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Solving Linear Fred Holm Fuzzy Integral Equations of the Second Kind by Modified Trapezoidal Method

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Abstract

One of the methods for solving definite integrals is modified trapezoid method, which is obtained by using Hermitian interpolation (see e.g. [12]). In this article, we have used modified trapezoid quadrature method and Generalized differential to solve the Fredholm fuzzy integral equations of the second kind. This method leads to solve fuzzy linear system. Finally the proposed method is illustrated by solving some numerical examples.

Keywords: Modified Trapezoid Method, Fuzzy Linear System, Generalized Differential.

1 Introduction

Topics of fuzzy integral equations (FIE) which growing interest for some time, in particular in relation to fuzzy control, have been rapidly developed in resent years.
Prior to discuss fuzzy integral equations and their associated numerical algorithms, it is necessary to present an appropriate brief introduction to preliminary topics such as fuzzy number and fuzzy calculus.
The concept of integration of fuzzy functions was first introduced by Dubois and Prade [2] and investigated by Goestschel and Voxman [7], Kaleva [1], Nanda [8] and others. One of the first applications of fuzzy integration was given by Wu and Ma who investigated the fuzzy Fredholm integral equation of the second kind (FFIE-2). The numerical solutions of FFIE-2 are introduced by Allahviranloo et.al. in[13,14,15].
In this work we concentrate on numerical procedure for solving FFIE-2. In Section 2 we bring some basic definitions of fuzzy subsets and distance between fuzzy numbers. In Section 3 we explain modified trapezoidal method as a numerical method for solving the system of FFIE-2. In Section 4, we have some examples to illustrate the mentioned method and the conclusion is drawn in Section 5.

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2 Preliminaries

We begin this section with defining the notation we will use in the paper. Let us denote by $\Re_f$ the class of fuzzy subsets of the real axis $u:\Re \to [0,1]$, satisfying the following properties:

(i) u is normal, i.e. $\exists x_0 \in \Re$ with $u(x_0) = 1$;

(ii) u is convex fuzzy set(i.e. $u(tx+(1-t)y \geq \min\{u(x),u(y)\}, \forall t \in [0,1], x, y \in \Re$);

(iii) u is upper semi continuous on $\Re$;

(iv) $\{x \in \Re : u(x) > 0\}$ is compact, where $\overline{A}$ denotes the closure of A.

Then $\Re_f$ is called the space of fuzzy numbers (see e.g.[4]). Obviously $\Re \subset \Re_f$.
Here $\Re \subset \Re_f$ is understood as $\Re = \{\chi_{\{x\}};$ x is usual real number$\}$. For $0 < r \leq 1$, denote $[u]^r = \{x \in \Re; u(x) \geq r\}$ and $[u]^0 = \{x \in \Re; u(x) > 0\}$. Then it is well-known that for any $r \in [0,1]$, $[u]^r$ is a bounded closed interval. For $u, v \in \Re_f$, and $\lambda \in \Re$, the sum $u \oplus v$ and the product $\lambda \otimes u$ are defined by $[u \oplus v]^r = [u]^r + [v]^r$, $[\lambda \otimes u]^r = \lambda [u]^r, \forall r \in [0,1]$, where $[u]^r + [v]^r = \{x + y : x \in [u]^r, y \in [v]^r\}$ means the usual addition of two intervals (subsets) of $\Re$ and $\lambda[u]^r = \{\lambda x : x \in [u]^r\}$ means the usual product between a scalar and a subset of $\Re$ (see e.g. [4,5]).
Let $D : \Re_f \times \Re_f \to \Re_+ \cup \{0\}$, $D(u,v) = \sup_{r\in[0,1]} \max\{|u_-^r - v_-^r|, |u_+^r - v_+^r|\}$ be the Hausdorff distance between fuzzy numbers, where $[u]^r = [u_-{}^r, u_+{}^r]$, $[v]^r = [v_-{}^r, v_+{}^r]$.
The following properties are well-known (see e.g. [5,6]):

$$D(u \oplus w, v \oplus w) = D(u,v), \forall u, v, w \in \Re_f,$$
$$D(k \otimes u, k \otimes v) = |k| D(u,v), \forall k \in \Re, u, v \in \Re_f,$$
$$D(u \oplus v, w \oplus e) \leq D(u,w) + D(v,e), \forall u, v, w, e \in \Re_f$$

and $(\Re_f, D)$ is a complete metric space.
Also are known the following results and concepts.

Definition 2.1 Let $F:(a,b) \to \Re_f$ and $t_\circ \in (a,b)$. We say that F is differentiable at $t_\circ$; if we have 2 forms as follows:
1) It exists an element $F'(t_0) \in \Re_f$ such that, for all h>0 sufficiently near to 0, there are $F(t_\circ + h) - F(t_\circ), F(t_\circ) - F(t_\circ - h)$ and the limits:

$$\lim_{h\to 0^+} \frac{F(t_\circ + h) - F(t_\circ)}{h} = \lim_{h\to 0^+} \frac{F(t_\circ) - F(t_\circ - h)}{h} = F'(t_\circ) \qquad (1)$$

2) It exists an element $F'(t_0) \in \Re_f$ such that, for all $h < 0$ sufficiently near to 0, there are

$$\lim_{h\to 0^-} \frac{F(t_\circ + h) - F(t_\circ)}{h} = \lim_{h\to 0^-} \frac{F(t_\circ) - F(t_\circ - h)}{h} = F'(t_\circ) \qquad (2)$$

Theorem 2.1 Let $F:(a,b)\rightarrow \Re_f$ be a function and denote $[F(t)]^{\alpha}=[f_-^{\alpha}(t), f_+^{\alpha}]$, for each $\alpha \in [0,1]$. Then

(i) If F is differentiable in the first form 1, then f_{α} and g_{α} are differentiable functions and

$$[F'(t)]^{\alpha}=[f_-^{\alpha'}, f_+^{\alpha'}] \tag{3}$$

(ii) If F is differentiable in the second form 2, then f_{α} and g_{α} are differentiable functions and

$$[F'(t)]^{\alpha}=[f_+^{\alpha'}, f_-^{\alpha'}] \tag{4}$$

Proof: see [9].

Definition 2.2 (see e.g. [10]). Let $x, y \in \Re_f$, if there exists $z \in \Re_f$ such that $x = y \oplus z$, then z is called the H-difference of x and y and it is denoted by $x - y$.

Definition 2.3 Let f: $(a, b)\rightarrow \Re_f$ and $x_0 \in (a, b)$. We say that f is strongly generalized differentiable at x_0, if there exists an element $f'(x_0) \in \Re_f$, such that

(i) for all $h > 0$ sufficiently small, $\exists f(x_0+h)-f(x_0), f(x_0)-f(x_0-h)$ and the limits (in the metric D)

$$\lim_{h\rightarrow 0}\frac{f(x_0+h)-f(x_0)}{h}=\lim\frac{f(x_0)-f(x_0-h)}{h}=f'(x_0), \tag{5}$$

or

(ii) for all $h > 0$ sufficiently small, $\exists f(x_0)-f(x_0+h), f(x_0-h)-f(x_0)$ and the limits

$$\lim_{h\rightarrow 0}\frac{f(x_0)-f(x_0+h)}{(-h)}=\lim\frac{f(x_0-h)-f(x_0)}{(-h)}=f'(x_0), \tag{6}$$

or

(iii) for all $h > 0$ sufficiently small, $\exists f(x_0+h)-f(x_0), f(x_0)-f(x_0-h)$ and the limit

$$\lim_{h\rightarrow 0}\frac{f(x_0+h)-f(x_0)}{h}=\lim\frac{f(x_0-h)-f(x_0)}{(-h)}=f'(x_0), \tag{7}$$

or

(iv) for all $h > 0$ sufficiently small, $\exists f(x_0)-f(x_0+h), f(x_0)-f(x_0-h)$ and the limits

$$\lim_{h\rightarrow 0}\frac{f(x_0)-f(x_0+h)}{(-h)}=\lim\frac{f(x_0)-f(x_0-h)}{h}=f'(x_0). \tag{8}$$

(h and (-h) at denominators mean $\frac{1}{h}\otimes$ and $-\frac{1}{h}\otimes$, respectively).

Before we clarify further, we define following notation for simplicity:

$$f_i = f(s_i), \qquad K_{ij} = K(s_i, s_j),$$
$$f_i' = f'(s_i), \qquad H_{ij} = H(s_i, s_j),$$

$J_{ij} = J(s_i, s_j), \qquad x_i = x(s_i),$
$x'_i = x'(s_i).$

3 Linear Fredholm fuzzy integral equations of the second kind

The general form of these equations is as follows:

$$x(s) = f(s) + \int_a^b k(s,t)x(t)dt, \tag{9}$$

where $K(s,t)$ is a arbitrary given fuzzy function over the square $a \le s,t \le b$ and $f(s)$ is a given fuzzy function of $s \in [a,b]$. If for solving (9), we approximate the right-hand integral (9), with the repeated modified method, we have

$$x(s) = f(s) + \frac{h}{2}K(s,s_0)x_0 + h\sum_{j=1}^{n-1} K(s,s_j)x_j + \frac{h}{2}K(s,s_n)x_n \tag{10}$$
$$+\frac{h^2}{12}\left[J(s,s_0)x_0 + K(s,s_0)x'_0 - J(s,s_n)x_n + K(s,s_n)x'_n\right]$$

where $J(s,t) = \frac{\partial K(s,t)}{\partial t}$. By using *r*-cuts form of fuzzy functions we consider three cases. In all of the cases we suppose that J_{ij} is positive for all *i,j* .

Case 1 If $x'_\circ$ and x'_n are differentiable in the first form 1 and
$K_{ij} \ge 0, j = 0,\dots,k, K_{ij} \le 0, j = k+1,\dots,n:$

$$\left[\underline{x}(s_i,r),\overline{x}(s_i,r)\right] = \left[\underline{f}(s_i,r),\overline{f}(s_i,r)\right] + \frac{h}{2}K_{i0}\left[\underline{x}(s_0,r),\overline{x}(s_0,r)\right]$$
$$+h\sum_{j=1}^{k}K_{ij}\left[\underline{x}(s_j,r),\overline{x}(s_j,r)\right] + h\sum_{j=k+1}^{n-1}K_{ij}\left[\overline{x}(s_j,r),\underline{x}(s_j,r)\right] +$$
$$+\frac{h}{2}K_{in}\left[\overline{x}(s_n,r),\underline{x}(s_n,r)\right] + \frac{h^2}{12}[J_{i0}\left[\underline{x}(s_0,r),\overline{x}(s_0,r)\right] + K_{i0}\left[\underline{x'}(s_0,r),\overline{x'}(s_0,r)\right]$$
$$-J_{in}\left[\underline{x}(s_n,r),\overline{x}(s_n,r)\right] + K_{in}\left[\overline{x'}(s_n,r),\underline{x'}(s_n,r)\right]]$$

Then we have: (11)

$$\underline{x}(s_i,r) = \underline{f}(s_i,r) + \left[\frac{h}{2}K_{i0} + \frac{h^2}{12}J_{i0}\right]\underline{x}(s_0,r) + h\sum_{j=1}^{k}K_{ij}\underline{x}(s_j,r)$$
$$+h\sum_{j=k+1}^{n-1}K_{ij}\overline{x}(s_j,r) + [\frac{h}{2}K_{in} - \frac{h^2}{12}J_{in}]\overline{x}(s_n,r) + \frac{h^2}{12}[K_{i0}\underline{x}'(s_0,r) + K_{in}\overline{x'}(s_n,r)]$$

$$\overline{x}(s_i,r)=\overline{f}(s_i,r)+\left[\frac{h}{2}K_{i0}+\frac{h^2}{12}J_{i0}\right]\overline{x}(s_0,r)+h\sum_{j=1}^{k}K_{ij}\overline{x}(s_j,r)$$
$$+h\sum_{j=k+1}^{n-1}K_{ij}\underline{x}(s_j,r)+[\frac{h}{2}K_{in}-\frac{h^2}{12}J_{in}]\underline{x}(s_n,r)+\frac{h^2}{12}[K_{i0}\overline{x}'(s_0,r)+K_{in}\underline{x}'(s_n,r)]. \tag{12}$$

Case 2 If $x'_\circ$ and x'_n are differentiable in the second form 2 and $K_{ij}\geq 0, j=0,\ldots,k, K_{ij}\leq 0, j=k+1,\ldots,n$ then

$$\left[\underline{x}(s_i,r),\overline{x}(s_i,r)\right]=\left[\underline{f}(s_i,r),\overline{f}(s_i,r)\right]+\frac{h}{2}K_{i0}\left[\underline{x}(s_0,r),\overline{x}(s_0,r)\right]$$
$$+h\sum_{j=1}^{k}K_{ij}\left[\underline{x}(s_j,r),\overline{x}(s_j,r)\right]+h\sum_{j=k+1}^{n-1}K_{ij}\left[\overline{x}(s_j,r),\underline{x}(s_j,r)\right]+$$
$$+\frac{h}{2}K_{in}\left[\overline{x}(s_n,r),\underline{x}(s_n,r)\right]+\frac{h^2}{12}[J_{i0}\left[\underline{x}(s_0,r),\overline{x}(s_0,r)\right]+K_{i0}\left[\overline{x'}(s_0,r),\underline{x'}(s_0,r)\right]$$
$$-J_{in}\left[\underline{x}(s_n,r),\overline{x}(s_n,r)\right]+K_{in}\left[\underline{x'}(s_n,r),\overline{x'}(s_n,r)\right]].$$

Also we have:

$$\underline{x}(s_i,r)=\underline{f}(s_i,r)+\left[\frac{h}{2}K_{i0}+\frac{h^2}{12}J_{i0}\right]\underline{x}(s_i,r)+h\sum_{j=1}^{k}K_{ij}\underline{x}(s_j,r)+h\sum_{j=k+1}^{n-1}K_{ij}\overline{x}(s_j,r)$$
$$+[\frac{h}{2}K_{in}-\frac{h^2}{12}J_{in}]\overline{x}(s_n,r)+\frac{h^2}{12}[K_{i0}\overline{x'}(s_0,r)+K_{in}\underline{u}'(x_n,r)] \tag{13}$$

$$\overline{x}(s_i,r)=\overline{f}(s_i,r)+\left[\frac{h}{2}K_{i0}+\frac{h^2}{12}J_{i0}\right]\overline{x}(s_i,r)+h\sum_{j=1}^{k}K_{ij}\overline{x}(s_j,r)+h\sum_{j=k+1}^{n-1}K_{ij}\underline{x}(s_j,r)$$
$$+[\frac{h}{2}K_{in}-\frac{h^2}{12}J_{in}]\underline{x}(s_n,r)+\frac{h^2}{12}[K_{i0}\underline{x}'(s_0,r)+K_{in}\overline{x'}(s_n,r)]. \tag{14}$$

Case 3 If $x'_\circ$ is differentiable in the first form 1 and x'_n is differentiable in the second form 2 and $K_{ij}\geq 0, j=0,\ldots,k, K_{ij}\leq 0, j=k+1,\ldots,n$ then

$$\left[\underline{x}(s_i,r),\overline{x}(s_i,r)\right]=\left[\underline{f}(s_i,r),\overline{f}(s_i,r)\right]+\frac{h}{2}K_{i0}\left[\underline{x}(s_0,r),\overline{x}(s_0,r)\right]$$
$$+h\sum_{j=1}^{k}K_{ij}\left[\underline{x}(s_j,r),\overline{x}(s_j,r)\right]+h\sum_{j=k+1}^{n-1}K_{ij}\left[\overline{x}(s_j,r),\underline{x}(s_j,r)\right]+$$
$$+\frac{h}{2}K_{in}\left[\overline{x}(s_n,r),\underline{x}(s_n,r)\right]+\frac{h^2}{12}[J_{i0}\left[\underline{x}(s_0,r),\overline{x}(s_0,r)\right]+K_{i0}\left[\underline{x'}(s_0,r),\overline{x'}(s_0,r)\right]$$
$$-J_{in}\left[\underline{x}(s_n,r),\overline{x}(s_n,r)\right]+K_{in}\left[\underline{x'}(s_n,r),\overline{x'}(s_n,r)\right]].$$

Also we have:

$$\underline{x}(s_i,r)=\underline{f}(s_i,r)+\left[\frac{h}{2}K_{i0}+\frac{h^2}{12}J_{i0}\right]\underline{x}(s_0,r)+h\sum_{j=1}^{k}K_{ij}\underline{x}(s_j,r)+h\sum_{j=k+1}^{n-1}K_{ij}\overline{x}(s_j,r)$$
$$+[\frac{h}{2}K_{in}-\frac{h^2}{12}J_{in}]\overline{x}(s_n,r)+\frac{h^2}{12}[K_{i0}\underline{x}'(s_0,r)+K_{in}\underline{x}'(s_n,r)] \tag{15}$$

$$\overline{x}(s_i,r)=\overline{f}(s_i,r)+\left[\frac{h}{2}K_{i0}+\frac{h^2}{12}J_{i0}\right]\overline{x}(s_0,r)+h\sum_{j=1}^{k}K_{ij}\overline{x}(s_j,r)+h\sum_{j=k+1}^{n-1}K_{ij}\underline{x}(s_j,r)$$
$$+[\frac{h}{2}K_{in}-\frac{h^2}{12}J_{in}]\underline{x}(s_n,r)+\frac{h^2}{12}[K_{i0}\overline{x'}(s_0,r)+K_{in}\overline{x'}(s_n,r)]. \tag{16}$$

By taking derivative form Eq. (9) and setting $H(s,t)=\dfrac{\partial K(s,t)}{\partial s}$ we obtain

$$x'(s)=f'(s)+\int_a^b H(s,t)x(t)dt,,\qquad a\le s\le b \tag{17}$$

We note that if x be a solution of (9) it is a solution of (17) too.

Consider the partial derivative $\dfrac{\partial K^2(s,t)}{\partial s\partial t}$ does not exist.
In this case, we solve Eq. (17) with repeated trapezoid method, so we have

$$x'(s)=f'(s)+\frac{h}{2}H(s,s_0)x_0+h\sum_{j=1}^{n-1}H(s,s_j)x_j+\frac{h}{2}H(s,s_n)x_n.$$

Now by using r-cuts form of fuzzy functions we consider three cases:

Case 1 If x' and f' are differentiable in the first form 1 and
$H_{ij}\ge 0, j=0,\ldots,k, H_{ij}\le 0, j=k+1,\ldots,n$:

$$\left[\underline{x}'(s_i,r),\overline{x}'(s_i,r)\right]=\left[\underline{f}'(s_i,r),\overline{f}'(s_i,r)\right]+\frac{h}{2}H_{i0}\left[\underline{x}(s_0,r),\overline{x}(s_0,r)\right]$$

$$+h\sum_{j=1}^{k}H_{ij}\left[\underline{x}(s_j,r),\overline{x}(s_j,r)\right]+h\sum_{j=k+1}^{n-1}H_{ij}\left[\overline{x}(s_j,r),\underline{x}(s_j,r)\right]+\frac{h}{2}H_{in}\left[\overline{x}(s_n,r),\underline{x}(s_n,r)\right]$$

Then we have

$$\underline{x}'(s_i,r)=\underline{f}'(s_i,r)+\frac{h}{2}H_{i0}\underline{x}(s_0,r)+h\sum_{j=1}^{k}H_{ij}\underline{x}(s_j,r)+h\sum_{j=k+1}^{n-1}H_{ij}\overline{x}(s_j,r)+\frac{h}{2}H_{in}\overline{x}(s_n,r) \quad (18)$$

$$\overline{x}'(s_i,r)=\overline{f}'(s_i,r)+\frac{h}{2}H_{i0}\overline{x}(s_0,r)+h\sum_{j=1}^{k}H_{ij}\overline{x}(s_j,r)+h\sum_{j=k+1}^{n-1}H_{ij}\underline{x}(s_j,r)+\frac{h}{2}H_{in}\underline{x}(s_n,r). \quad (19)$$

Case 2 If x' and f' are differentiable in the second form 2 and $H_{ij}\geq 0, j=0,\ldots,k, H_{ij}\leq 0, j=k+1,\ldots,n$:

$$\left[\overline{x}'(s_i,r),\underline{x}'(s_i,r)\right]=\left[\overline{f}'(s_i,r),\underline{f}'(s_i,r)\right]+\frac{h}{2}H_{i0}\left[\underline{x}(s_0,r),\overline{x}(s_0,r)\right]$$

$$+h\sum_{j=1}^{k}H_{ij}\left[\underline{x}(s_j,r),\overline{x}(s_j,r)\right]+h\sum_{j=k+1}^{n-1}H_{ij}\left[\overline{x}(s_j,r),\underline{x}(s_j,r)\right]+\frac{h}{2}H_{in}\left[\overline{x}(s_n,r),\underline{x}(s_n,r)\right]$$

Then we have

$$\overline{x}'(s_i,r)=\overline{f}'(s_i,r)+\frac{h}{2}H_{i0}\underline{x}(s_i,r)+h\sum_{j=1}^{k}H_{ij}\underline{x}(s_j,r)+h\sum_{j=k+1}^{n-1}H_{ij}\overline{x}(s_j,r)+\frac{h}{2}H_{in}\overline{x}(s_n,r) \quad (20)$$

$$\underline{x}'(s_i,r)=\underline{f}'(s_i,r)+\frac{h}{2}H_{i0}\overline{x}(s_i,r)+h\sum_{j=1}^{k}H_{ij}\overline{x}(s_j,r)+h\sum_{j=k+1}^{n-1}H_{ij}\underline{x}(s_j,r)+\frac{h}{2}H_{in}\underline{x}(s_n,r)]. \quad (21)$$

Case 3 If x' is differentiable in the first form 1 and f' is differentiable in the second form (2) and $H_{ij}\geq 0, j=0,\ldots,k, H_{ij}\leq 0, j=k+1,\ldots,n$

$$\left[\underline{x}'(s_i,r),\overline{x}'(s_i,r)\right]=\left[\overline{f}'(s_i,r),\underline{f}'(s_i,r)\right]+\frac{h}{2}H_{i0}\left[\underline{x}(s_0,r),\overline{x}(s_0,r)\right]$$

$$+h\sum_{j=1}^{k}H_{ij}\left[\underline{x}(s_j,r),\overline{x}(s_j,r)\right]+h\sum_{j=k+1}^{n-1}H_{ij}\left[\overline{x}(s_j,r),\underline{x}(s_j,r)\right]+\frac{h}{2}H_{in}\left[\overline{x}(s_n,r),\underline{x}(s_n,r)\right]$$

Then we have:

$$\underline{x}'(s_i,r)=\overline{f}'(s_i,r)+\frac{h}{2}H_{i0}\underline{x}(s_0,r)+h\sum_{j=1}^{k}H_{ij}\underline{x}(s_j,r)+h\sum_{j=k+1}^{n-1}H_{ij}\overline{x}(s_j,r)+\frac{h}{2}H_{in}\overline{x}(s_n,r) \quad (22)$$

$$\overline{x}'(s_i,r)=\underline{f}'(s_i,r)+\frac{h}{2}H_{i0}\overline{x}(s_0,r)+h\sum_{j=1}^{k}H_{ij}\overline{x}(s_j,r)+h\sum_{j=k+1}^{n-1}H_{ij}\underline{x}(s_j,r)+\frac{h}{2}H_{in}\underline{x}(s_n,r) \quad (23)$$

From system (18), we obtain two equations for i=0,n. These two equations with system (11) make the following system:

$$\begin{cases} \underline{x}(s_i,r)=\underline{f}(s_i,r)+\left[\frac{h}{2}K_{i0}+\frac{h^2}{12}J_{i0}\right]\underline{x}(s_0,r)+h\sum_{j=1}^{k}K_{ij}\underline{x}(s_j,r) \\ +h\sum_{j=k+1}^{n-1}K_{ij}\overline{x}(s_j,r)+[\frac{h}{2}K_{in}-\frac{h^2}{12}J_{in}]\overline{x}(s_n,r)+\frac{h^2}{12}[K_{i0}\underline{x'}(s_0,r)+K_{in}\overline{x'}(s_n,r)] \\ \underline{x'}(s_0,r)=\underline{f'}(s_0,r)+\frac{h}{2}H_{00}\underline{x}(s_0,r)+h\sum_{j=1}^{k}H_{0j}\underline{x}(s_j,r)+h\sum_{j=k+1}^{n-1}H_{0j}\overline{x}(s_j,r)+\frac{h}{2}H_{0n}\overline{x}(s_n,r) \\ \underline{x'}(s_n,r)=\underline{f'}(s_n,r)+\frac{h}{2}H_{n0}\underline{x}(s_0,r)+h\sum_{j=1}^{k}H_{nj}\underline{x}(s_j,r)+h\sum_{j=k+1}^{n-1}H_{nj}\overline{x}(s_j,r)+\frac{h}{2}H_{nn}\overline{x}(s_n,r) \end{cases} \quad (24)$$

Also from system (19), we obtain two equations for i=0,n. These two equations with system (12) make the following system:

$$\begin{cases} \overline{x}(s_i,r)=\overline{f}(s_i,r)+\left[\frac{h}{2}K_{i0}+\frac{h^2}{12}J_{i0}\right]\overline{x}(s_i,r)+h\sum_{j=1}^{k}K_{ij}\overline{x}(s_j,r) \\ +h\sum_{j=k+1}^{n-1}K_{ij}\underline{x}(s_j,r)+[\frac{h}{2}K_{in}-\frac{h^2}{12}J_{in}]\underline{x}(s_n,r)+\frac{h^2}{12}[K_{i0}\overline{x'}(s_0,r)+K_{in}\underline{x}'(s_n,r)] \\ \overline{x'}(s_0,r)=\overline{f'}(s_0,r)+\frac{h}{2}H_{00}\overline{x}(s_0,r)+h\sum_{j=1}^{k}H_{0j}\overline{x}(s_j,r)+h\sum_{j=k+1}^{n-1}H_{0j}\underline{x}(s_j,r)+\frac{h}{2}H_{0n}\underline{x}(s_n,r) \\ \overline{x'}(s_n,r)=\overline{f'}(s_n,r)+\frac{h}{2}H_{n0}\overline{x}(s_0,r)+h\sum_{j=1}^{k}H_{nj}\overline{x}(s_j,r)+h\sum_{j=k+1}^{n-1}H_{nj}\underline{x}(s_j,r)+\frac{h}{2}H_{nn}\underline{x}(s_n,r) \end{cases} \quad (25)$$

Where by composition system (24) and system (25) we obtain a system with 2(n+3) unknowns. Using matrix notation we get

$$\begin{bmatrix} A & B \\ B & A \end{bmatrix}\begin{bmatrix} \underline{\underline{x}} \\ \overline{x} \end{bmatrix}+\begin{bmatrix} -\underline{\underline{x}} \\ -\overline{x} \end{bmatrix}=\begin{bmatrix} \underline{\underline{f}} \\ \overline{f} \end{bmatrix} \quad (26)$$

Where

$$A=\begin{bmatrix} \frac{h}{2}K_{00}+\frac{h^2}{12}J_{00} & hK_{01} & \cdots & hK_{0k} & 0 & \cdots & 0 & 0 & \frac{h^2}{12}K_{00} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{h}{2}K_{n0}+\frac{h^2}{12}J_{n0} & hK_{n1} & \cdots & hK_{nk} & 0 & \cdots & 0 & 0 & \frac{h^2}{12}K_{n0} & 0 \\ \frac{h}{2}H_{00} & hH_{01} & \cdots & hH_{0k} & 0 & \cdots & 0 & 0 & 0 & 0 \\ \frac{h}{2}H_{n0} & hH_{n1} & \cdots & hH_{nk} & 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B=\begin{bmatrix} 0 & 0 & \cdots & 0 & hK_{0,k+1} & \cdots & hK_{0,n-1} & \frac{h}{2}K_{0n}-\frac{h^2}{12}J_{0n} & 0 & \frac{h^2}{12}K_{0n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & hK_{n,k+1} & \cdots & hK_{n,n-1} & \frac{h}{2}K_{nn}-\frac{h^2}{12}J_{nn} & 0 & \frac{h^2}{12}K_{nn} \\ 0 & 0 & \cdots & 0 & hH_{0,k+1} & \cdots & hH_{0,n-1} & \frac{h}{2}H_{0n} & 0 & 0 \\ 0 & 0 & \cdots & 0 & hH_{n,k+1} & \cdots & hH_{n,n-1} & \frac{h}{2}H_{nn} & 0 & 0 \end{bmatrix}$$

$$\underline{x}=\begin{bmatrix} \underline{x}(s_0,r) \\ \vdots \\ \underline{x}(s_k,r) \\ \underline{x}(s_{k+1},r) \\ \vdots \\ \underline{x}(s_n,r) \\ \underline{x}'(s_0,r) \\ \underline{x}'(s_n,r) \end{bmatrix}, \overline{x}=\begin{bmatrix} \overline{x}(s_0,r) \\ \vdots \\ \overline{x}(s_k,r) \\ \overline{x}(s_{k+1},r) \\ \vdots \\ \overline{x}(s_n,r) \\ \overline{x'}(s_0,r) \\ \overline{x'}(s_n,r) \end{bmatrix}, \quad \underline{f}=\begin{bmatrix} -\underline{f}(s_0,r) \\ \vdots \\ -\underline{f}(s_k,r) \\ -\underline{f}(s_{k+1},r) \\ \vdots \\ -\underline{f}(s_n,r) \\ -\underline{f'}(s_0,r) \\ -\underline{f'}(s_n,r) \end{bmatrix}, \quad \overline{f}=\begin{bmatrix} -\overline{f}(s_0,r) \\ \vdots \\ -\overline{f}(s_k,r) \\ -\overline{f}(s_{k+1},r) \\ \vdots \\ -\overline{f}(s_n,r) \\ -\overline{f'}(s_0,r) \\ -\overline{f'}(s_n,r) \end{bmatrix}$$

We have:

$$\begin{bmatrix} A-I & B \\ B & A-I \end{bmatrix}\begin{bmatrix} \underline{x} \\ \overline{x} \end{bmatrix}=\begin{bmatrix} \underline{f} \\ \overline{f} \end{bmatrix} \tag{27}$$

Related to properties above system is given [11]. Using system (3.19) the approximate solution of equation (10), in case 1 is achieved. Also from system (20), we obtain two equations for i=0,n and from system (21), we obtain two Other equations for i=0,n

where this four equations with two systems (13), (14) all together consist a system with 2(*n*+3) equations and 2(*n*+3) unknowns.

The above results satisfying in systems (13), (14), (15),(16). By using these systems we obtain the approximate solutions for two case 2, 3.

4 Numerical examples

Now we apply the method to solve two examples. We solve these examples by MATLAB.

Example 4.1 Consider the following fuzzy Fred Holm integral equation

$$\underline{f}(s, r) = 1 + 0.5(r - 1),$$

$$\underline{f}(s, r) = 1 + 0.5(r - 1)$$

and Kernel

$$K(s,t) = \frac{1}{1 + (s-t)^2}, \quad -1 \le s,\ t \le 1, \quad \lambda = 1$$

and a=1, b= - 1.

For $n = 3,\ h = \frac{1}{2}, s_0 = -1, s_1 = 1,\ s_2 = \frac{1}{2},\ s_3 = 1$ we obtain Matrix $\begin{bmatrix} A - I & B \\ B & A - I \end{bmatrix}$ as follow:

-0.7500	0.2500	0.1538	0	0.0208	0	0	0	0.1538	0.0533	0	0.0042
0.1354	-0.5000	0.4000	0	0.0104	0	0	0	0.4000	0.1354	0	0.0104
0.0828	0.4000	-0.5000	0	0.0064	0	0	0	0.5000	0.2133	0	0.0167
0.0533	0.2500	0.4000	-1.0000	0.0042	0	0	0	0.4000	0.2500	0	0.0208
0	0.2500	0	0	-1.0000	0	0	0	0.1420	0.0800	0	0
-0.0400	-0.2500	0	0	0	-1.0000	0	0	-0.3200	0	0	0
0	0	0.1538	0.0533	0	0.0042	-0.7500	0.2500	0.1538	0	0.0208	0
0	0	0.4000	0.1354	0	0.0104	0.1354	-0.5000	0.4000	0	0.0104	0
0	0	0.5000	0.2133	0	0.0167	0.0828	0.4000	-0.5000	-1.0000	0.0064	0
0	0	0.4000	0.2500	0	0.0208	0.0533	0.2500	0.4000	-1.0000	0.0042	0
0	0	0.1420	0.0800	0	0	0	0.2500	0	0	-1.0000	0
0	0	-0.3200	0	0	0	-0.0400	-0.2500	0	0	0	-1.0000

Also obtained solution for 10 *r*-level as *r*=0, 0.1, 0.2, …., 1 are given in table 1.

Table1. Numerical results of Example 4.1

	r=0	r=0.1	r=0.2	r=0.3	r=0.4	r=0.5	r=0.6	r=0.7	r=0.8	r=0.9	r=1
$\underline{x}(s_0;r)$	-1.2074	-1.1074	-1.0075	-0.9076	-0.8077	-0.7078	-0.6078	-0.5079	-0.4080	-0.3081	-0.2081
$\underline{x}(s_1;r)$	-2.8665	-2.7569	-2.6473	-2.5378	-2.4282	-2.3186	-2.2091	-2.0995	-1.9899	-1.8804	-1.7708
$\underline{x}(s_2;r)$	-2.9816	-2.8939	-2.8062	-2.7185	-2.6307	-2.5430	-2.4553	-2.3676	-2.2799	-2.1922	-2.1045
$\underline{x}(s_3;r)$	-2.1127	-2.0471	-1.9815	-1.9159	-1.8503	-1.7847	-1.7191	-1.6535	-1.5819	-1.5223	-1.4567
$\underline{x}'(s_0;r)$	-0.4550	-0.3953	-0.3356	-0.2759	-0.2162	-0.1565	-0.0968	-0.0372	0.0225	0.0822	0.1416
$\underline{x}'(s_3;r)$	1.6576	1.7043	1.7510	1.7977	1.8444	1.8910	1.9377	1.9844	2.0311	2.0778	2.1244
$\overline{x}(s_0;r)$	0.7911	0.6912	0.5912	0.4913	0.3914	0.2915	0.1916	0.0916	-0.0083	-0.1082	-0.2081
$\overline{x}(s_1;r)$	-0.6751	-0.7847	-0.8943	-1.0038	-1.1134	-1.2230	-1.3325	-1.4421	-1.5517	-1.6612	-1.7708
$\overline{x}(s_2;r)$	-1.2273	-1.3150	-1.4027	-1.4904	-1.5782	-1.6659	-1.7536	-1.8413	-1.9290	-2.0167	-2.1045
$\overline{x}(s_3;r)$	-0.8007	-0.8663	-0.9319	-0.9955	-1.0631	-1.1287	-1.1943	-1.2599	-1.3255	1.8911	-1.4567
$\overline{x}'(s_0;r)$	0.7388	0.6791	0.6194	0.5597	0.5000	0.4403	0.3807	0.3210	0.2613	0.2016	0.1419
$\overline{x}'(s_3;r)$	2.5912	2.5446	2.4979	2.4512	2.4045	2.3578	2.3112	2.2645	2.2178	2.7711	2.1244

Example 4.2 Consider the following fuzzy Fredholm integral equation

$$\underline{f}(s,r) = rs\,,$$

$$\underline{f}(s,r) = (2-r)s$$

and Kernel

$$K(s,t) = \sqrt{s-t}, \quad 0 \le s,t \le 2, \quad \lambda = 1$$

and a=0,b=2.

For $n=3,\ h=\frac{1}{3}, S_0=0, S_1=\frac{1}{3},\ S_2=\frac{6}{5},\ S_3=2$ we get:

-1.0000	0.1925i	0.3651i	0	0	0	0	0	0.3651i	0.0015+002357i	0	0.0131i
0.1012	-1.0000	0.3103i	0	0.0053	0	0	0	0.3103i	0.0022+0.2152i	0	0.0120i
0.1863	0.3103	-1.0000	0	0.0101	0	0	0	0	0.0055+0.1491i	0	0.0083i
0.2372	0.4303	0.2981	-1.0000	0.0131	0	0	0	0.2981	0	0	0
0	0.1800	0	0	-1.0000	0	0	0	0.1344	0.0533	0	0
-0.0267	-0.0779	0	0	0	-1.0000	0	0	-0.1983	0	0	0
0	0	0.3651i	0.0015+0.2357i	0	0.0131i	-1.0000	0.1925i	0.3651i	0	0	0
0	0	0.3103i	0.0022+0.2152i	0	0.0120i	0.1012	-1.0000	0.3103i	0	0.0053	0
0	0	0	0.0055+0.1491i	0	0.0083i	0.1863	0.3103	-1.0000	0	0.0101	0
0	0	0.2981	0	0	0	0.2372	0.4303	0.2981	-1.0000	0.0131	0
0	0	0.1344	0.0533	0	0	0	0.1800	0	0	-1.0000	0
0	0	-0.1983	0	0	0	-0.0267	-0.0779	0	0	0	-1.0000

And obtained Solution for *r*-level as *r*=0, 0.2, 0.4, 0.6, 0.8 are given in table 2.

Table2. Numerical results of Example 4.2

	r=0	*r* =0.2	*r* =0.4	*r* =0.6	*r* =0.8
$\underline{x}(s_0;r)$	-1.0383+1.3798i	-1.0349+1.2865i	-1.0314+1.11933i	-1.0280+1.1000i	-1.0245+1.0067i
$\underline{x}(s_1;r)$	-0.7479+1.5007i	-0.6955+1.3964i	-0.6431+1.2926i	-0.5907+1.1889i	-0.5383+1.0851i
$\underline{x}(s_2;r)$	-0.5130+1.3620i	-0.2669+1.2437i	-0.0208+1.1327i	0.2252+1.0181i	0.4713+0.9034i
$\underline{x}(s_3;r)$	-0.1358+1.4476i	0.2896+1.3808i	0.7149+1.3141i	1.1403+1.2473i	1.5657+1.1805i
$\underline{x}'(s_0;r)$	0.33468+0.3406i	0.5004+0.3409i	0.6541+0.3412i	0.8078+0.3415i	0.9614+0.1805i
$\underline{x}'(s_3;r)$	-0.3003-0.1963i	-0.0557-0.2085i	0.1889-0.2207i	0.4336-0.2328i	0.6782-0.2450i
$\overline{x}(s_0;r)$	-1.0039+0.4470i	-1.0073+0.5403i	-1.0108+0.6336i	-1.0142+0.7269i	-1.0777+0.8207i
$\overline{x}(s_1;r)$	-0.2238+0.4626i	-0.2762+0.5664i	-0.3287+0.6701i	-0.3811+0.7739i	-0.4335+0.8776i
$\overline{x}(s_2;r)$	1.9477+0.2156i	1.7016+0.3302i	1.4556+0.4448i	1.2095+0.5595i	0.9634+0.6741i
$\overline{x}(s_3;r)$	4.1180+0.7799i	3.6926+0.8467i	3.2672+0.9135i	2.8418+0.9802i	2.4165+1.0470i
$\overline{x}'(s_0;r)$	1.8835+0.3435i	1.7299+0.3432i	1.5762+0.3429i	1.4225+0.3426i	1.2688+0.3423i
$\overline{x}'(s_3;r)$	2.1459-0. 3181i	1.9073-0.3058i	1.6567-0.2937i	1.4120-0.2815i	1.7674-0.2693i

5 Conclusions

In this paper we presented the generalized differentiability for linear integral equations. We illustrated a numerical algorithm for solving linear fuzzy Fredholm integral equations of the second kind, using modified trapezoid quadrature method. It feels that this work which presents applicable computational methods may help to narrow the existing gap between the theoretical research on fuzzy integral equations and the practical applications already used in the design of various fuzzy dynamical systems.

References

[1] O.kaleva, Fuzzy differential equations, Fuzzy Sets Sestems 24 (1987) 301-317.

[2] D. Dubois, H. Prade, Towards fuzzy differential calculus, Fuzzy Sets Sestems 8 (1982) 1-7.

[3] B. Bede, S.G. Gal , Generalizations of the differentiability of fuzzy- with applications to fuzzy differential equations, Fuzzy 151(2005)581.

[4] D. Dubois, H. Prade, Fuzzy numbers: an overview, in : J. Bezdek (Ed.), Analysis of Fuzzy Information, CRC press, 1987, pp. 112- 148.

[5] C. Wu, Z.Gong,On Henstock integral of fuzzy number-valued functions, I, Fuzzy sets and systems 120(2001) 523-532.

[6] S. G. Gal, Approximation theory in fuzzy setting , in: G.A. Annastassiou (Ed.), Handbook of Analytic-Computational Methods in Applied Mathematics, chapman & Hall/CRC press, 2000, pp. 617-666.

[7] R. Goetschel, W. Voxman, Elementary calculus, Fuzzy Sets Sestems 18 (1986) 31-43.

[8] S. Nanda, On integration of fuzzy mapping, Fuzzy Sets Sestems 32 (1989) 95-101.

[9] Y. Chalco-Cano, Roman-Flores, on new solutions of fuzzy differential equations, Chaos Solutions and Fractals, 2006,pp.1016-1043.

[10] M. Puri, D. Ralescu, Differentials of fuzzy functions, Journal of Mathematical Analysis and Applications 91(1983)552-558.

[11] M. Friedman, M. Ming, A. Kandel, Fuzzy linear systems, Fuzzy Sets and Systems 96 (1998) 201-209.

[12] J. Stoer, R. Bulirsch, Introduction to Numerical linear Analysis, Second Edition, Springer- Verlag, 1993.

[13] S.Abbasbandy,T. Allah Viranloo, The Adomian Decomposition Method Applied to the fuzzy System of the second kind , in Journal of International Journal of Uncertaioty , Fuzziness, and Knowledge Based Systems (IJUFKS). Vol . 14 (1), 2006, pp. 101-10 (ISI) .

[14] T. Allahviranloo , M. otadi , Numerical Solution of fuzzy integral eqution , in Journal of Applied Mathematical Sciences, Vol .2,2008,no . 1,33 – 46.

[15] T. Allahviranloo , S. Hashemzadei , The homotopy perturbation method for fuzzy Fredholm integral equtions , in Journal of Applied Mathematics IAU, Vol .5, no . 19, Winter 2008, 1-13.