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ORIGINAL RESEARCH

Bending analysis of a general cross-ply laminate using 3D elasticity solution and layerwise theory

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Abstract In this study, the analytical solution of interlaminar stresses near the free edges of a general (symmetric and unsymmetric layups) cross-ply composite laminate subjected to pure bending loading is presented based on Reddy's layerwise theory (LWT) for the first time. First, the reduced form of displacement field is obtained for a general cross-ply composite laminate subjected to a bending moment by elasticity theory. Then, first-order shear deformation theory of plates and LWT is utilized to determine the global and local deformation parameters appearing in the displacement fields, respectively. One of the main advantages of the developed solution based on the LWT is exact prediction of interlaminar stresses at the boundary layer regions. To show the accuracy of this solution, three-dimensional elasticity bending problem of a laminated composite is solved for special set of boundary conditions as well. Finally, LWT results are presented for edge-effect problems of several symmetric and unsymmetric cross-ply laminates under the bending moment. The obtained results indicate high stress gradients of interlaminar stresses near the edges of laminates.

Keywords Interlaminar stresses · General cross-ply laminate · Elasticity formulation · First-order shear deformation plate theory · Layerwise theory

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Introduction

With the ever-increasing application of laminated composite, especially in aerospace industries which require strong, stiff, and lightweight structural components; however, interlaminar stresses play a significant role in the analysis and design of composite structures since they can lead to catastrophic failure modes like delamination. It has already been shown that the classical lamination theory is incapable of accurately predicting three-dimensional stress state in region near the edges of laminate known as boundary layer regions. Because of substantial importance of boundary layer phenomenon in practical usage enormous research works have been concentrated on the study of interlaminar stresses at free edges of composite laminates. Since no exact elasticity solution to this problem is yet known to exist, various approximate analytical and numerical methods are presented in the past three decades to determine interlaminar stresses in the boundary layer region. Also pure bending loading in laminated composite is important issue for analyzing and designing of composite structures.

The most important methods in this area alter from the approximate elasticity solutions by Pipes and Pagano (1974), higher-order plate theory by Pagano (1974), the boundary layer theory by Tang and Levy (1975), the perturbation technique by Hsu and Herakovich (1977), finite difference by Pipes and Pagano (1970) to finite elements by Wang and Crossman (1977) and Whitcomb et al. (1972). A relatively comprehensive discussion of the literature on the interlaminar stresses problem is given in the survey paper by Kant and Swaminathan (2000). Savoia and Reddy (1992) employed a displacement separation of variable approaches and the minimization of the total potential energy and obtained the three-dimensional elasticity

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solutions for antisymmetric angle-ply rectangular laminates under sinusoidal transverse loading. Wu and Kuo (1992) predicted interlaminar stresses in composite laminates under cylindrical bending. They used a local higher-order lamination theory. Kim and Atluri (1994) analyzed interlaminar stresses near straight free edges of beam-type composite laminates under out-of-planes shear/bending using an approximate method based on equilibrated stress representations and the principle of minimum complementary energy. They included the longitudinal degrees of freedom in the stress distributions. They obtained that interlaminar stresses under the shear/bending might exhibit substantially different characteristics than subjected to uniaxial loading or under pure bending. Robbins and Reddy (1993) developed a layerwise finite element model of laminated composite plates. They exhibited that their model is capable of computing interlaminar stresses and other localized effects with the same level of accuracy as a conventional three-dimensional finite element model. Shau and Soldatos (2000) determined stress distributions in angle-ply laminated plates, subjected to the cylindrical bending with different sets of edge boundary conditions. Huang et al. (2002), using a partially hybrid stress element with interlaminar continuity, analyzed bending of composite laminated plates. Matsunaga (2002) also obtained stress and displacement distributions of simply supported cross-ply laminated composite and sandwich plates subjected to lateral pressure using a global higher-order plate theory. Mittelstedt and Becker (2008) utilized Reddy's layerwise laminate plate theory to obtain the closed-form analysis of free-edge effects in layered plates of arbitrary non-orthotropic layups. The approach consists of the subdivision of the physical laminate layers into an arbitrary number of mathematical layers through the plate thickness. Jin Na (2008) used a finite element model based on the layerwise theory (LWT) and the von Kármán type nonlinear strains are used to analyze damage in laminated composite beams. In the formulation, the Heaviside step function is employed to express the discontinuous interlaminar displacement field at the delaminated interfaces. Recently, the LWT and improved first-order shear deformation theory (IFSDT) are employed by Nosier and Maleki (2008) to analyze free-edge stresses in general composite laminates under extension loads. Kim et al. (2012) analyzed interlaminar stresses near free edges in composite laminates by considering interface modeling. This interface modeling provided not only nonsingular stresses but also concentrated finite interlaminar stresses using the principle of complementary virtual work and the stresses that satisfy the traction-free conditions not only at the free edges but also at the top and bottom surfaces of laminates were obtained. Lee et al. (2011) analyzed the interlaminar stresses of a laminated composite patch using a stress-based equivalent single-layer model under a bending load. The adhesive stresses were obtained by solving the equilibrium equations. Ahn et al. (2011) presented efficient modeling technique using multi-dimensional method for prediction of free-edge stresses in laminate plates. The results obtained by this proposed model were compared with those available in literatures. The present models using the p-convergent transition element are demonstrated to be more practical and economical than previously p-version FEM using only single element type.

From the literature survey, it appears that the failure of structural components because of bending loads much more often than the in-plane loads; little work effort has been dedicated so far for development of the theoretical or numerical models for predicting boundary layer effects of bending of structural laminates. For this reason, the present study deals with the analytical solution of interlaminar stresses near free edges of a general cross-ply composite laminate subjected to a bending moment. Beginning from the general form of the displacement field for long laminates and making logical hypotheses in joining with physical behavior of a cross-ply laminate, the displacement field is significantly decreased. A LWT is utilizing to analyze the bending problem of a cross-ply laminate with free edges by employing the simplified displacement field. The first-order shear deformation theory (FSDT) is then employed to determine the unknown coefficients in the reduced displacement field. To show the accuracy of the LWT results, the elasticity solution of this problem is developed for a special set of boundary conditions.

Theoretical formulation

It is intended here to determine the interlaminar stresses in a general (symmetric and unsymmetric layups) cross-ply laminate subjected to the bending moment at $x = a$ and $x = -a$. Laminate geometry and coordinate system are shown in Fig. 1. The formulation is limited to linear elastic material behavior and small strain and displacement. First, the reduced form of displacement field is obtained for a general cross-ply composite laminate subjected to a bending moment by elasticity theory. To determine global and local deformation parameters appearing in the reduced displacement fields of elasticity formulation, the first-order shear deformation theory of plates (FSDT) and LWT is utilized, respectively.

Elasticity formulation

According to Fig. 1, a long laminate with length $2a$, thickness h, and width $2b$ is considered to be loaded under pure bending at $x = a$ and $x = -a$. Upon integration of the

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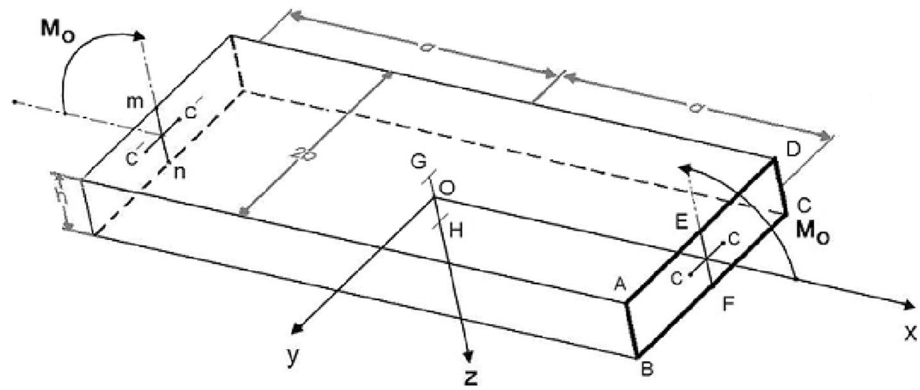

Fig. 1 Laminate geometry and coordinate system

strain–displacement relation, where all strain components are function of y and z only. The most general form of the displacement field of the kth layer within an arbitrary laminate can be shown to be as follows(Lekhnitskii 1981):

$$u_1^{(k)}(x,y,z) = B_2x + B_6xz + u^{(k)}(y,z) \tag{1a}$$

$$u_2^{(k)}(x,y,z) = -B_1xz + v^{(k)}(y,z) \tag{1b}$$

$$u_3^{(k)}(x,y,z) = B_1xy - \frac{1}{2}B_6x^2 + w^{(k)}(y,z). \tag{1c}$$

For a general cross-ply laminate based on physical grounds, the following restrictions will, furthermore, hold (see Fig. 1):

$$u_1^{(k)}(-x,y,z) = -u_1^{(k)}(-x,y,z) \tag{2a}$$

$$u_2^{(k)}(x,y,z) = u_2^{(k)}(-x,y,z). \tag{2b}$$

Upon imposing (2a) on (1a), it is concluded that $u^{(k)}(y,z) = 0$. Also, from Eqs. (2b) and (1b), it is found that $B_1 = 0$. Thus, for cross-ply laminates the most general form of the displacement field is given as follows:

$$\begin{aligned} u_1^{(k)}(x,y,z) &= B_6xz + B_2x \\ u_2^{(k)}(x,y,z) &= v^{(k)}(y,z) \\ u_3^{(k)}(x,y,z) &= -\frac{1}{2}B_6x^2 + w^{(k)}(y,z) \end{aligned} \tag{3}$$

It is next noted that if the loading conditions at $x = a$ and $x = -a$ are identical, the following conditions along the line GOH must hold:

$$u_1^{(k)}(x=0,y=0,z) = 0 \quad \text{and} \quad u_2^{(k)}(x=0,y=0,z) = 0. \tag{4}$$

From these conditions, it is concluded from Eq. (3) that $v^{(k)}(y=0,z) = 0$ and therefore

$$u_1^{(k)}(x,y=0,z) = B_6xz + B_2x \tag{5a}$$

$$u_2^{(k)}(x,y=0,z) = 0. \tag{5b}$$

The second parameter in relation (5a) indicates that lines such as AB, EF, and DC within the plan ADCB will remain straight after deformation and, furthermore, B_2 is the uniform axial strain of the laminate in the x-direction due to the bending moment. Denoting the axial displacement of the line EF by a $\bar{L}$ and that of MN by –a $\bar{L}$, it is then concluded that $B_2 = \bar{L}$. The first parameter in relation (5a), on the other hand, indicates that the plane ADCB rotates about the line cc (in the y-direction) and B_6 is the angle of bending γ per unit length. Denoting the angle of the bending of the line EF about the line cc by θ, it is, therefore, concluded that $B_6 = \gamma = \frac{\theta}{a}$.

From the preceding discussions, it is evident that either B_2 and B_6 or M_0 must be specified at the ends of the laminate. These latter conclusions may also be arrived at by the application of the principle of minimum total potential energy. Substituting the displacement field (3) into the linear strain–displacement relations of elasticity (Fung 1965), the following results will be obtained:

$$\begin{aligned} &\varepsilon_x^{(k)} = B_6z + B_2,\ \varepsilon_y^{(k)} = v^{(k)},\ \varepsilon_z^{(k)} = w_{,z}^{(k)} \\ &\gamma_{yz}^{(k)} = v_{,z}^{(k)} + v_{,y}^{(k)},\ \gamma_{xy}^{(k)} = 0,\ \gamma_{xz}^{(k)} = 0 \end{aligned}, \tag{6}$$

where a comma followed by a variable indicates partial differentiation with respect to that variable. The constitutive relations for the kth orthotropic lamina, with fiber orientations of 0° and 90° only, are given by (Herakovich 1998):

$$\{\sigma\}^{(k)} = [\bar{C}]^{(k)}\{\varepsilon\}^{(k)}, \tag{7}$$

where $[\bar{C}]$ is called the transformed (or off-axis) stiffness matrix. The local equilibrium equations of elasticity are known to be (Fung 1965)

$$\sigma_{ij,j} = 0 \quad i = 1,2,3, \tag{8}$$

where body forces are considered to be negligible. Also, the repeated index in Eq. (8) indicates summation from one to three. The displacement equilibrium equations within

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the kth ply are achieved by substituting Eqs. (6) into (7) and the subsequent results into Eq. (5). It is to be noted that the displacement equilibrium equation for $i = 1$ is identically satisfied. Therefore, the results are

$$\begin{aligned} &\bar{C}_{22}^{(k)} v_{,yy}^{(k)} + \bar{C}_{44}^{(k)} v_{,zz}^{(k)} + \left(\bar{C}_{23}^{(k)} + \bar{C}_{44}^{(k)}\right) w_{,yz}^{(k)} = 0 \\ &\left(\bar{C}_{44}^{(k)} + \bar{C}_{23}^{(k)}\right) v_{,zy}^{(k)} + \bar{C}_{44}^{(k)} w_{,yy}^{(k)} + \bar{C}_{33}^{(k)} w_{,zz}^{(k)} = -\bar{C}_{13}^{(k)} B_6 \end{aligned} . \quad (9)$$

The laminate plate is assumed to have free edges at $y = b$ and $y = -b$, solutions of Eqs. (9) must satisfy the following traction-free boundary conditions:

$$\sigma_y^{(k)} = \sigma_{yz}^{(k)} = 0 \quad \text{at y} = \pm \text{b}. \quad (10)$$

Unfortunately, however, no analytical solution seems to exist for such a boundary value problem. For this reason, in the present work, attention is devoted to technical plate theories. It is noted that the parameters B_2 and B_6 in Eq. (3) correspond to the global deformation of the laminate and, therefore, the unknown constants B_2 and B_6 may be determined from the simple first-order shear deformation plate theory (FSDT). The remaining terms (i.e., $v^{(k)}$ and $w^{(k)}$) in Eq. (3), on the other hand, belong to the local deformations of laminate within the laminate and will be determined using the layerwise laminated plate theory (LWT).

First-order shear deformation plate theory

In addition to their inherent simplicity and low computational cost, the equivalent single layer theories (ESL) often provide sufficiently accurate illustration of global responses for thin to moderately thick laminates. Among the ESL theories, FSDT which is also known as the Mindlin–Reissner theory, seems to provide the best compromise as far as the solution accuracy and simplicity are involved. The theory assumes that the displacement component of any point within the laminate can be suggested as (Reddy 2003):

$$\begin{aligned} u_1(x,y,z) &= u(x,y) + z\psi_x(x,y) \\ u_2(x,y,z) &= v(x,y) + z\psi_y(x,y) \\ u_3(x,y,z) &= w(x,y) \end{aligned} . \quad (11)$$

By comparing Eq. (11) with the reduced elasticity displacement field in (3), it is concluded that the displacement field of FSDT Eq. (11) must have the following simple form:

$$\begin{aligned} u_1(x,y,z) &= B_6 xz + B_2 x \\ u_2(x,y,z) &= v(y) + z\psi_y(y) \\ u_3(x,y,z) &= -\frac{1}{2} B_6 x^2 + w(y) \end{aligned} . \quad (12)$$

By employing the principle of minimum total potential energy (Fung 1965) and the displacement field in (11), the equilibrium equations within FSDT can be obtained to be as follows:

$$\delta v: \; N_y' = 0 \quad (13a)$$

$$\delta w: \; Q_y' = 0 \quad (13b)$$

$$\delta \psi_y: \; Q_y - M_y' = 0 \quad (13c)$$

$$\delta B_2: \; \int_{-b}^{+b} N_x \mathrm{d}y = 0 \quad (14a)$$

$$\delta B_6: \; \int_{-b}^{+b} M_x \mathrm{d}y - \mathrm{M}_0 = 0. \quad (14b)$$

Here a prime in (13) indicates ordinary differentiation with respect to the variable y. The following boundary conditions must be imposed at the free edges of the laminate (i.e., at $y = \pm b$):

$$N_y = M_y = Q_y = 0 \text{ at y} = \pm \text{b}. \quad (15)$$

In Eqs. (13)–(15), the stress and moment resultants are found to be as follows (Reddy 2003):

$$\left(M_x, M_y, N_x, N_y, Q_y\right) = \int_{-h/2}^{h/2} \left(\sigma_x z, \sigma_y z, \sigma_x, \sigma_y, \sigma_{yz}\right) \mathrm{d}z. \quad (16)$$

Based on the displacement field in (11) for general cross-ply laminates, these stress and moment resultants are readily found to be as follows:

$$\begin{aligned} \left(N_x, N_y\right) &= (B_{11}, B_{12})B_6 + (A_{11}, A_{12})B_2 + (A_{12}, A_{22})V' \\ &\quad + (B_{12}, B_{22})\psi_y' \\ \left(M_x, M_y\right) &= (D_{11}, D_{12})B_6 + (B_{11}, B_{12})B_2 + (B_{12}, B_{22})V' \\ &\quad + (D_{12}, D_{22})\psi_y' \\ Q_y &= k_4^2 A_{44}\left(\psi_y + W'\right) \end{aligned} \quad (17)$$

where A_{ij}, B_{ij}, and D_{ij} are the stretching, bending-stretching coupling, and bending stiffness of composite laminates (Reddy 2003). Also, in Eq. (17), k_4 is the shear correction factor. The displacement equilibrium equations are found by substituting Eq. (17) into Eqs. (13) and (14). Solving these equations under the boundary conditions in (15) will yield the displacement functions $v(y)$, $\psi_y(y)$, and $w(y)$ and the unknown constants B_2 and B_6 appearing in (12). The parameters B_2 and B_6, which are needed in (4), are determined to be as follows:

$$\begin{aligned} B_6 &= \frac{\bar{A}_{11}}{\bar{A}_{11}\bar{D}_{11} - \bar{B}_{11}^2} \frac{M_0}{2b} \\ B_2 &= \frac{\bar{B}_{11}}{\bar{A}_{11}\bar{D}_{11} - \bar{B}_{11}^2} \frac{M_0}{2b} \end{aligned} . \quad (18)$$

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The constant parameters in (18) are listed in Appendix 1. In the continuing of this paper, the following loading cases are used:

(a) *Loading case 1:*

$$B_2 = -\frac{\bar{B}_{11}}{\bar{A}_{11}\bar{D}_{11} - \bar{B}_{11}^2}\frac{M_0}{2b} = \bar{L}$$
$$\text{and } B_6 = \frac{\bar{A}_{11}}{\bar{A}_{11}\bar{D}_{11} - \bar{B}_{11}^2}\frac{M_0}{2b}. \tag{19}$$

(b) *Loading case 2:*

$$B_2 = -\frac{\bar{B}_{11}}{\bar{A}_{11}\bar{D}_{11} - \bar{B}_{11}^2}\frac{M_0}{2b} = \bar{L} \quad \text{and} \quad B_6 = 0. \tag{20}$$

In both cases, the specimen is stretched due to the bending moment. In loading case (*a*), the cross-ply laminate is allowed to freely rotate about the *y*-axis but in loading case (*b*) the rotation about the *y*-axis is restricted (consider the specimen that the lower and upper surfaces parallel to *xy* are fixed between two jaws, thus the specimen can stretch in *x*-direction, but cannot freely rotate about the *y*-axis. In this form, the specimen was subjected to the bending moment at its two edges.

Layerwise laminated plate theory of Reddy

Due to the existence of local high stress gradient and the three-dimensional nature of the boundary layer phenomenon, the interlaminar stresses in the boundary layer region cannot be determined accurately by the FSDT theory. Thus, Reddy's LWT that is capable of modeling localized three-dimensional effects is employed here to carry out the bending interlaminar stress analysis in general cross-ply laminates with free edges. Based on the results in Eq. (4) which is obtained within the elasticity formulation, the appropriate displacement field within LWT will be simplified to be as follows:

$$\begin{aligned} u_1(x,y,z) &= B_6xz + B_2x \\ u_2(x,y,z) &= V_k(y)\phi_k(z) \quad k = 1,2,\ldots,N+1 \\ u_3(x,y,z) &= -\frac{1}{2}B_6x^2 + W_k(y)\phi_k(z) \end{aligned}. \tag{21}$$

In Eq. (21), u_1, u_2, and u_3 represent the displacement components in the *x*, *y*, and *z* directions, respectively, of a material point initially located at (*x*, *y*, *z*) in the undeformed laminate. Also, B_2x, B_6xz and $-\frac{1}{2}B_6x^2$ denote global terms that signify the general behavior of the laminate and $V_k(y)$ and $W_k(y)$ ($k = 1, 2, \ldots, N + 1$) represent the local displacement components of all points located on the *k*th surface in the undeformed state (Reddy 2003; Nosier et al. 1993). In relation (21), *N* marks the total number of numerical layers considered in a laminate. Furthermore, the linear global interpolation function $\phi_k(z)$ is defined as follows:

$$\phi_k(z) = \begin{cases} 0 & z \le z_{k-1} \\ \psi_{k-1}^2(z) & z_{k-1} \le z \le z_k \\ \psi_k^1(z) & z_k \le z \le z_{k+1} \\ 0 & z \ge z_{k+1} \end{cases} (k = 1,2,\ldots,N+1), \tag{22}$$

where $\psi_k^j (j = 1,2)$ are the local Lagrangian linear interpolation functions within the *k*th layer which are defined as follows:

$$\psi_k^1(z) = \frac{1}{h_k}(z_{k+1} - z) \quad \text{and} \quad \psi_k^2(z) = \frac{1}{h_k}(z - z_k) \tag{23}$$

with h_k being the thickness of the *k*th layer. Upon substituting of into the linear strain–displacement relations of elasticity (Fung 1965) will be obtained as follows:

$$\begin{aligned} &\varepsilon_x = B_6z + B_2, \varepsilon_y = V_k'\phi_k, \varepsilon_z = W_k\phi_k', \gamma_{yz} = V_k\phi_k', W_k'\phi_k \\ &\gamma_{xz} = 0, \gamma_{xy} = 0 \end{aligned}. \tag{24}$$

By utilizing the principle of minimum total potential energy, the equilibrium equations within LWT are found. The results are 2(*N* + 1) local equilibrium equations corresponding to 2(*N* + 1) unknowns V_k and W_k and two global equilibrium equations corresponding to B_2 and B_6 can be shown to be as follows:

$$\delta V_k : Q_y^k - \frac{\mathrm{d}M_y^k}{\mathrm{dy}} = 0 \quad k = 1,2,\ldots,N+1 \tag{25a}$$

$$\delta W_k : Q_y^k - \frac{\mathrm{d}M_y^k}{\mathrm{dy}} = 0 \quad k = 1,2,\ldots,N+1 \tag{25b}$$

$$\delta B_2 : \int_{-b}^{b} N_x \mathrm{dy} = 0 \tag{25c}$$

$$\delta B_6 : \int_{-b}^{b} M_x \mathrm{dy} - M_0 = 0. \tag{25d}$$

Also, the boundary conditions at the edges parallel to *x*-axis (i.e., at *y* = −*b*, *b*) involve the specifications of either V_k or M_y^k and W_k or R_y^k. The generalized stress resultants in Eqs. (25) are defined as follows:

$$\begin{aligned} \left(M_y^k, R_y^k\right) &= \int_{-h/2}^{h/2} \left(\sigma_y, \sigma_{yz}\right)\phi_k \mathrm{dz} \\ \left(N_z^k, Q_y^k\right) &= \int_{-h/2}^{h/2} \left(\sigma_z, \sigma_{yz}\right)\phi_k' \mathrm{dz}\,. \\ \left(M_x, N_x\right) &= \int_{-h/2}^{h/2} \left(\sigma_x z, \sigma_x\right) \mathrm{dz} \end{aligned} \tag{26}$$

The generalized stress resultants in (26) are expressed in terms of the unknown displacement functions by substituting

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of (24) into (8) and the subsequent results into (26). The results are obtained which can be represented as follows:

$$\begin{aligned}\left(M_x^k, N_x^k, M_y^k, N_z^k\right) &= \left(D_{11}, B_{11}, D_{12}^k, \bar{B}_{13}^k\right)B_6 \\ &+ \left(B_{11}, A_{11}, B_{12}^k, A_{13}^k\right)B_2 \\ &+ \left(D_{12}^k, B_{12}^k, D_{22}^{kj}, B_{23}^{jk}\right)V_j' \\ &+ \left(\bar{B}_{13}^k, A_{13}^k, B_{23}^{kj}, A_{33}^{kj}\right)W_j \\ \left(R_y^k, Q_y^k\right) &= \left(B_{44}^{kj}, A_{44}^{kj}\right)V_j + \left(D_{44}^{kj}, B_{44}^{jk}\right)W_j'\end{aligned} \tag{27}$$

where in (27) the laminate rigidities are defined as follows:

$$\begin{aligned}\left(A_{pq}^{kj}, B_{pq}^{kj}, D_{pq}^{kj}\right) &= \sum_{i=1}^{N}\int_{-h/2}^{h/2}\bar{C}_{pq}^{(i)}\left(\phi_k'\phi_j', \phi_k\phi_j', \phi_k\phi_j\right)\mathrm{d}z \\ \left(A_{pq}^k, B_{pq}^k, \bar{B}_{pq}^k, D_{pq}^k\right) &= \sum_{i=1}^{N}\int_{-h/2}^{h/2}\bar{C}_{pq}^{(i)}\left(\phi_k', \phi_k, \phi_k' z, \phi_k z\right)\mathrm{d}z \\ (k, j &= 1, 2, \ldots, N+1)\end{aligned} \tag{28}$$

The integrations in (28) carried out the final expressions of the rigidities are for convenience presented in Appendix 2. The local equilibrium equations are expressed in terms of the displacement functions by substituting (27) into Eqs. (25a), (25b). The results are as follows:

$$\begin{aligned}&\delta V_k: A_{44}^{kj}V_j - D_{22}^{kj}V_j'' + \left(B_{44}^{jk} - B_{23}^{kj}\right)W_j' = 0 \\ &\quad k = 1, 2, \ldots, N+1 \\ &\delta W_k: A_{33}^{kj}W_j - D_{44}^{kj}W_j'' + \left(B_{23}^{jk} - B_{44}^{kj}\right)V_j' = -\bar{B}_{13}^k B_6 - A_{13}^k B_2 \\ &\quad k = 1, 2, \ldots, N+1\end{aligned} \tag{29}$$

Finally, by substituting (8) into (25c), (25d) the global equilibrium conditions are expressed in terms of the displacement functions in the following form:

$$\begin{aligned}&\delta B_2: B_{11}B_6 + A_{11}B_2 + \frac{B_{12}^k}{b}V_k(b) + \frac{A_{13}^k}{2b}\int_{-b}^{b}W_k\mathrm{d}y = 0 \\ &\delta B_6: D_{11}B_6 + B_{11}B_2 + \frac{D_{12}^k}{b}V_k(b) + \frac{\bar{B}_{13}^k}{2b}\int_{-b}^{b}W_k\mathrm{d}y = \frac{M_0}{2b}\end{aligned}. \tag{30}$$

Analytical solution

In this section, the procedures for solving the displacement equations of equilibrium within LWT and elasticity theory are debated for a cross-ply laminate subject to the bending moment M_0.

LWT solution

The system of equations appearing in Eq. (29) presents $2(N+1)$ coupled second-order ordinary differential equations with constant coefficients which may be introduced in a matrix form as follows:

$$[M]\{\eta''\} + [K]\{\eta\} = [T]\{B\}.y, \tag{31}$$

where

$$\begin{aligned}\{\eta\} &= \left\{\{V\}^T, \{\bar{W}\}^T\right\}^T \\ \{V\} &= \{V_1, V_2, \ldots, V_{N+1}\}^T \\ \{\bar{W}\} &= \{\bar{W}_1, \bar{W}_2, \ldots, \bar{W}_{N+1}\}^T \\ \{B\} &= \{B_2, B_6\}^T\end{aligned} \tag{32a}$$

and

$$\bar{W}_j = \int^{y} W_j\mathrm{d}y \tag{32b}$$

The coefficient matrices $[M]$, $[K]$, and $[T]$ are listed in Appendix 2. The general solution of Eq. (31) may be written as follows:

$$\{\eta\} = [\psi][\sinh(\lambda y)]\{H\} + [K]^{-1}[T]\{B\}.y. \tag{33}$$

The detailed solution can be found in (Nosier and Maleki 2008). The following traction-free boundary conditions must be imposed within LWT:

$$M_y^k = R_y^k = 0 \text{ at y } = \pm\text{b}. \tag{34}$$

Elasticity solution

As previously mentioned, no analytical solution seems to exist for Eqs. (11) subjected to the traction-free boundary conditions in Eq. (15). It is, however, noted here that if the bending rotation is impeded by the end grips (i.e., $B_6 = 0$), while the laminate is being extended in the x-direction, then it is possible to determine an analytical solution for Eq. (11) for the following boundary conditions:

$$\sigma_y^{(k)} = u_3^{(k)} = 0 \text{ at y } = \pm\text{b}. \tag{35}$$

Such an analytical solution is developed here only to appraise the accuracy of the LWT. With $B_6 = 0$, the displacement field in (4) is reduced and the elasticity equilibrium equations in (10) are simplified. In terms of the reduced displacement functions appearing, the following conditions in (35) will be given as below:

$$\begin{aligned}&w^{(k)}(y, z) = 0 \\ &\bar{C}_{12}^{(k)}L + \bar{C}_{22}^{(k)}v_{,y}^{(k)} + \bar{C}_{23}^{(k)}w_{,z}^{(k)} = 0 \quad \text{at y} = \pm\text{ b}\end{aligned}. \tag{36}$$

Next within any layer, it is assumed that

$$\begin{aligned}v^{(k)}(y, z) &= V^{(k)}(y, z) + \bar{v}^{(k)}(y) \\ w^{(k)}(y, z) &= W^{(k)}(y, z) + \bar{w}^{(k)}(y)\end{aligned}. \tag{37}$$

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Upon substituting of Eq. (37) into the reduced governing equations of equilibrium two set of equations will be obtained. The first contains $\bar{v}^{(k)}$ and $\bar{w}^{(k)}$ as follows:

$$\begin{aligned} &\bar{C}_{22}^{(k)}\,\bar{v}_{,yy}^{(k)} = 0 \\ &\bar{C}_{44}^{(k)}\,\bar{w}_{,yy}^{(k)} = 0 \end{aligned}. \tag{38}$$

The second set of equations contains $V^{(k)}$ and $W^{(k)}$ as follows:

$$\begin{aligned} &\bar{C}_{22}^{(k)}\,V_{,yy}^{(k)} + \bar{C}_{44}^{(k)}\,V_{,zz}^{(k)} + \left(\bar{C}_{23}^{(k)} + \bar{C}_{44}^{(k)}\right) W_{,yz}^{(k)} = 0 \\ &\left(\bar{C}_{44}^{(k)} + \bar{C}_{23}^{(k)}\right) V_{,yz}^{(k)} + \bar{C}_{44}^{(k)}\,W_{,yy}^{(k)} + \bar{C}_{33}^{(k)}\,W_{,zz}^{(k)} = 0 \end{aligned}. \tag{39}$$

Similarly, substitution of Eq. (37) into the relevant boundary conditions, Eq. (36), at $y = -b$ and $y = b$ yields

$$\begin{aligned} &\bar{w}^{(k)}(y) = 0 \\ &\bar{C}_{22}^{(k)}\,\bar{v}_{,y}^{(k)} + \bar{C}_{12}^{(k)}\bar{L} = 0 \end{aligned} \tag{40}$$

and

$$\begin{aligned} &W^{(k)}(y) = 0 \\ &\bar{C}_{22}^{(k)}\,V_{,y}^{(k)} + \bar{C}_{23}^{(k)} W_{,z}^{(k)} = 0 \end{aligned}. \tag{41}$$

Next it is noted that from the solutions of the ordinary differential equations in (36) and the boundary conditions in (40) it can be concluded that

$$\begin{aligned} &\bar{v}^{(k)} = -\frac{\bar{C}_{12}^{(k)}}{\bar{C}_{22}^{(k)}}\bar{L}y = \frac{\bar{C}_{12}^{(k)}}{\bar{C}_{22}^{(k)}}\frac{\bar{B}_{11}}{\bar{A}_{11}\bar{D}_{11} - \bar{B}_{11}^{2}}\frac{M_0}{2b}y \\ &\bar{w}^{(k)} = 0 \end{aligned}. \tag{42}$$

It remains to solve Eqs. (39) with the boundary conditions in (41), it is noted that the boundary conditions in (41) are identically satisfied by assuming the following solution representations:

$$\begin{aligned} &V^{(k)}(y,z) = \sum_{m=0}^{\infty} V_m^{(k)}(z)\sin(\alpha_m y) + V_0^{(k)}(z) \\ &W^{(k)}(y,z) = \sum_{m=0}^{\infty} W_m^{(k)}(z)\cos(\alpha_m y) \end{aligned}, \tag{43}$$

where $\alpha_m = (2m+1)\frac{\pi}{2b}$.

Next, upon substitution of (43) into (39), two set of ordinary differential equations is obtained as follows:

$$V_0^{(k)\prime\prime}(z) = 0 \tag{44}$$

and

$$\begin{aligned} &-\alpha_m^2\bar{C}_{22}^{(k)}\,V_m^{(k)}(z) + \bar{C}_{44}^{(k)}\,V_m^{(k)\prime\prime}(z) - \alpha_m\left(\bar{C}_{23}^{(k)} + \bar{C}_{44}^{(k)}\right)\,W_m^{(k)\prime} = 0 \\ &\alpha_m\left(\bar{C}_{44}^{(k)} + \bar{C}_{23}^{(k)}\right)\,V_m^{(k)\prime}(z) - \alpha_m^2\bar{C}_{44}^{(k)}\,W_m^{(k)} + \bar{C}_{33}^{(k)}\,W_m^{(k)\prime\prime} = 0 \end{aligned}. \tag{45}$$

Equation (44) has the following solution:

$$V_0^{(k)}(z) = E_k z + F_k, \tag{46}$$

where E_k and F_k are unknown constants introduced in the remaining of the present work. Next, in order to solve Eqs. (45), it is assumed that

$$\begin{aligned} &V_m^{(k)}(z) = B_{km}e^{\lambda_{km}z} \\ &W_m^{(k)}(z) = C_{km}e^{\lambda_{km}z} \end{aligned}. \tag{47}$$

This upon substitution into (45) yields the following algebraic equations:

$$\begin{bmatrix} -\alpha_m^2\bar{C}_{22}^{(k)} + \lambda_{km}^2\bar{C}_{44}^{(k)} & -\alpha_m\lambda_{km}\left(\bar{C}_{23}^{(k)} + \bar{C}_{44}^{(k)}\right) \\ \alpha_m\lambda_{km}\left(\bar{C}_{44}^{(k)} + \bar{C}_{23}^{(k)}\right) & -\alpha_m^2\bar{C}_{44}^{(k)} + \lambda_{km}^2\bar{C}_{33}^{(k)} \end{bmatrix} \times \begin{Bmatrix} B_{km} \\ C_{km} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \tag{48}$$

For a nontrivial solution, the determinant of the coefficient matrix in (48) must vanish. This way, a fourth-order polynomial equation in λ_{km} is obtained as follows:

$$\left[\bar{C}_{33}^{(k)}\bar{C}_{44}^{(k)}\right]\lambda_{km}^4 + \alpha_m^2\left[\left(\bar{C}_{44}^{(k)} + \bar{C}_{23}^{(k)}\right)^2 - \bar{C}_{44}^{(k)^2} - \bar{C}_{22}^{(k)}\bar{C}_{33}^{(k)}\right]\lambda_{km}^2 + \alpha_m^4\bar{C}_{22}^{(k)}\bar{C}_{44}^{(k)} = 0 \tag{49}$$

Equation (49) has four distinct roots which may, in general, be complex. Therefore, the general solutions of Eqs. (45) may be presented as follows:

$$\begin{aligned} &V_m^{(k)}(z) = \sum_{i=1}^{4} B_{kmi}e^{\lambda_{kmi}z} \\ &W_m^{(k)}(z) = \sum_{i=1}^{4} \bar{C}_{kmi}B_{kmi}e^{\lambda_{kmi}z} \end{aligned}, \tag{50}$$

where $\bar{C}_{kmi} = \frac{\bar{C}_{44}^{(k)}\lambda_{km}^2 - \alpha_m^2\bar{C}_{22}^{(k)}}{\alpha_m\lambda_{km}\left(\bar{C}_{23}^{(k)} + \bar{C}_{44}^{(k)}\right)}$. Next, with $\alpha_m = (2m+1)\frac{\pi}{2b}$, the following Fourier sine expansion for y is used in (42):

$$y = \sum_{m=0}^{\infty} a_m \sin(\alpha_m y), \tag{52a}$$

where

$$a_m = \frac{8b}{\pi^2}\frac{(-1)^m}{(2m+1)^2}. \tag{52b}$$

Thus, the displacement components within the kth layer of the laminate are given by

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$$u_1^{(k)}(x,y,z) = -\frac{\bar{B}_{11}}{\bar{A}_{11}\bar{D}_{11} - \bar{B}_{11}^2}\frac{M_0}{2b}x$$

$$\begin{aligned} u_2^{(k)}(x,y,z) &= \sum_{m=1}^{\infty} B_{km}\sin(\alpha_m y) + E_k z + F_k \\ &\quad + \sum_{m=0}^{\infty}\sum_{i=1}^{4} B_{kmi}e^{\lambda_{kmi}z}\sin(\alpha_m y)u_3^{(k)}(x,y,z) \\ &= \sum_{m=0}^{\infty}\sum_{i=1}^{4} \bar{C}_{kmi}B_{kmi}e^{\lambda_{kmi}z}\cos(\alpha_m y), \end{aligned} \quad (53a)$$

where

$$B_{km} = \frac{\bar{C}_{12}^{(k)}}{\bar{C}_{22}^{(k)}}\frac{\bar{B}_{11}}{\bar{A}_{11}\bar{D}_{11} - \bar{B}_{11}^2}\frac{M_0}{2b}a_m. \quad (53b)$$

It is to be noted that the constant F_k being in Eq. (53) is a part of rigid body translations and can, therefore, be ignored. The remaining unknown constants in (53) (i.e., B_{kmi} and E_k) are obtained by imposing the traction-free boundary conditions at the top and bottom surface of the laminate, the displacement continuity conditions at the interfaces and the stress equilibrium conditions at the interfaces. In order to be able to impose the conditions stated before, the parameter $\bar{L}$ can be expended in Fourier cosine series as follows:

$$\bar{L} = \sum_{m=0}^{\infty} \bar{L}b_m \cos(\alpha_m y), \quad (54a)$$

where $\alpha_m = (2m+1)\frac{\pi}{2b}$ and

$$b_m = \frac{4}{\pi}\frac{(-1)^m}{2m+1}. \quad (54b)$$

For a general cross-ply laminate with N layers, satisfying the conditions stated before generates 4 N algebraic equations which upon solving will produce the 4 N unknown constants of integrations B_{kmi} appearing in (53) for each Fourier integer m. The remaining unknown constant (i.e., E_k) will be obtained to be equal to zero. As a result, the strain and stress components will ready to be determined from the strain–displacement relations in (7) and the Hooke's law in (8), respectively.

In fact, according to the solution found here, the interlaminar normal stress σ_z will vanish at points located on the edges of the laminate at $y = \pm b$. This is, of course, not a correct result since an exact elasticity solution would yield nonzero values for σ_z on these edges. The exact value of σ_z on these edges, however, is determined by considering the following three-dimensional Hooke's law (Herakovich 1998):

$$\begin{aligned} \varepsilon_x^{(k)} &= \bar{S}_{11}^{(k)}\sigma_x^{(k)} + \bar{S}_{12}^{(k)}\sigma_y^{(k)} + \bar{S}_{13}^{(k)}\sigma_z^{(k)} \\ \varepsilon_z^{(k)} &= \bar{S}_{13}^{(k)}\sigma_x^{(k)} + \bar{S}_{23}^{(k)}\sigma_y^{(k)} + \bar{S}_{33}^{(k)}\sigma_z^{(k)} \end{aligned}, \quad (57)$$

Table 1 Engineering properties of unidirectional graphite/epoxy

E_1 (GPa)	$E_2 = E_3$ (GPa)	$G_{12} = G_{13}$ (GPa)	G_{23} (GPa)	$\upsilon_{12} = \upsilon_{13}$	υ_{23}
132	10.8	5.65	3.38	0.24	0.59

where $\bar{S}_{ij}^{(k)}$'s are the transformed compliances of the kth layer. At the edges of laminate $u_3^{(k)}$ is specified to vanish [see Eq. (35)]. Therefore, at all points on these edges (expect for points located at the intersections of these edges with interfaces, bottom surface, and top surface of the laminate) the following result can be concluded:

$$\varepsilon_z \equiv \frac{\partial u_3^{(k)}}{\partial z} = 0 \quad \text{at y} = \pm\text{b}. \quad (58)$$

Next, it is noted that the substitution of Eqs. (34) and (58) into Eq. (57) results in

$$\bar{L} = \bar{S}_{11}^{(k)}\sigma_x^{(k)} + \bar{S}_{13}^{(k)}\sigma_z^{(k)} \quad (59a)$$

$$0 = \bar{S}_{13}^{(k)}\sigma_x^{(k)} + \bar{S}_{33}^{(k)}\sigma_z^{(k)}. \quad (59b)$$

Solving Eqs. (59) yields the exact value of $\sigma_z^{(k)}$ found to be as follows:

$$\sigma_z^{(k)} = \frac{\bar{S}_{13}^{(k)}\bar{L}}{\bar{S}_{13}^{(k)^2} - \bar{S}_{11}^{(k)}\bar{S}_{33}^{(k)}}, \quad (60)$$

where $\bar{L} = -\frac{\bar{B}_{11}}{\bar{A}_{11}\bar{D}_{11} - \bar{B}_{11}^2}\frac{M_0}{2b}$. This relation marks that the interlaminar normal stress has a constant value at the edges of each lamina and, furthermore, this constant value becomes different from one layer to another (adjacent) layer because of changes in fiber direction.

Results and discussion

In this section, several examples are provided for general cross-ply laminates under the bending moment M_0. The mechanical properties of unidirectional graphite/epoxy T300/5208 are shown in Table 1 (Herakovich 1998). Thickness of each ply is assumed to be 0.5 mm and the value 5/6 is used for the shear correction factor in the FSDT.

All the numerical results shown in what follows are presented by means of $\bar{B}_j = \frac{B_j}{M_0}$ ($j = 2,\ 6$) and $\bar{\sigma}_{ij} = \frac{\sigma_{ij}}{\sigma_0}$ ($\sigma_0 = \frac{M_0}{bh^2}$). Also, for obtaining accurate results within LWT, each physical lamina is divided into, unless otherwise mentioned, twelve sub-layers (i.e., $p = 12$). To closely study the accuracy of FSDT in estimating B_6, numerical results for the ratio $\bar{B}_6$ according to FSDT and LWT are obtained and are presented in Table 2 for different width-to-thickness ratios and various general cross-

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Table 2 Numerical values of $\bar{B}_6$ for various laminates according to FSDT and LWT

Layup	Theory	2*b*/*h*		
		5	10	20
[90°/90°/90°/0°]	FSDT	0.9725	0.4863	0.2431
	LWT	0.9172	0.4590	0.2292
[0°/0°/0°/90°]	FSDT	0.6744	0.3372	0.1686
	LWT	0.6494	0.3250	0.1625
[90°/90°/0°/0°]	FSDT	0.8514	0.4257	0.2129
	LWT	0.8110	0.4056	0.2027
[90°/0°/90°/0°]	FSDT	1.2196	0.6098	0.3049
	LWT	1.1553	0.5775	0.2884

ply laminates. Close agreements are found between the results of the two theories, particularly for thin to moderately thick laminates. Numerical studies indicate that the terms involving B_6 in Eq. (21) have unimportant effects on distribution of interlaminar stresses in various laminates, even for thick laminate. It is, therefore, concluded that the formula obtained for B_6 according to FSDT may always be utilized for various cross-ply laminates under bending within other theories such as LWT and elasticity theory [see Eq. (10)].

Next, in order to assess the accuracy of LWT, the results of LWT are compared here within those of elasticity solution as developed in the present study for loading case (*b*) [see Eq. (20) and the boundary conditions in Eq. (35)]. The boundary conditions used in LWT equivalent to those in the elasticity solution [see Eq. (35)] are as follows:

$$M_y^k = W_k = 0 \quad \text{at } y = \pm b. \tag{61}$$

The interlaminar stresses $\sigma_z^{(k)}$ and $\sigma_{yz}^{(k)}$ are calculated in LWT by integrating the local elasticity equations of equilibrium. In order to find correct value of interlaminar normal stress within LWT at exactly $y = \pm b$, procedure similar to what is done here within the elasticity solution is employed. Toward this goal, it is noted that using the boundary conditions (61) in the laminate constitutive relations (27), the following relation is achieved:

$$D_{22}^{kj} V_j' = -\ B_{12}^k \bar{L} \text{ at } y = \pm b. \tag{62}$$

The quantity V_j' at $y = \pm$ b is obtained from Eq. (62). Next, upon substitution of this quantity into the strain–displacement relations in (24) and the subsequent results into (8), the interlaminar normal stress σ_z is obtained within LWT at the edges of laminate.

In the following, numerical results are developed for various general cross-ply laminates with free edges only and with width-to-thickness ratio (2*b*/*h*) equal to 5 according to LWT. Both loading cases defined in (19) and (20) will be considered. Figures 2 and 3 show the distribution of interlaminar normal and shear stresses along the width of [90°/90°/90°/0°] and [90°/0°/90°/0°] laminates, respectively.

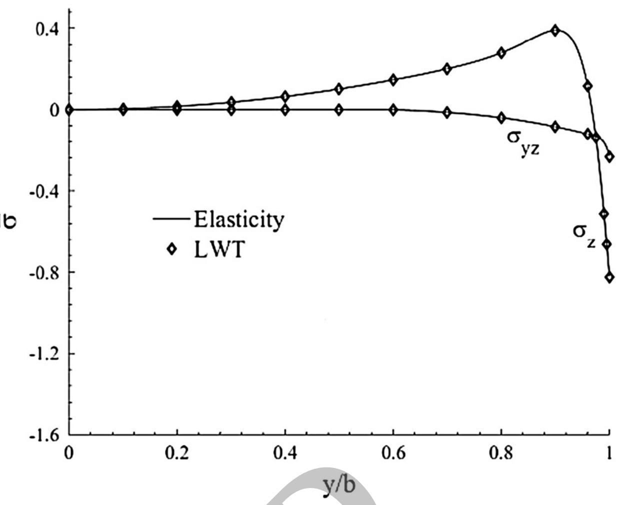

Fig. 2 Interlaminar stresses along the mid-plane of [90°/90°/90°/0°] layup

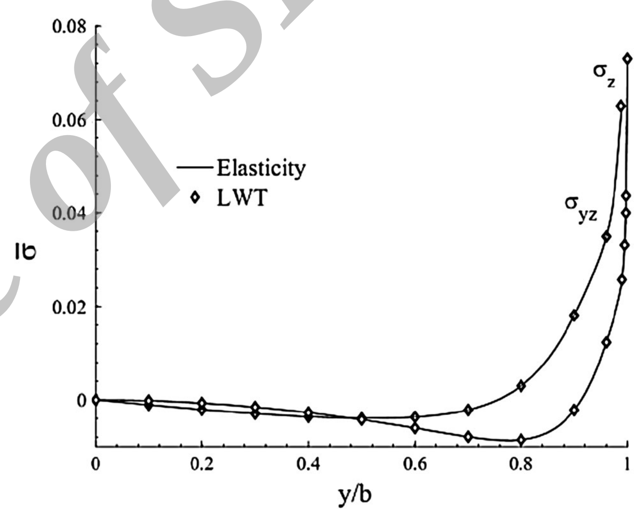

Fig. 3 Interlaminar stresses along the mid-plane of [90°/0°/90°/0°] layup

Excellent agreement between the layerwise solution and the elasticity solution is found. This close agreement verifies the accuracy of the LWT. It is reminded that these results are obtained only for loading case (*b*). To study the convergence of the stresses near free edges, two simple laminates [0°/90°/0°/90°] and [0°/90°/90°/0°] subjected to the bending moment M_0 are considered. Since, except exactly at $y = b$, the difference in σ_z with various p values at the laminate interfaces and through the thickness in the boundary layer region is small, the value of σ_z at $y = b$ is used in the convergence study. Figure 4 shows the

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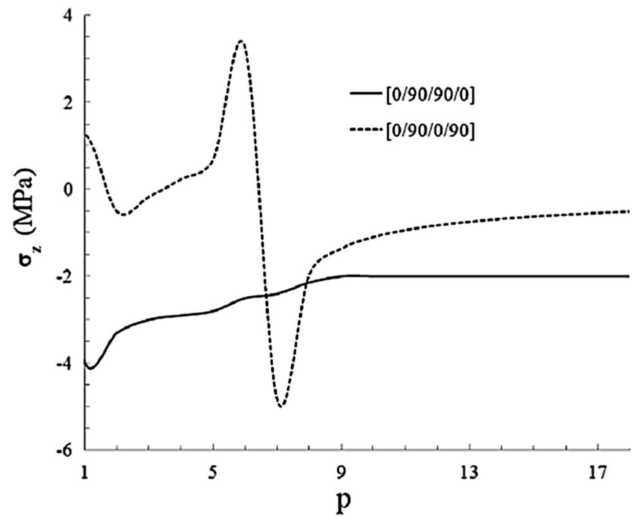

Fig. 4 Convergence of σ_z at $y = b$ and mid-plane of [0°/90°/0°/90°] and [0°/90°/90°/0°] layups for different values of subdivisions (p)

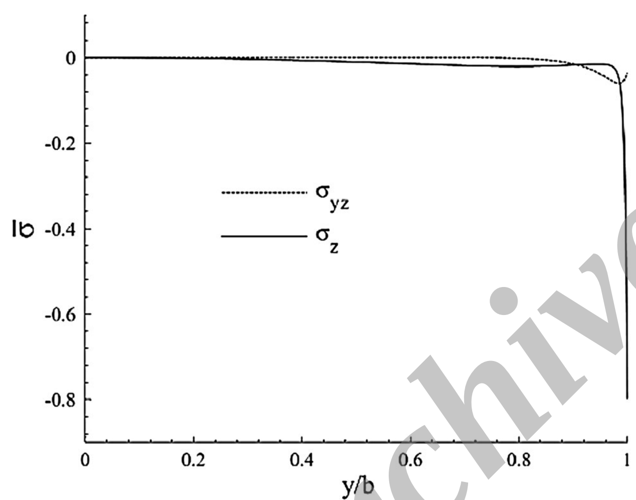

Fig. 5 Interlaminar stresses along the 90°/0° interface of [90°/90°/90°/0°] layup

numerical value of σ_z at exactly $y = b$ versus p value for both [0°/90°/0°/90°] and [0°/90°/90°/0°] laminates for loading case (a). In the unsymmetric layup [0°/90°/0°/90°] (dash line in Fig. 4), it is seen that the value of σ_z is more noticeably dependent on the number of subdivisions (p) than the symmetric laminate [0°/90°/90°/0°]. In the symmetric laminate, the numerical values of σ_z remain constant for $p > 9$, while they remain constant for $p > 12$ in unsymmetric layup.

Figure 5 displays the distribution of the interlaminar stresses along the 90°/0° interface of [90°/90°/90°/0°] layup for loading case (a). It is observed that the interlaminar normal stress σ_z grows rapidly in the vicinity of the free edges, while being zero in the interior region of

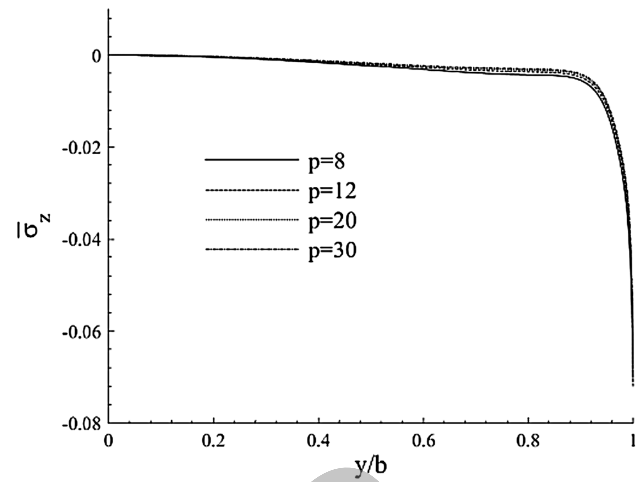

Fig. 6 Distribution of $\bar{\sigma}_z$ along the 90°/90° interface of [0°/0°/90°/90°] laminate for different values of layer subdivisions

the laminate. On the other hand, σ_{yz} rises toward the free edge and decreases rather abruptly to zero at free edge. It is also seen that the magnitude of the maximum of the transverse normal stress σ_z is greater than that of transverse shear stress σ_{yz}. By rising the number of layers in each lamina, σ_{yz} becomes slimly closer to zero but it may be never become zero. This is, most likely, due to the fact that within LWT the generalized stress resultant R_y^k, rather than σ_{yz}, is forced to disappear at the free edge [see Eqs. (38)].

Interlaminar normal stress along the 90°/90° interface of [0°/0°/90°/90°] laminate for loading case (a) is displayed in Fig. 6. It is observed that increasing the number of layer subdivisions, p, has no significant effect on the numerical value of interlaminar stress σ_z within the boundary layer region of the laminate, especially at the free edge (i.e., $y = b$) because of the interface-edge junction of similar layers (i.e., 90°/90°). It is significant to note that increasing the number of subdivisions results in no convergence for σ_z at interface-edge junction of two dissimilar layers such as 0°/90°, and the numerical value of this stress component continues to grow as the subdivision values is increased. On the contrary, at the interface-edge junction of similar layers such as 90°/90°, the values of σ_z remain constant as the subdivision values within each physical layer are increased. Through the thickness distribution of the interlaminar normal stress σ_z for [90°/90°/0°/0°] layup is displayed in Fig. 7 for loading case (a). It is seen that the maximum negative value of σ_z happens within the bottom 90° layer and the maximum positive value of σ_z occurs within top 0° layer both near the middle surface of the laminate at the free edge (i.e., $y = b$). It is also seen that σ_z diminishes away from the free edge as the interior region of the laminate is approached.

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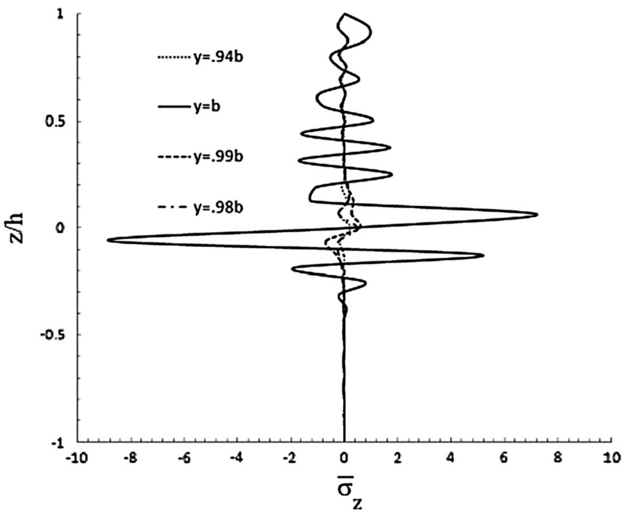

Fig. 7 Interlaminar normal stress $\bar{\sigma}_z$ through the thickness of [90°/90°/0°/0°] layup

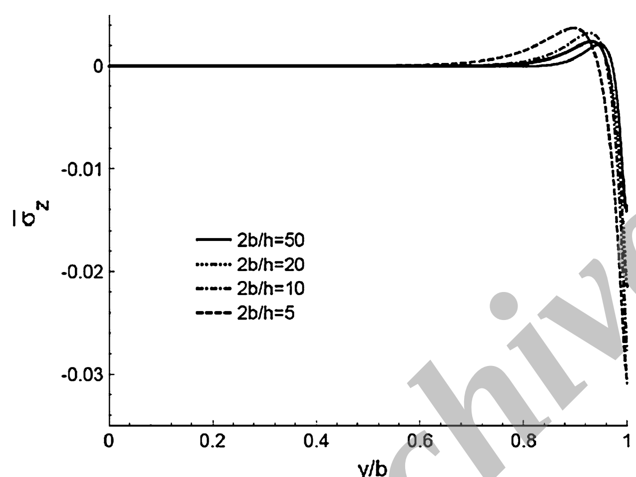

Fig. 8 Interlaminar stress along the mid-plane of [0°/90°/90°/0°] layup for various width-to-thickness ratios

The effect of the laminate width-to-thickness ratio on the interlaminar stress due to the loading case (*a*) is examined in Fig. 8 in the [0°/90°/90°/0°] laminate. It is found that the width of the boundary layer regions always remains almost equal to the thickness of the laminate. That is, a thickness away from the edges of the laminate the interlaminar stresses approach zero.

Conclusions

An elasticity formulation is developed for the displacement field of the long cross-ply laminate under the bending moment. The first-order shear deformation theory (FSDT) is then employed to determine the unknown constant coefficients appearing in the relevant displacement fields when the laminate is subjected to bending. Next, Reddy's LWT is utilized to examine the edge effect interlaminar stresses. For special boundary conditions, an analytical elasticity solution is developed to verify the accuracy of the LWT in describing the interlaminar stresses. Excellent agreement is seen to exist between the results of the LWT and those of the elasticity theory. Therefore, LWT can be used for all boundary conditions. The unknown constants B_2 and B_6 appearing in the displacement field are also determined within LWT and it is found that FSDT is very adequate in predicting these constants. Several numerical results according to LWT are then developed for the interlaminar stresses through the thickness and across the interfaces of the different cross-ply laminates. A convergence study is performed to determine suitable subdivisions to be used within each lamina for accurate results in LWT. It is revealed that a moderately large number of numerical layers must be employed within the laminate and in general, this number is dependent on fiber directions and the stacking sequences of the plies within the laminate.

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Appendix 1

The constant coefficients appearing in Eq. (18) are defined as follows:

$$\begin{aligned}\bar{A}_{11} &= A_{11} - A_{12}\bar{a}_1 - B_{12}\bar{b}_2 \\ \bar{B}_{11} &= B_{11} - A_{12}\bar{b}_1 - B_{12}\bar{a}_2\,, \\ \bar{D}_{11} &= D_{11} - B_{12}\bar{b}_1 - D_{12}\bar{a}_2\end{aligned} \tag{63}$$

where

$$\begin{aligned}(\bar{a}_1, \bar{a}_2, \bar{b}_1, \bar{b}_2) = \frac{1}{A_{22}D_{22} - B_{22}^2}&[(A_{12}D_{22} - B_{12}B_{22}), \\ &(A_{22}D_{12} - B_{12}B_{22}),(B_{12}D_{22} - B_{22}D_{12}), \\ &(A_{22}B_{12} - A_{12}B_{22})]\end{aligned} \tag{64}$$

also

$$\left(A_{ij}, B_{ij}, D_{ij}\right) = \int_{-h/2}^{h/2} \bar{Q}_{ij}^{(k)}\left(1, z, z^2\right) \mathrm{d}z \tag{65}$$

are the rigidities in the first-order shear deformation theory and $\bar{Q}_{ij}^{(k)}$/s are the transformed (i.e., off-axis) reduced stiffnesses of the kth layer.

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Appendix 2

The laminate rigidities, being in (28), upon integration, are presented in (Nosier and Maleki 2008). The coefficient matrices $[M]$, $[K]$, and $[T]$ appearing in Eq. (31) are given as follows:

$$[M] = \begin{bmatrix} [D_{22}] & [B_{23}] - [B_{44}]^T \\ [0] & [D_{44}] \end{bmatrix},$$
$$[K] = \begin{bmatrix} -([A_{44}] + [\alpha]) & [0] \\ [B_{44}] - [B_{23}]^T & -([A_{33}] + [\alpha]) \end{bmatrix}, \qquad (66)$$
$$[T] = \begin{bmatrix} \{0\} & \{0\} \\ \{A_{13}\} & \{\bar{B}_{13}\} \end{bmatrix}$$

where $[A_{pq}], [B_{pq}]$, and $[D_{pq}]$ are $(N + 1) \times (N + 1)$ square matrices containing A_{pq}^{kj}, B_{pq}^{kj} and D_{pq}^{kj}, respectively, and the vectors $\{A_{pq}\}, \{B_{pq}\}$, and $\{\bar{B}_{pq}\}$ are $(N + 1) \times 1$ column matrices containing A_{pq}^{k}, B_{pq}^{k}, and $\bar{B}_{pq}^{k}$, respectively. Also, $[0]$ is $(N + 1) \times (N + 1)$ square zero and $\{0\}$ is a zero vector with N + 1 rows. The artificial matrix $[\alpha]$ is also a $(N + 1) \times (N + 1)$ square matrix whose elements are given by

$$\alpha^{kj} = \alpha \int_{-h/2}^{h/2} \phi_k \phi_j \, dz \qquad (67)$$

with α being a relatively small parameter in comparison with the rigidity constants $A_{pq}^{kj} (pq = 33, 44, 55)$. It is to be noted that the inclusion of $[\alpha]$ in the matrix $[K]$ makes the eigenvalues of the matrix $\left(-[M]^{-1}[K]\right)$ be all distinct.

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