The Homotopy Perturbation Method for Fuzzy Fredholm Integral Equations

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Abstract

In this paper Homotopy Perturbation Method (HPM) to solve fuzzy Fredholm integral equations is proposed. The method is discussed in details and it is illustrated by solving some numerical examples.

Keywords: Homotopy, Perturbation Method, Fredholm Integral Equations.

1 Introduction

HPM has been recently intensively studied by scientists and engineers and used for solving nonlinear problems. This method [7,8] was proposed first by He which is, in fact, a coupling of the traditional perturbation method and homotopy in topology. HPM yields a very rapid convergence of the solution series in most cases, usually only few iterations leading to very accurate solution. This method has been applied to many problems [2,3,17,18].

Consider the Fredholm integral equation:

$$\gamma(x) = f(x) + \int_a^b k(x,t)\gamma(t)dt, \qquad c \le x \le d$$

Let

$$L(u) = u(x) - f(x) - \int_a^b k(x,t)\gamma(t)dt = 0, \tag{1}$$

with solution $u(x) = \gamma(x)$, we define the homotopy $H(u, p)$ by

$$H(u,0) = F(u), \qquad H(u,1) = L(u)$$

where $F(u)$ is a functional operator with solution, say, u_0, which can be obtained easily. We may choose a convex homotopy

$$H(u, p) = (1-p)F(u) + pL(u) = 0, \tag{2}$$

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and continuously trace an implicitly defined curve from a starting point $H(u_0,0)$ to a solution $H(\gamma,1)$. The embedding parameter p monotonically increases from 0 to 1 as the trivial problem $F(u)=0$ continuously deformed to the original problem $L(u)=0$. The parameter p can be considered as an expanding parameter [8,11,13]. In fact HPM uses the homotopy parameter p as an expanding parameter [9,13] to obtain

$$u = u_0 + pu_1 + p^2u_2 + \cdots, \tag{3}$$

when $p \to 1$, (3) corresponds to (2) and gives an approximation to the solution of (1) as follows:

$$\gamma = \lim_{p\to 1} u_2 + \cdots, \tag{4}$$

The series (4) converges in most cases, and the rate of convergence [12] depends on $L(u)$.

The paper is organized as follows:

In section 2, some basic definitions and fuzzy background is brought. In section3, fuzzy integral equation is introduced. In section4, the HPM for solving fuzzy fredholm integral equation is proposed, and examples are brought in section5, finally conclusion are drown in section6.

2 Preliminaries

In this section the most basic notation used in fuzzy calculus are introduced. We start by defining a fuzzy number.

Definition 1 A fuzzy number is a fuzzy set $u : R^1 \to I = [0,1]$ which satisfies

i. u is upper semi continuous.
ii. $u(x)=0$ outside some interval $[c,d]$.
iii. There are real numbers $a,b : c \le a \le b \le d$ for which
  1. $u(x)$ is monotonic increasing on $[c,d]$,
  2. $u(x)$ is monotonic decreasing on $[c,d]$,
  3. $u(x)=1, a \le x \le b$.

The set of all fuzzy numbers (as given by Definition 1) is denoted by E^1. An alternative definition or parametric form of a fuzzy number which yields the same E^1 is given by Kaleva [10].

Definition 2 A fuzzy number u is a pair $(\underline{u},\overline{u})$ of functions $\underline{u}(r),\overline{u}(r); 0 \le r \le 1$ which satisfying the following requirements:

i. $\underline{u}(r)$ is bounded monotonic increasing left continuous function,
ii. $\overline{u}(r)$ is bounded monotonic decreasing left continuous function,
iii. $\underline{u}(r) \le \overline{u}(r), 0 \le r \le 1$.

For arbitrary $u = (\underline{u}, \overline{u})$, $v = (\underline{v}, \overline{v})$ and $k > 0$ we define addition $(u+v)$ and multiplication by k as

$$\left(\underline{u+v}\right)(r) = \underline{u}(r) + \underline{v}(r),$$

$$\left(\overline{u+v}\right)(r) = \overline{u}(r) + \overline{v}(r) \tag{5}$$

$$ku(r) = \begin{cases} \left(k\underline{u}(r), k\overline{u}(r)\right), & k \geq 0 \\ \left(k\overline{u}(r), k\underline{u}(r)\right), & k < 0 \end{cases} \tag{6}$$

The collection of all the fuzzy numbers with addition and multiplication as defined by Eqs. (1) and (2) is denoted by E^1 and is a convex cone. It can be shown that Eqs. (1) and (2) are equivalent to the addition and multiplication as defined by using the α-cut approach [5] and the extension principles, [14]. We will next defined the fuzzy function notation and a metric D in E^1, [5].

Definition 3 For arbitrary fuzzy numbers $u = (\underline{u}, \overline{u})$ and $v = (\underline{v}, \overline{v})$ the quantity

$$D(u,v) = \max\left\{ \sup_{0 \leq r \leq 1} \left|\underline{u}(r) - \underline{v}(r)\right|, \ \sup_{0 \leq r \leq 1} \left|\overline{u}(r) - \overline{v}(r)\right| \right\} \tag{7}$$

is the distance between u and v .

This metric is equivalent to the one used by Puri and Ralescu [15] and Kaleva [10]. It is shown [16] that (E^1, D) is a complete metric space. We now follow Goetschel and Voxman [5] and define the integral of a fuzzy function using the Riemann integral concept .

Let $f : [a,b] \to E^1$. For each partition $p = \{t_0, t_1, \ldots, t_n\}$ of $[a,b]$ with $h = \max|t_i - t_{i-1}|$ and for arbitrary $\xi_i : t_{i-1} \leq \xi_i \leq t_i, 1 \leq i \leq n$ let

$$R_p = \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}). \tag{8}$$

The definite integral of $f(x)$ over $[a,b]$ is

$$\int_a^b f(x)dx = \lim R_p, h \to 0. \tag{9}$$

Provided that this limit exists in the metric D .

If the fuzzy function $f(x)$ is continuous in the metric D, its definite integral exists, [5] . furthermore,

$$\underline{\left(\int_a^b f(x,\alpha)dt\right)} = \int_a^b \underline{f}(x,\alpha)dx,$$

$$\overline{\left(\int_a^b f(x,\alpha)dt\right)} = \int_a^b \overline{f}(x,\alpha)dx \qquad (10)$$

It should be noted that the fuzzy integral can be also defined using the Lebesgue-type approach [10]. However, if $f(x)$ is continuous, both approaches yield the same value. Moreover, the representation of the fuzzy integral using Eqs. (4) and (5) is more convenient for numerical calculations. More details about the properties of the fuzzy integral are given in [5,10].

3 Fuzzy Integral Equations

The integral equations which are discussed in this section are the Fredholm equations . The fredholm integral equation of the second kind is, [6]

$$\gamma(x) = f(x) + \int_a^b k(x,t)\gamma(t)dt, \qquad (11)$$

where $k(x,t)$ is an arbitrary kernel function over the square $a \le x,t \le b$ and $f(x)$ is a function of $x : a \le x \le b$. If $f(x)$ is a crisp function then the solutions of Eq. (7) are crisp as well. However, if $f(x)$ is a fuzzy function these equations may only possess fuzzy solutions. Sufficient conditions for the existence of a unique solution to the Fuzzy Fredholm Integral Equation of the second kind (FFIE-2), i.e. to Eq. (7) where $f(x)$ is a fuzzy function, are given in[1].

Now, we introduce parametric form of a FFIE-2 with respect to Definition 2. Let $\left(\underline{f}(x,\alpha), \overline{f}(x,\alpha)\right)$ and $\left(\underline{u}(x,\alpha), \overline{u}(x,\alpha)\right), 0 \le \alpha \le 1$ are parametric form of $f(x)$ and $u(x)$ for $x \in [a,b]$, respectively then, the parametric form of FFIE-2 is as follows.

$$\underline{\gamma}(x,\alpha) = \underline{f}(x,\alpha) + \int_a^b k_1\left(x,t,\underline{\gamma}(t,\alpha), \overline{\gamma}(t,\alpha)\right) dt,$$

$$\overline{\gamma}(x,\alpha) = \overline{f}(x,\alpha) + \int_a^b k_2\left(x,t,\underline{\gamma}(t,\alpha), \overline{\gamma}(t,\alpha)\right) dt, \qquad (12)$$

where

$$k_1\left(x,t,\underline{\gamma}(t,\alpha), \overline{\gamma}(t,\alpha)\right) = \begin{cases} k(x,t)\underline{\gamma}(t,\alpha), & k(x,t) \ge 0 \\ k(x,t)\overline{\gamma}(t,\alpha), & k(x,t) < 0 \end{cases}$$

and

$$k_2\left(x,t,\underline{\gamma}(t,\alpha), \overline{\gamma}(t,\alpha)\right) = \begin{cases} k(x,t)\overline{\gamma}(t,\alpha), & k(x,t) \ge 0 \\ k(x,t)\underline{\gamma}(t,\alpha), & k(x,t) < 0 \end{cases}$$

For each $0 \le \alpha \le 1$ and $a \le t \le b$.

4 Homotopy Perturbation Method for Solving Fuzzy Fredholm Integral Equations:

Consider the fuzzy Fredholm integrall equation:

$$\gamma(x) = f(x) + \int_a^b k(x,t)\gamma(t)\,dt$$

where $f(x) \in E^1$. Let

$$\begin{cases} L(\underline{u}) = \underline{u}(x,\alpha) - \underline{f}(x,\alpha) - \int_a^b k_1\left(x,t,\underline{\gamma}(t,\alpha),\overline{\gamma}(t,\alpha)\right) dt = 0 \\ \\ L(\overline{u}) = \overline{u}(x,\alpha) - \overline{f}(x,\alpha) - \int_a^b k_2\left(x,t,\underline{\gamma}(t,\alpha),\overline{\gamma}(t,\alpha)\right) dt = 0 \end{cases} \quad (13)$$

with solution $\underline{u}(x,\alpha) = \underline{\gamma}(x,\alpha)$, $\overline{u}(x,\alpha) = \overline{\gamma}(x,\alpha)$, we define the homotopy $H(\underline{u},p)$, $H(\overline{u},p)$ by

$$\begin{cases} H(\underline{u},0) = F(\underline{u}), \quad H(\underline{u},1) = L(\underline{u}) \\ H(\overline{u},0) = F(\overline{u}), \quad H(\overline{u},1) = L(\overline{u}) \end{cases}$$

where $F(\underline{u})$, $F(\overline{u})$ are functional operators with solutions, say, $\underline{u}_0$, $\overline{u}_0$, which can be obtained easily. We may choose a convex homotopy

$$\begin{cases} H(\underline{u},p) = (1-p)F(\underline{u}) + pL(\underline{u}) = 0 \\ H(\overline{u},p) = (1-p)F(\overline{u}) + pL(\overline{u}) = 0 \end{cases} \quad (14)$$

and continuously trace an implicitly defined curve from a starting points $H(\underline{u}_0,0)$, $H(\overline{u}_0,0)$ to a solutions $H(\underline{\gamma},1)$, $H(\overline{\gamma},1)$. The embedding parameter p monotonically increases from 0 to 1 as the trivial problem $F(\underline{u}) = 0$, $F(\overline{u}) = 0$ continuously deformed to the original problem $L(\underline{u}) = 0$, $L(\overline{u}) = 0$. The parameter p can be considered as an expanding parameter. In fact HPM uses the homotopy parameter p as an expanding parameter to obtain

$$\begin{cases} \underline{u} = \underline{u}_0 + p\underline{u}_1 + p^2\underline{u}_2 + \cdots \\ \overline{u} = \overline{u}_0 + p\overline{u}_1 + p^2\overline{u}_2 + \cdots \end{cases} \quad (15)$$

when $p \to 1$, (11) corresponds to (10) and gives an approximation to the solution of (9) as follows:

$$\begin{cases} \underline{\gamma} = \lim_{p\to 1} \underline{u}_2 + \cdots \\ \overline{\gamma} = \lim_{p\to 1} \overline{u}_2 + \cdots \end{cases} \quad (16)$$

The series (12) converges in most cases, and the rate of convergence depends on $L(\underline{u}), L(\overline{u})$.

HMP for solving Fuzzy Fredholm Integral Equation of the first kind (FFIE-1)
Consider fuzzy Fredholm integral equation of first kind

$$f(x)=\int_a^b k(x,t)\gamma(t)\,dt,$$

The parametric form of FFIE-1 is as follows:

$$\begin{cases}\underline{f}(x,\alpha)=\int_a^b k_1\left(x,t,\underline{\gamma}(t,\alpha),\overline{\gamma}(t,\alpha)\right)dt\\[2ex] \overline{f}(x,\alpha)=\int_a^b k_2\left(x,t,\underline{\gamma}(t,\alpha),\overline{\gamma}(t,\alpha)\right)dt\end{cases}$$

taking

$$\begin{cases}F(\underline{u})=\underline{u}(x,\alpha)\\ F(\overline{u})=\overline{u}(x,\alpha)\end{cases}$$

$$\begin{cases}L(\underline{u})=\int_a^b k_1\left(x,t,\underline{\gamma}(t,\alpha),\overline{\gamma}(t,\alpha)\right)dt-\underline{f}(x,\alpha)=0\\[2ex] L(\overline{u})=\int_a^b k_2\left(x,t,\underline{\gamma}(t,\alpha),\overline{\gamma}(t,\alpha)\right)dt-\overline{f}(x,\alpha)=0\end{cases}$$

and the convex homotopy:

$$\begin{cases}H(\underline{u},p)=(1-p)\underline{u}(x,\alpha)+p[\int_a^b k_1\left(x,t,\underline{u}(t,),\overline{u}(t,\alpha)\right)dt-\underline{f}(x,\alpha)]=0\\ H(\overline{u},p)=(1-p)\overline{u}(x,\alpha)+p[\int_a^b k_2\left(x,t,\underline{u}(t,\alpha),\overline{u}(t,\alpha)\right)dt-\overline{f}(x,\alpha)]=0\end{cases}$$

with the starting points $H(\underline{0},0),H(\overline{0},0)$ and the solution $H(\underline{\gamma},1),\ H(\overline{\gamma},1)$, by similar operations as above, we obtain

$$\begin{cases}p^0:\underline{u}_0(x,\alpha)=0\\ p^0:\overline{u}_0(x,\alpha)=0\end{cases}$$

$$\begin{cases}p^1:\underline{u}_1=\underline{f}(x,\alpha)\\ p^1:\overline{u}_1=\overline{f}(x,\alpha)\end{cases}$$

$$\begin{cases}p^2:\underline{u}_2=\underline{u}_1+\int_a^b k_1\left(x,t,\underline{u}_1,\overline{u}_1\right)dt\\ p^2:\overline{u}_2=\overline{u}_1+\int_a^b k_2\left(x,t,\underline{u}_1,\overline{u}_1\right)dt\end{cases}$$

$$\vdots$$

and in general we have

$$\begin{cases} \underline{u}_0(x,\alpha)=0, \quad \underline{u}_1(x,\alpha)=\underline{f}(x,\alpha) \\ \underline{u}_{n+1}(x,\alpha)=\underline{u}_n(x,\alpha)-\int_a^b k_1\left(x,t,\underline{u}_n(t,\alpha),\overline{u}_n(t,\alpha)\right)\,dt \end{cases}$$

$$\begin{cases} \overline{u}_0(x,\alpha)=0, \quad \overline{u}_1(x,\alpha)=\overline{f}(x,\alpha) \\ \overline{u}_{n+1}(x,\alpha)=\overline{u}_n(x,\alpha)-\int_a^b k_2\left(x,t,\underline{u}_n(t,\alpha),\overline{u}_n(t,\alpha)\right)\,dt \end{cases}$$

Then if $k(x,t)\geq 0$ we have

$$\begin{cases} \underline{u}_0(x,\alpha)=0, \quad \underline{u}_1(x,\alpha)=\underline{f}(x,\alpha) \\ \underline{u}_{n+1}(x,\alpha)=\underline{u}_n(x,\alpha)-\int_a^b k_1\left(x,t\right)\underline{u}_n(t,\alpha)\,dt \end{cases}$$

$$\begin{cases} \overline{u}_0(x,\alpha)=0, \quad \overline{u}_1(x,\alpha)=\overline{f}(x,\alpha) \\ \overline{u}_{n+1}(x,\alpha)=\overline{u}_n(x,\alpha)-\int_a^b k_2\left(x,t\right)\,\overline{u}_n(t,\alpha)\,dt \end{cases}$$

and if $k(x,t)<0$ we have

$$\begin{cases} \underline{u}_0(x,\alpha)=0, \quad \underline{u}_1(x,\alpha)=\underline{f}(x,\alpha) \\ \underline{u}_{n+1}(x,\alpha)=\underline{u}_n(x,\alpha)-\int_a^b k_1\left(x,t\right)\,\overline{u}_n(t,\alpha)\,dt \end{cases}$$

$$\begin{cases} \overline{u}_0(x,\alpha)=0, \quad \overline{u}_1(x,\alpha)=\overline{f}(x,\alpha) \\ \overline{u}_{n+1}(x,\alpha)=\overline{u}_n(x,\alpha)-\int_a^b k_1\left(x,t\right)\,\underline{u}_n(t,\alpha)dt \end{cases}$$

HMP for solving Fuzzy Fredholm Integral Equation of the second kind (FFIE-2)

Taking

$$\begin{cases} F(\underline{u})=\underline{u}(x,\alpha)-\underline{f}(x,\alpha) \\ F(\overline{u})=\overline{u}(x,\alpha)-\overline{f}(x,\alpha) \end{cases}$$

and substituting (11) in (10)

$$\begin{cases} H(\underline{u},p)=\underline{u}(x,\alpha)-\underline{f}(x,\alpha)-p\int_a^b k_1\left(x,t,\underline{u}(t,),\overline{u}(t,\alpha)\right)\,dt=0 \\ H(\overline{u},p)=\overline{u}(x,\alpha)-\overline{f}(x,\alpha)-p\int_a^b k_2\left(x,t,\underline{u}(t,\alpha),\overline{u}(t,\alpha)\right)\,dt=0 \end{cases}$$

and equating the terms with identical power of p, we obtain

$$\begin{cases} p^0:\underline{u}_0-\underline{f}(x,\alpha)=0\Rightarrow\underline{u}_0=\underline{f}(x,\alpha) \\ p^0:\overline{u}_0-\overline{f}(x,\alpha)=0\Rightarrow\overline{u}_0=\overline{f}(x,\alpha) \end{cases}$$

$$\begin{cases} p^1:\underline{u}_1-\int_a^b k_1\left(x,t,\underline{u}_0,\overline{u}_0\right)\,dt=0\Rightarrow\underline{u}_1=\int_a^b k_1\left(x,t,\underline{u}_0,\overline{u}_0\right)\,dt \\ p^1:\overline{u}_1-\int_a^b k_2\left(x,t,\underline{u}_0,\overline{u}_0\right)\,dt=0\Rightarrow\overline{u}_1=\int_a^b k_2\left(x,t,\underline{u}_0,\overline{u}_0\right)\,dt \end{cases}$$

and in general we have

$$\begin{cases} \underline{u}_0(x,\alpha) = \underline{f}(x,\alpha) \\ \underline{u}_{n+1}(x,\alpha) = \int_a^b k_1\left(x,t,\underline{u}_n(t,\alpha),\overline{u}_n(t,\alpha)\right)\, dt, \quad n = 1,2,\ldots \end{cases}$$

$$\begin{cases} \overline{u}_0(x,\alpha) = \overline{f}(x,\alpha) \\ \overline{u}_{n+1}(x,\alpha) = \int_a^b k_2\left(x,t,\underline{u}_n(t,\alpha),\overline{u}_n(t,\alpha)\right)\, dt, \quad n = 1,2,\ldots \end{cases}$$

Then if $k(x,t) \geq 0$ we have

$$\begin{cases} \underline{u}_0(x,\alpha) = \underline{f}(x,\alpha) \\ \underline{u}_{n+1}(x,\alpha) = \int_a^b k\left(x,t\right)\, \underline{u}_n(t,\alpha)dt, \qquad n = 1,2,\ldots \end{cases}$$

$$\begin{cases} \overline{u}_0(x,\alpha) = \overline{f}(x,\alpha) \\ \overline{u}_{n+1}(x,\alpha) = \int_a^b k\left(x,t\right)\, \overline{u}_n(t,\alpha)\, dt, \qquad n = 1,2,\ldots \end{cases}$$

and if $k(x,t) < 0$ we have

$$\begin{cases} \underline{u}_0(x,\alpha) = \underline{f}(x,\alpha) \\ \underline{u}_{n+1}(x,\alpha) = \int_a^b k\left(x,t\right)\, \overline{u}_n(t,\alpha)\, dt, \qquad n = 1,2,\ldots \end{cases}$$

$$\begin{cases} \overline{u}_0(x,\alpha) = \overline{f}(x,\alpha) \\ \overline{u}_{n+1}(x,\alpha) = \int_a^b k\left(x,t\right)\, \underline{u}_n(t,\alpha)\, dt, \qquad n = 1,2,\ldots \end{cases}$$

5 Numerical Results

In this section, we are going to apply the HPM for solving several examples. The results are compared with exact solutions by using defined metric in Definition 3.

Example 5.1 Consider the fuzzy Fredholm integral equation with

$$\underline{f}(x,\alpha) = (\frac{\alpha+3}{4})e^{3x} - \frac{1}{9}\left(2e^3+1\right)x$$

$$\overline{f}(x,\alpha) = (\frac{5-\alpha}{4})e^{3x} - \frac{1}{9}\left(2e^3+1\right)x$$

and kernel

$$k(x,t) = xt, \qquad 0 \leq x,t \leq 1$$

and $a = 0, b = 1$. The exact solution in this cases is given by

$$\underline{u}(x,\alpha) = (\frac{\alpha+3}{4})e^{3x},$$

$$\overline{u}(x,\alpha) = (\frac{5-\alpha}{4})e^{3x}.$$

We apply $H(u,p)$ method to approximate the solutions:
we have

$$\begin{cases} \underline{u}_0(x,\alpha) = \underline{f}(x,\alpha) \\ \underline{u}_{n+1}(x,\alpha) = \int_a^b k(x,t)\ \underline{u}_n(t,\alpha)\, dt, & n = 1,2,\ldots \end{cases}$$

$$\begin{cases} \overline{u}_0(x,\alpha) = \overline{f}(x,\alpha) \\ \overline{u}_{n+1}(x,\alpha) = \int_a^b k(x,t)\ \overline{u}_n(t,\alpha)\, dt, & n = 1,2,\ldots \end{cases}$$

So we obtain

$$\underline{u}_0(x,\alpha) = (\frac{\alpha+3}{4})e^{3x} - \frac{1}{9}(2e^3+1)x,$$

$$\underline{u}_1(x,\alpha) = \int_0^1 xt\underline{u}_0(t,\alpha)dt = x\int_0^1 t\left[\frac{\alpha+3}{4}e^{3t} - \frac{1}{9}(2e+1)t\right]dt = \left[\frac{1}{9} + \frac{1}{27}\left[(\alpha+3)e^3 - 1\right]\right]x,$$

$$\underline{u}_2(x,\alpha) = \frac{1}{3}\left[\frac{1}{9} + \frac{1}{27}\left[(\alpha+3)e^3 - 1\right]\right]x,$$

$$\underline{u}_3(x,\alpha) = \frac{1}{3^2}\left[\frac{1}{9} + \frac{1}{27}\left[(\alpha+3)e^3 - 1\right]\right]x,$$

$$\underline{u}_4(x,\alpha) = \frac{1}{3^3}\left[\frac{1}{9} + \frac{1}{27}\left[(\alpha+3)e^3 - 1\right]\right]x,$$

$$\vdots$$

And

$$\overline{u}_0(x,\alpha) = (\frac{5-\alpha}{4})e^{3x} - \frac{1}{9}(2e^3+1)x,$$

$$\overline{u}_1(x,\alpha) = \int_0^1 xt\overline{u}_0(t,\alpha)dt = x\int_0^1 t\left[\frac{5-\alpha}{4}e^{3t} - \frac{1}{9}(2e+1)t\right]dt = \left[\frac{1}{9} + \frac{1}{27}\left[(5-\alpha)e^3 - 1\right]\right]x,$$

$$\overline{u}_2(x,\alpha) = \frac{1}{3}\left[\frac{1}{9} + \frac{1}{27}\left[(5-\alpha)e^3 - 1\right]\right]x,$$

$$\overline{u}_3(x,\alpha) = \frac{1}{3^2}\left[\frac{1}{9} + \frac{1}{27}\left[(5-\alpha)e^3 - 1\right]\right]x,$$

$$\overline{u}_4(x,\alpha) = \frac{1}{3^3}\left[\frac{1}{9} + \frac{1}{27}\left[(5-\alpha)e^3 - 1\right]\right]x,$$

$\vdots$

Hence the solution will be as follows:

$$\underline{u}(x,\alpha)=\underline{u}_0+\underline{u}_1+\underline{u}_2+\cdots=(\frac{\alpha+3}{4})e^{3x}-\frac{1}{9}(2e^3+1)x+\left(1+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\cdots\right)\left[\frac{1}{9}+\frac{1}{27}\left[(\alpha+3)e^3-1\right]\right]x$$

$$=(\frac{\alpha+3}{4})e^{3x}$$

$$\overline{u}(x,\alpha)=\overline{u}_0+\overline{u}_1+\overline{u}_2+\cdots=(\frac{5-\alpha}{4})e^{3x}-\frac{1}{9}(2e^3+1)x+\left(1+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\cdots\right)\left[\frac{1}{9}+\frac{1}{27}\left[(5-\alpha)e^3-1\right]\right]x$$

$$=(\frac{5-\alpha}{4})e^{3x}$$

$$u(x)=\left(\underline{u}(x,\alpha),\overline{u}(x,\alpha)\right)=\left(\frac{\alpha+3}{4}e^{3x},\frac{5-\alpha}{4}e^{3x}\right)=\left(\frac{\alpha+3}{4},\frac{5-\alpha}{4}\right)e^{3x}=\tilde{1}.e^{3x}$$

Example 5.2 [4] **.** Consider the fuzzy Fredholm integral equation with

$$\underline{f}(x,\alpha)=\sin(x/2)(13/15(\alpha^2+\alpha)+2/15(4-\alpha^3-\alpha)),$$

$$\bar{f}(x,\alpha)=\sin(x/2)(2/15(\alpha^2+\alpha)+13/15(4-\alpha^3-\alpha)),$$

and kernel

$$k(x,t)=0.1\sin(t)\sin(x/2),\qquad 0\le x,t\le 2\pi,$$

and $a=0, b=2\pi$. The exact solution in this cases is given by

$$\underline{u}(x,\alpha)=(\alpha^2+\alpha)\sin(x/2),$$

$$\overline{u}(x,\alpha)=(4-\alpha^3-\alpha)\sin(x/2).$$

We apply $H(u,p)$ method to approximate the solution:
we have

$$\underline{u}_0(x,\alpha)=1/15\sin(1/2x)(13\alpha^2+11\alpha+8-2\alpha^3),$$

$$\underline{u}_1(x,\alpha)=22/225\sin(1/2x)(\alpha^2+2\alpha-4+\alpha^3),$$

$$\underline{u}_2(x,\alpha)=88/3375\sin(1/2x)(\alpha^2+2\alpha-4+\alpha^3),$$

$$\underline{u}_3(x,\alpha)=352/50625\sin(1/2x)(\alpha^2+2\alpha-4+\alpha^3),$$

$\vdots$

and

$$\overline{u}_0(x,\alpha) = 1/15\sin(1/2x)(2\alpha^2 + 11\alpha + 52 - 13\alpha^3),$$

$$\overline{u}_1(x,\alpha) = -22/225\sin(1/2x)(\alpha^2 + 2\alpha - 4 + \alpha^3),$$

$$\overline{u}_2(x,\alpha) = -88/3375\sin(1/2x)(\alpha^2 + 2\alpha - 4 + \alpha^3),$$

$$\overline{u}_3(x,\alpha) = -352/50625\sin(1/2x)(\alpha^2 + 2\alpha - 4 + \alpha^3),$$

$\vdots$

then we approximate

$$\underline{u}(x,\alpha) = -\frac{1}{50625}(50497\alpha^3 - 128\alpha^2 + 50369\alpha - 201988)\times\sin(1/2x),$$

$$\overline{u}(x,\alpha) = -\frac{1}{50625}(128\alpha^3 - 50497\alpha^2 - 50369\alpha - 512)\times\sin(1/2x).$$

6 Conclusion

In this paper we illustrated homotopy perturbation method for solving fuzzy Fredholm integral equation. This method is a simple and very effective tool for calculating the exact solutions.

References

[1] C. Wu, M. Ma, On the integrals, series and integral equations of fuzzy set-valued functions, J. Harbin Inst. Technol. 21 (1990) 11-19.
[2] Chun C. Integration using He s homotopy perturbation method. Chaos, Solitons & Fractals 2007; 34(4):1130-4.
[3] Cveticanin L. Homotopy perturbation method for pure nonlinear differential equation. Chaos, Solitons & Fractals 2006; 30(5):1221-30.
[4] M. Friedman, M. Ma, A. Kandel, Numerical solution of fuzzy differential and integral equations, Fuzzy Sets Sy stems 106 (1999) 35-48.
[5] R. Goestscel, W. Voxman, Elementary calculus, Fuzzy Sets Systems 18 (1986) 31-34.
[6] H. Hochstadt, Integral Equations, Wiley, New York, 1973.
[7] He J-H. A coupling method of homotopy technique and perturbation technique for nonlinear problems. Int Nonlinear Mech 2000:35(1):37-43.
[8] He J-H. homotopy perturbation technique. Comput Methods Appl Mech Eng 1999;173(3-4):257-62.
[9] Hillermeier C. Generalized homotopy approach to multiobjective optimization. Int J Optim Theory Appl 2001;110(3):557-83.
[10] O. Kaleva, Fuzzy differential equations, Fuzzy Sets Systems 24 (1987) 301-317.
[11] Liao SJ. An approximate solution technique not depending on small parameters: a special example. Int J Non-Linear Mech 1995;30(3):371-80.
[12] Liao SJ. Boundary element method for general nonlinear differential operators. Eng Anal Boundary Element 1997;20(2):91-9.
[13] Nayef AH. Problem in perturbation. New york: John Wiley; 1985.
[14] H.T. Nguyen, Anote on the extension principle for fuzzy sets, J. Math. Anal. Appl. 64 (1978) 369-380.
[15] M.L. Puri, D. Ralescu, Differential for fuzzy function, J. Math. Anal. Appl. 91 (1983) 552-558.
[16] M.L. Puri, D. Ralescu, fuzzy random variables, J. Math. Anal. Appl. 114 (1986) 409-422.

[17] Siddiqui AM, Mahmood R, Ghori QK. Homotopy perturbation method for thin film flow of a third fluid down an inclined plane. Chos, Solitons & Fractals 2008;35(1):140-7.
[18] Ozis T, Yildirim A. A note on He s homotopy perturbation method for van der Pol oscillator with very strong nonlinearity. Chaos, Solitons & Fractals 2007;34(3):989-91.