2\right]B = 0 \tag{32}$$

Following Ogden and Vinh [19], the non-trivial solution of Eqs. (31) and (32) requires

$$(c_1^2 + c_2^2 - \rho c^2)\sqrt{1 - \frac{\rho c^2}{c_3^2}} - \rho c^2 = 0 \tag{33}$$

which is the secular equation for wave speed. Eq. (33) may also be written as:

$$(4\mu_E - \rho c^2)\sqrt{1 - \frac{\rho c^2}{\mu_L}} - \rho c^2 = 0 \tag{34}$$

For a real solution for c of Eq. (34), the inequality $\rho c^2 \langle 4\mu_E$ must also hold along with inequality (30).With the help of Eqs. (7), (17), (28), (29), (31) and (32), it can be shown that the particles in a solid move in elliptical paths with the major axis of the ellipse perpendicular to the surface of the solid. The width of the elliptical path decreases with the increase in depth into the solid.

4 NUMERICAL RESULTS

Taking $x = \frac{\rho c^2}{\mu_L}$ and $R = \frac{4\mu_E}{\mu_L}$ into the Eq. (34) and squaring both sides, we can obtain

$$x = \frac{R^2 + 2Rx^2 - x^3}{2R + R^2} \tag{35}$$

Using iteration method, the Eq. (35) is solved for x in the range $0 \le R \le 10$. The non-dimensional speed $x = \frac{\rho c^2}{\mu_L}$ is plotted against the ratio $R = \frac{4\mu_E}{\mu_L}$ of material constants in Fig. 1. For small values of ratio R of material constants, the wave speed is very small. It increases sharply with the increase in value of ratio R. For $\mu_E = \mu_L (R = 4)$, the wave speed corresponds to isotropic case and is approximately 0.9126. For very large value of R, the non-dimensional speed $x = \frac{\rho c^2}{\mu_L}$ tends to one. It is necessary to mention here that the Fig. 1 is similar to that plotted by Ogden and Vinh [19] using an explicit formula for wave speed.

Taking $x = \frac{\rho c^2}{4\mu_E}$ and $R = \frac{4\mu_E}{\mu_L}$ into the Eq. (34) and squaring both sides, we can obtain

$$x = \frac{1}{R} + 2x^2 - x^3 \tag{36}$$

Using iteration method, the Eq. (36) is solved for x in the range $0.2 \le R \le 10$. The non-dimensional speed $x = \dfrac{\rho c^2}{4\mu_E}$ is plotted against the ratio $R = \dfrac{4\mu_E}{\mu_L}$ of material constants in Fig. 2. For small values of ratio R of material constants, the wave speed is very large. It decreases very sharply with the increase in value of ratio R. For $\mu_E = \mu_L (R = 4)$, the wave speed corresponds to isotropic case and is 0.25. At higher values of R , the non-dimensional speed $x = \dfrac{\rho c^2}{4\mu_E}$ decreases slowly.

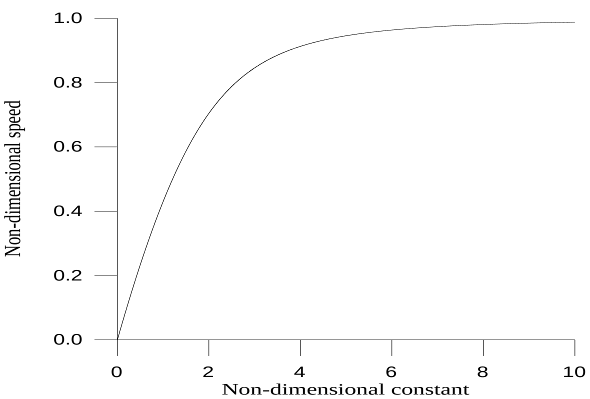

Fig.1
Variation of non-dimensional speed $\dfrac{\rho c^2}{\mu_L}$ of Rayleigh wave against non-dimensional constant $R = \dfrac{4\mu_E}{\mu_L}$.

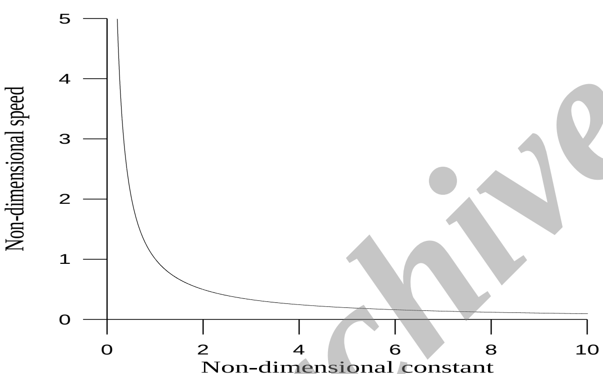

Fig.2
Variation of non-dimensional speed $\dfrac{\rho c^2}{4\mu_E}$ of Rayleigh wave against non-dimensional constant $R = \dfrac{4\mu_E}{\mu_L}$.

5 CONCLUSIONS

A problem on Rayeligh wave in an incompressible transversely isotropic fibre-reinforced elastic solid half-space is considered. The secular equation for Rayleigh wave speed is obtained. The numerical results show the dependence of wave speed on material parameters. The present theoretical and numerical results may provide important information to experimental seismologists for their further research on the subject.

REFERENCES

[1] Anderson D.L., 1961, Elastic wave propagation in layered anisotropic media , *Journal of Geophysical Research* **66**: 2953-2963.

[2] Belfield A. J., Rogers T. G., Spencer A.J.M., 1983, Stress in elastic plates reinforced by bres lying in concentric circles , *Journal of the Mechanics and Physics of Solids* **31**: 25-54.

[3] Bose S.K., Mal A.K., 1974, Elastic waves in a fiber reinforced composite , *Journal of the Mechanics and Physics of Solids* **22**: 217-229.

[4] Chadwick P., Smith G.D., 1977, Foundations of the theory of surface waves in anisotropic elastic materials, *Advances in Applied Mechanics* **17**: 303-376.
[5] Chadwick P., 1993, Wave propagation in incompressible transversely isotropic elastic media I. Homogeneous plane waves , *Proceedings of the Royal Irish Academy* **93A**: 231-253.
[6] Crampin S., Taylor D.B., 1971, The propagation of surface waves in anisotropic media , *Geophysical Journal of the Royal Astronomical Society* **25**: 71-87.
[7] Destrade M., 2001, Surface waves in orthotropic incompressible materials , *Acoustical Society of America* **110**:837-840.
[8] Destrade M., 2001, The explicit secular equation for surface acoustic waves in monoclinic elastic crystals , *Acoustical Society of America* **109**:1398-1402.
[9] Destrade M., 2001, Surface waves in orthotropic incompressible materials , *Acoustical Society of America* **110**: 837-840.
[10] Dowaikh M.A., Ogden R.W., 1990, On surface waves and deformations in a prestressed incompressible elastic solid , *The IMA Journal of Applied Mathematics* **44**: 261-284.
[11] Hashin Z., Rosen W. B., 1964, The elastic moduli of bre reinforced materials , *Journal of Applied Mechanics* **31**:223-232.
[12] Malischewsky P.G., 2000, A new formula for the velocity of Rayleigh waves *Wave Motion* **31**: 93-96.
[13] Markham M. F.,1970, Measurement of the elastic constants of fibre composites by ultrasonics, *Composites* **1**: 145-149.
[14] Mozhaev V.G.,1995, Some new ideas in the theory of surface acoustic waves in anisotropic media , *In IUTAM Symposium on Anisotropy, Inhomogeneity and Nonlinearity in Solid Mechanics* **39**:455-462.
[15] Musgrave M.J.P., 1959, The propagation of elastic waves in crystals and other anisotropic media , *Reports on Progress in Physics* **22**:74-96.
[16] Nair S., Sotiropoulos D.A., 1999, Interfacial waves in incompressible monoclinic materials with an interlayer , *Mechanics of Materials* **31**:225-233.
[17] Nkemzi D., 1997, A new formula for the velocity of Rayleigh waves , *Wave Motion* **26**:199-205.
[18] Rayleigh L., 1885, On waves propagated along the plane surface of an elastic solid , *Proceedings of the Royal Society of London Series A* **17**:4-11.
[19] Ogden R.W., Vinh P.C., 2004 , On Rayleigh waves in incompressible orthotropic elastic solids , *Acoustical Society of America* **115**:530-533.
[20] Ogden R.W., Singh B., 2011, Propagation of waves in an incompressible transversely isotropic elastic solid with initial stress: Biot revisited , *Journal of Mechanics of Materials and Structures* **6**: 453-477.
[21] Ogden R.W., Singh B., 2014, The effect of rotation and initial stress on the propagation of waves in a transversely isotropic elastic solid , *Wave Motion* **51**: 1108-1126.
[22] Royer D., Dieulesaint E., 1984, Rayleigh wave velocity and displacement in orthorhombic, tetragonal, hexagonal and cubic crystals , *Acoustical Society of America* **75**:1438-1444.
[23] Scott N.H., Hayes M., 1976, Small vibrations of a fibre reinforced composite , *Journal of Mechanics and Applied Mathematics* **29**:467-486.
[24] Scott N.H., 1992, Waves in a homogeneously prestrained incompressible, almost inex- tensible, fibre-reinforced elastic material , *Proceedings of the Royal Irish Academy* **92 A**: 9-36.
[25] Scott N.H., 1991, Small vibrations of prestrained constrained elastic materials: the idealised fibre-reinforced material , *International Journal of Solids and Structures* **27**:1969-1980.
[26] Sengupta P. R., Nath S., 2001, Surface waves in bre reinforced anisotropic elastic media , *Sadhana* **26**:363-370.
[27] Shams M., Ogden R.W., 2014, On Rayleigh-type surface waves in an initially stressed incompressible elastic solid , *The IMA Journal of Applied Mathematics* **79**: 360-372.
[28] Singh S. J. , 2002 , Surface waves in bre reinforced anisotropic elastic media, *Sadhana* **27**:1-3.
[29] Singh B., Singh S.J., 2004, Reection of plane waves at the free surface of a bre reinforced elastic half-space , *Sadhana* **29**(3):249-257.
[30] Singh B., 2007, Wave propagation in an incompressible transversely isotropic fibre reinforced elastic media , *Archive of Applied Mechanics* **77**:253-258.
[31] Singh B., Yadav A.K., 2013, Reflection of plane waves from a free surface of a rotating fibre reinforced elastic solid half-space with magnetic field , *Journal of Applied Mathematics and Mechanics* **9**:75-91.
[32] Stoneley R., 1963, The propagation of surface waves in an elastic medium with orthorhombic symmetry , *Geophysical Journal of the Royal Astronomical Society* **8**:176-186.
[33] Ting T.C.T., 2002, An explicit secular equation for surface waves in an elastic material of general anisotropy , *Journal of Mechanics and Applied Mathematics* **55**:297-311.
[34] Vinh P.C., Linh N.T.K., 2013, Rayleigh waves in an incompressible elastic half-space overlaid with a water layer under the effect of gravity, *Meccanica* **48**:2051-2060.