Optimal Selling Price, Marketing Expenditure and Order Quantity with Backordering

Samira Mohabbatdar [1] and Maryam Esmaeili [*1]

[1] Department of Engineering, Alzahra University, Tehran, Iran

Abstract

Demand is assumed constant in the classical economic order quantity (EOQ) model. However, in the real world, the demand is dependent on many factors such as the selling price, warranty of product and marketing effort. In addition pricing and ordering quantity decisions are interdependent for a seller when demand for the product is price sensitive in the inventory models. These types of models are very popular in the literature as joint pricing and order quantity models. Many researchers consider these models under some conditions such as quantity discount, trade credit and marketing effort. In this paper, we propose a new inventory model for the seller who conducts marketing effort. The marketing effort is the process of performing market research, selling products and/or services to customers and promoting them via advertising to further enhance sales. It is used to identify the customer, to satisfy the customer, and to keep the customer. This process will happen during the planning horizon; therefore the product will be demanded increasingly as time passes. This increasing in the demand leads to the backorder condition in the model. Since the marketing effort as a decision variable is dependent of the time, in this paper, the marketing effort is assumed a linear function of time which has an effect on the demand in addition of price in our model. The model would be included the backorder cost due to raising the shortage of inventory in addition, the purchasing, ordering and holding costs. An algorithm for finding the optimal solution for the selling price, marketing expenditure and the time length of positive stock are obtained when the seller's profit is maximized. To clarify the model more, numerical examples presented in this paper, including sensitivity analysis of some key parameter- the cost parameters and non-cost parameters- that will compare the obtained results of proposed model.

Introduction

The classical economic order quantity (EOQ) model introduced by Harris in 1913 is based on the unreal assumptions such as demand constant. Progressively, the concept of fixing demand is avoided therefore; some new inventory models are emerged (Abad, 1994; Lee, 1993; Lee et al., 1996; Kim and Lee, 1998; Jung and Klein, 2001, 2005). The demand is a function of price over a planning horizon in these models to maximize the firm's profit. In addition, some other models have been presented by assuming a general demand. In fact; the demand rate is assumed as a convex function of selling price (a linear or nonlinear function of price) under some conditions such as quantity discount or paying for freight (Papachristos and Skouri (2003), Abad (2006), Dye (2007)). Moreover, to avoid fixed demand in EOQ,

marketing expenditure would depend on demand over a planning horizon. For instance, Sahidual Islam (2008) has formulated a multi-objective marketing planning inventory model under the limitations of space capacity. The optimal order quantity, marketing expenditure and shortage amount are obtained by applying geometric programming. Similar approaches have also been used in cases where both marketing expenditure and price influence demand (Freeland, 1982; Lee and Kim, 1993, 1998; Esmaeili, 2007).

A significant shortcoming of all these models is, considering the marketing expenditure as a decision variable which is independent of the time. However in the real world, during marketing effort, the product observes increasing sales after gaining

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* Corresponding author: Tel: +98- 21- 88041469   Fax: +98- 21- 88617537   Email: esmaeili_m@alzahra.ac.ir

consumer acceptance. Ignoring the shortages of inventory is another key assumption in the classic EOQ model. Therefore, the concept of backorders (backlogged) captured the mind of inventory modelers to develop the classic EOQ models. Padmanabhan and Vrat (1990) have introduced the backlogging models to represent inventory shortage. They have not considered the necessary conditions for optimal solution, while Chu et al. (2004) used it in his model. In 2005, San Jose et al. have expanded the backlogging models, by considering lost sales penalties in their model. Considering deterministic demand is the common assumption for the above models. However, Zhou et al. (2004) have presented the partial backlogging model under time-varying demand. They emphasize the replenishment costs consist of both fixed and lot size dependent components. They have developed a numerical procedure for determining appropriate lot-sizing policies based on their exploration of the mathematical properties of the model with the sensitivity analysis. Some papers have also obtained optimal stocking policies under backorder and shortage assumption in a supply chain such as Emmett J. Lodree Jr. (2007) and Chun Jen Chung, Hui Ming Wee (2007). One of the common purposes in the mentioned backlogging models is de-termining the optimal lot size. Although the demand has a significant role in inventory models, they have ignored some factors such as price and marketing expenditure which influence on the demand.

Abad (2008) considers the pricing and lot-sizing model for a product subject to general rate of deterioration and backordering which is more realistic to compare to the previous models. The model is included all three costs-the lost sale, carrying backorders and the shortage cost. However, it is assumed that the demand is a function of one factor, price, in order to avoid the confounding effect of the demand function.

In this paper, we propose a novel model which will enable the sellers in making decisions regarding purchasing, selling price and marketing effort when the backorder occurs. The marketing effort in previous papers is considered static. However, the marketing effort will happen during the planning horizon. The marketing effort is the process of performing market research, selling product and/or services to customers and promoting them via advertising for further enhance sales. It is used to identify, satisfy and keep the customer. Therefore the product will be demanded increasingly as time passes. Unlike most of the models cited above, we use a new approach in including time in marketing effort. In fact, it is considered a linear function of time which has an effect on the demand in addition of price in our model. For that reason, the proposed model can lead to a realistic and distinguished inventory policy in comparison with the previous models. Since the marketing effort influence the demand increasingly as time passes, the seller encounters the shortage cost during the fix planning horizon. The length of the period with positive stock, selling price and marketing expenditure are considered as decision variables and the goal is to determine the optimal solution per period by using an optimization procedure. Logistic costs including purchasing, ordering, holding/carrying, and shortage costs are considered in the proposed model. Note, the order quantity and backorder level can be obtained by the demand over duration of inventory cycle.

The remainder of this paper is organized as follows. Notation, assumptions, decision variables and input parameters are provided in Section 2. A mathematical model and an algorithm for finding the optimal solution are given in Sections 3 and 4. Section 5 presents some computational results including numerical examples. Finally, the paper concludes in Section 6 with some suggestions for future work in this area.

1. Notation and problem formulation

This section introduces the notation and formulation of our model. Here, we state decision variables, input parameters and assumptions underlying the model.

1.1. Decision Variables

P Selling price ($/unit),

M Marketing expenditure per unit ($/unit),

T_1 Duration of the period with positive inventory ($T_1 \leq T$).

1.2. Input Parameters

$D(P)$ Demand rate (units/period),

$B(M)$ Marketing function,

$I(t)$ Net inventory level at time t,

$S(t)$ The shortage level at time t,

π The unit shortage cost per unit of time,

h Holding cost ($/per unit) ($/unit/period) ($\pi < h$),

k_0 The fixed ordering cost per order ($/order),

Q The order quantity,

b Backorder level,

T Duration of inventory cycle/cycle time,

K_1 Scaling constant for demand function,

α Price elasticity of demand function,

K Scaling constant for marketing function,
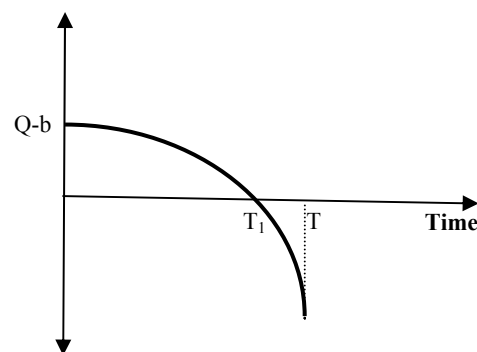
C Unit purchasing cost.

Figure 1: Inventory pattern over time

1.3. Assumptions

The proposed model is based on the following assumptions:

1. The planning horizon is infinite.
2. Duration of inventory cycle/cycle time is fixed.
3. Shortages are permitted and completely backordered.
4. The product is not perishable.
5. Demand is a function of price; for notational simplicity we let $D \equiv D(P)$

$$D = K_1 P^{-\alpha}; K_1 > 0 \qquad (1)$$

6. It is assumed that marketing effort is an increasing function of marketing expenditure and time. For notational simplicity we let $B \equiv B(M)$ such that

$$B = K + Mt; \quad 0 < t \leq T, \quad K > 0 \qquad (2)$$

The net inventory system of the seller is illustrated in Fig. 1. It can be described by the following equation. During $t \in (0 \; T_1]$,

$$\frac{dI(t)}{d(t)} = -DB, I(T_1) = 0 \qquad (3)$$

The solution to differential (3) is

$$I(t) = K_1 P^{-\alpha} (KT_1 + \frac{MT_1^2}{2} - (Kt + \frac{M}{2}t^2)) \qquad (4)$$

Where at time $t = 0$, $I(t) = I(0)$ such that Eq.(4) yields

$$I(0) = K_1 P^{-\alpha} (KT_1 + \frac{MT_1^2}{2}) \qquad (5)$$

The shortage level at time t can be described by the following equation. During $t \in [T_1 \quad T] \ t \in [T_1, T]$

$$\frac{dS(t)}{dt} = -DB, \qquad (6)$$

The solution to differential (6) is

$$S(t) = K_1 P^{-\alpha} (KT_1 - Kt + \frac{M}{2}(T_1^2 - t^2)) \qquad (7)$$

Where at time $t = T$, $S(t = T) = b$ that b is the maximum permitted backorder (shortage) level. In addition, the lot size will be obtained from (8)

$$Q = I(0) - b \qquad (8)$$

2. Mathematical models

The seller increases the level of profit over cycle time through marketing effort. It can be done by representing widely advertisement for the product at the sale locations. On one side, the seller is faced with the shortage cost due to raising level of demand by passing time. On the other side, the seller is interested in minimizing the inventory's costs that includes ordering, holding and purchasing costs. Therefore, the seller maximizes the profit by considering the holding, shortage, marketing, ordering and purchasing costs simultaneously. In other words, P, M, T_1 are determined by maximizing the annual seller's profit as follows:
Seller's Profit = Sales Revenue - Marketing Cost - Ordering Cost - Purchasing cost - Holding Cost - Shortage cost, during the inventory cycle of time span (o T] such that:

Sales revenue=

$$\int_0^{T_1} PDB(M)dt + \int_{T_1}^{T} PDB(M)dt$$
$$= PDKT + \frac{PDMT^2}{2}. \qquad (9)$$

Marketing cost=

$$\int_0^{T_1} MDB(M)dt + \int_{T_1}^{T} MDB(M)dt$$
$$= MDKT + \frac{DM^2T^2}{2}. \qquad (10)$$

The present worth of the holding cost in the period (0 T_1] is
Holding cost=

$$h\int_0^{T_1} I(t)dt = h(K_1 P^{-\alpha} \frac{KT_1^2}{2} + \frac{MT_1^3}{3}). \qquad (11)$$

As it is seen in Fig.1, the inventory level gradually decreases to meet demand. By this process the inventory level reaches zero level at time T_1 and then shortages are allowed to occur. It can also be shown in a similar manner that the present worth of the shortage cost during the period $[T_1 \ T]$
Shortage cost=

$$\pi\int_{T_1}^{T} S(t)dt = \pi K_1 P^{-\alpha} (K(T_1 T - \frac{T_1^2}{2} - \frac{T^2}{2}) +$$
$$M(\frac{TT_1^2}{2} - \frac{T_1^3}{3} - \frac{T^3}{6})). \qquad (12)$$

The fixed Ordering Cost=K_0 $\qquad$ (13)

The present worth of the purchase cost (denoted by PC) in the cycle time (0 T] is:
PC=

$$\int_0^{T_1} CDB(M)dt + \int_{T_1}^{T} CDB(M)dt$$
$$= CDKT + \frac{CDMT^2}{2}. \qquad (14)$$

Profit during time-span (0 T] is as follows:

$$F(P,M,T_1) = PDKT + \frac{PDMT^2}{2} - MDKT$$
$$- \frac{M^2DT}{2} - K_0 - \frac{hDKT_1^2}{2} - \frac{hDMT_1^3}{3} - \pi DKT_1 T$$
$$- \frac{\pi DMT_1^2 T}{2} + \frac{\pi DKT_1^2}{2} + \frac{\pi DKT^2}{2} + \frac{\pi DMT_1^3}{3}$$
$$+ \frac{\pi DMT^3}{6} - CDKT - \frac{CDMT^2}{2}, \qquad (15)$$

Or the annual profit

$$\Pi(P,M,T_1) = PD(K + \frac{MT}{2}) - MD(K + \frac{MT}{2})$$

$$- \frac{K_0}{T} - \frac{hD}{T}(\frac{KT_1^2}{2} + \frac{MT_1^3}{3}) - CD(K + \frac{MT}{2})$$

$$- \pi D(KT_1 - \frac{KT_1^2}{2T} - \frac{KT}{2} + \frac{MT_1^2}{2} - \frac{MT_1^3}{3T}$$

$$- \frac{MT^2}{6}).$$

(16)

To maximize the seller's profit we have

$$Max\Pi(P;M;T_1)$$

Subject to P, M $\geq$ 0, 0 < T$_1 \leq$ T   (17)

3. An algorithm for finding the optimal solution

According to (17) the problem is to determine the selling price, P, marketing expenditure, M, and the length of the period with positive stock of the item (T_1) such that the annual profit is maximized. However, $\Pi(P, M, T_1)$ as defined in (17) cannot easily be proven to be a concave function. For this reason, theoretically (17) can have multiple local maximum. Therefore, the optimal solution can be obtained by a line search. However, we will use another procedure similar to that used in Esmaeili's paper (2009). In this procedure, the selling price P is assumed to be fixed. Since P is fixed, the objective function $t \in (0, T]$ given in (17) will be denoted as $\Pi(M,T_1|P)$ then the optimization problem would be

$$Max\Pi(M,T_1|P)$$ (18)

Subject to M $\geq$ 0, 0 < T$_1 \leq$ T   (19)

It can be shown that $\Pi(M,T_1|P)$ is a strictly concave function (refer to Appendix A) for $T_1 > 0$, $M \geq 0$ with respect to M, T_1 for fixed P. Since (18) and (19) are concave and linear in sequence, it would be a unique global maximum for the model. The described procedure for determining optimal M and T_1 with a fixed P is considered as the first step. In the next step, (16) is defined by $\Pi(P|T_1,M)$ when M and T_1 are fixed. $\Pi(P|T_1,M)$ is as an

unconstrained problem, which can be maximized locally by using a standard non-linear programming software. In other words, $\Pi(P|T_1,M)$ can be improved by starting with a current solution in each iteration. The following procedure explains the used algorithm.

3.1. Solution

1. Let $P = P_0$, where P_0 is some arbitrary starting value for P,

2. For the current P, solve (18) and (19) and let the optimal solution to (18) be denoted as M_0^* and $T_{1_0}^*$,

3. Let $M = M_0^*$ and $T_1 = T_{1_0}^*$ and maximize $\Pi(P|T_1,M)$ locally, let the value of P that maximizes $\Pi(P|T_1,M)$ be the current P.

Given concavity of $\Pi(M,T_1|P)$, when $\Pi(M,T_1|P)$ is solved for current P, $\Pi(M,T_1|P)$ should improve and consequently $\Pi(P,M,T_1)$ should improve in Step 3. The computational procedure given above would converge to a local maximum of $\Pi(P,M,T_1)$ by repeating Steps 2 and 3. Since there could be multiple local maxima, the above procedure should be repeated with different values of P_0 to identify the global maximum. Note that only a starting value in one-dimensional space P rather than the three-dimensional space P, M and T_1 is required. Therefore, the search for the global maximum would not be time consuming and can be carried out by using standard non-linear programming software.

4. Numerical Examples

In this section we explain our model by presenting two examples including the optimal solutions. Consider the seller needs to do marketing effort to increase the profit with many substitutes in a very competitive market. Let the scenario be as follows. $h = 2$ ($/per unit) ($/unit/period), duration of inventory cycle/cycle time is defined as $T =$

7.The order cost and the purchase cost are given by k_0 = 700000 ($/*order*) and C = 20 in sequence. In addition, the marketing effort is set at B = 20+Mt which K = 20 and $t \in (0 \ T]$. $t \in (0 \ T]$ We solve the proposed model with starting value P_0 = 31 by following the procedure in section 4.1.

Example 1: Assume, the demand for the product is defined as D = 700000P$^{-1.5}$ which K_1 = 700000, α=1.5 and the unit shortage cost per unit of time is defined as π =1.5. The seller would like to determine an optimal policy on selling price, marketing expenditure, the length of the period with positive stock of the item, order quantity and backorder level. By solving the proposed model, P^* = 212.71 per unit, M^* = 1.87, $Q^* - b^*$= 763, b^* = -342, T_1^* = 5.169 and maximum profit of the seller is 1117385 unit. Also, the seller's marketing effort is B = 30.61472 with requested demand D^* = 225.6347.

Example 2: Suppose that the demand is set at D = 2000000P$^{-1.6}$ and π =1.1 while the other parameters are the same as the previous example. Therefore, by solving the proposed model, the seller obtains maximum 2326938 maximum profit by doing marketing effort B = 27.8 with requested demand D^* D^*= 974 and T_1^*= 2.86. In addition, the seller should choose the selling price, marketing expenditure, order quantity and back order level 117, 1.46, 4997 and -3200 consequently for the optimal policy. The demand's scaling constant is less in the first example, in contrast to the second one. Therefore, the demand increases in the second example which causes an increasing in the shortage level. Thus, the seller has to reduce the shortage cost to maximize the profit. In such a situation the seller tries to have less marketing expenditure and selling price.

4.1. Sensitivity Analysis

Sellers need to understand how varying key parameters affect the optimal solutions, where this helps them to improve their current policy. We investigate the effect of cost parameters (h, π) and non-cost parameters *(α, K)* on, P^*, M^*, b^*, D^* and Π^* in the model through a sensitivity analysis. We will fix *K_1 = 700000, K_0 = 2010, C = 20, T = 7* as in the previous example 1 but allow α, π, h, K to vary. Results of the sensitivity analysis are summarized in Tables 1~4.

α	1.5	1.6	1.7	1.8	1.9
M	1.87	1.75	1.68	1.63	1.5
T_1	5.169	3.59	2.58	1.91	0.04
P	212.71	113.7	74.16	55.62	35.192
Q	1104.2	1032.5	909.72	768.76	8238.6
b	341.2	562.9	621.5	792.1	8278.2
D	225.6	359.8	463.2	505.4	806
Π	1117385	881438	672793	497524	378258

Table 1: Sensitivity analysis of the model for the demand with respect to α

K	1	5	10	15	30
M	1.9	1.75	1.64	1.58	1.5
T_1	5.6	3.5	2.06	1.12	0.2
P	191	88	56	44.7	37
Q	129	456	1093	2173	5874
b	45	288	854	1903	5716
D	54	219	490	748.79	1003
Π	53163	160400	296688	424381	759817

Table 2: Sensitivity analysis of the model for the demand with respect to K

h	2	3	4	5	6
M	1.63	1.58	1.55	1.54	1.53
T_1	1.9	0.29	0.13	0.08	0.06
P	55	39	38.13	38	37.8
Q	176	500	601.32	628.2	643
b	592	14239	39535	63821	82883
D	505	948.8	997	1009	1012
Π	497523	540824	541516	541598	541618

Table 3: Sensitivity analysis of the model for the demand with respect to h

| π	1.1	1.2	1.3	1.4	1.5
M	1.3	1.38	1.46	1.54	1.63
T_1	1.56	1.58	1.63	1.73	1.91
P	58	56	55.5	55	54
Q	1334	1259	1140	975	768
b	1079	1017	915	772	592
D	465	489	507	514	550
Π	452226	466156	478859	479741	497523

Table 4: Sensitivity analysis of the model for the demand with respect to π

As is seen in figure 2 (a, b, c, d), when price elasticity, α, increases the selling price decreases. Therefore, the seller is faced with the huge demand which increases the shortage level. In such a situation, the seller decreases the marketing expenditure. Due to increasing shortage level, the shortage cost goes up which reduces the seller's profit. It seems that the model is very sensitive to α such as the real world. The seller should have more attention on α because the customer is very aware.

According to Eq. (2), scaling constant for marketing function (K) explains elasticity of the product in market without marketing effort (from the entry time of the product). By increasing (K), the seller does not need to increase the marketing expenditure. Because at the entry time of the product into the market (t = 0), product's demand would be high. Therefore, the product shortage occurs and the seller has to decline price to keep customers. Once more, we observe again that the proposed model is compatible with the real world.

As is seen in figure 2, when the holding cost (h) increases the seller is interested in reducing the amount of order therefore the seller encounters with more shortage cost compared to the holding cost. In this situation, the product will be faced with falling demand because of dissatisfaction of customer. Therefore, the seller tries to reduce the price to keep the customer and increase customer's satisfaction. According to Eq.(1),

the demand increases by decreasing the price. Moreover, the seller does not prefer to raise the marketing expenditure when the shortage cost increases. Totally, the result of proposed model shows that the seller chooses the right strategy.
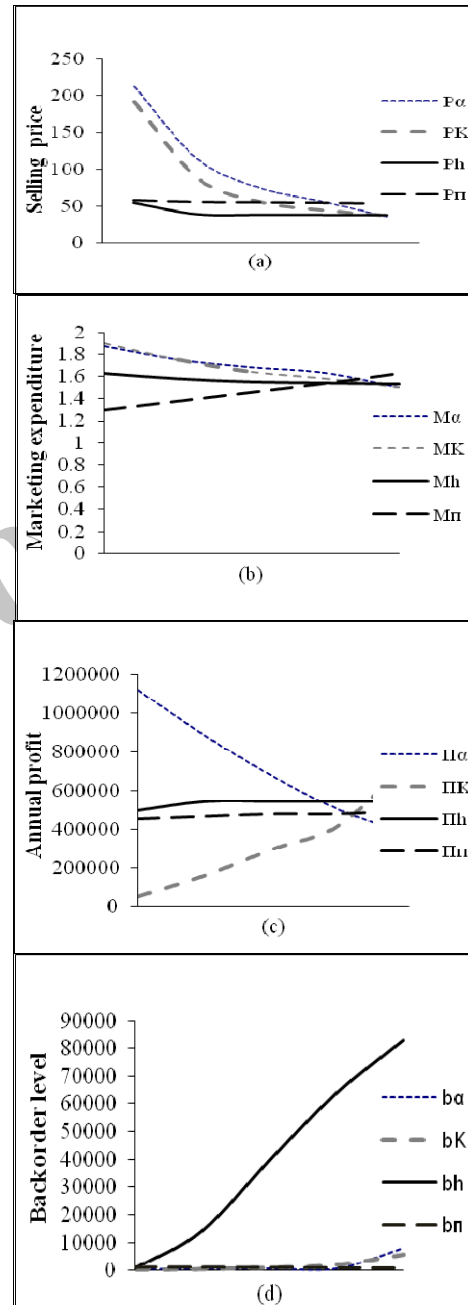
Figure 2: The effect of parameters α, K, h and π, on (a) Pi, (b) Mi, (c) Πi, (d) bi; i = α, K, h, π, for demand

When the unit shortage cost (π) increases, the seller prefers to face with less shortage. However, it is assumed that the unit shortage cost is less than shortage cost (π < h).

Therefore the seller has to make a balance between the shortage and holding cost to increase his/her profit. Thus, the seller reduces the price to increase the demand according the above mentioned reason. The sensitivity analysis of model shows that the proposed model is more realistic and distinguishes than other models.

5. Conclusions

In this paper, a new inventory model is presented subject to the impact of marketing effort. In the previous models, marketing effort was considered independent of time however; it is related to the time in the real world. Therefore, it is assumed the marketing effort is a linear function of time. The seller's profit is maximized while the demand is sensitive to the selling price. With the marketing effort the product will be demanded increasingly as time passes. The increasing in the demand leads to the backorder condition in the model. Therefore, the model would be included with the backorder cost due to raising of the inventory shortage as well as, the purchasing, ordering and holding costs. We have shown that the seller's objective function is a concave function of selling price, marketing expenditure and length of the period with positive stock as the decision variables. The optimal solution is obtained by considering the purchasing, ordering, holding/carrying, and shortage costs. Numerical examples are also presented, which are aimed to illustrate the model. There is more scope in extending the present work. For example, parameters and decision variables can be considered random or even fuzzy. Other parameters of a distributed system that was not included in this paper, such as perishability of product could be added to the model. Finally, the proposed model can be made even more realistic by considering the model in the supply chain.

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Appendix A:

The appendix contains the proof of the strict concavity of Π ($P,M,\ T_1$) with respect to M and T_1 for a fixed P, $\Pi(M,T_1|P)$. $\Pi(M,T_1|P)$ The terms in $\Pi(M,T_1|P)$ $\Pi(M,T_1|P)$ are separable such that:

$$\Pi(M,T_1|P) = \frac{PDMT}{2} - MDK - \frac{M^2DT}{2}$$
$$+ \frac{\pi DMT^2}{6} - \frac{CDMT}{2} \tag{20}$$

$$\frac{\pi DKT_1^2}{2T} - \pi DKT_1 - \frac{hDKT_1^2}{2T} \tag{21}$$

$$+ \frac{\pi DMT_1^3}{3T} - \frac{\pi DMT_1^2}{2} - \frac{hDMT_1^3}{3T} \tag{22}$$

In order to prove strictly concavity of $\Pi(M,T_1|P)$ $\Pi(M,T_1|P)$, it suffices to show that (20), (21) and (22) are concave. Concavity of (20):

$$\frac{d^2\Pi(M,T_1|P)}{dM^2} = -DT$$

Then (20) is concave. Concavity of (21):

$$\frac{d^2\Pi(M,T_1|P)}{dT_1^2} = \frac{\pi DK}{T} - \frac{hDK}{T}$$

Since $\pi < h$ then (21) is concave. Concavity of (22):

$$\frac{d^2\Pi(M,T_1|P)}{dM^2} = 0$$

$$\frac{\delta^2\Pi(M,T_1|P)}{\delta M \delta T_1} = -\pi DT_1 + \frac{\pi DT_1}{T} - \frac{hDT_1^2}{T}$$

Therefore, the Jacobian is

$$\frac{d^2\Pi(M,T_1|P)}{dT_1^2}\frac{d^2\Pi(M,T_1|P)}{dM^2} - \left(\frac{\delta^2\Pi(M,T_1|P)}{\delta M \delta T_1}\right)^2$$

$$= (-\pi DT_1 + \frac{\pi DT_1^2}{T} - \frac{hDT_1^2}{T})^2 < 0$$

Which implies $\Pi(M,T_1|P)$ is concave [17].