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RESEARCH PAPER

Applications of q-Homotopy Analysis with Laplace Transform and Padé- Approximate Method for Solving Magneto Hydrodynamic Boundary-Layer Equations

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Abstract

The magnetohydrodynamic (MHD) boundary layer flow under a magnetic field's effect was analyzed using a new analytical method named q-HALPM. This technique is based on integrating the q-homotopy analysis method with the Laplace transform, and Padé approximation methods. The flow velocity and heat transfer were investigated under the impact of the magnetic field. The results show that when the magnetic field increases from 2000 to 20000, then the velocity and heat decrease. Moreover, the results establish that q-HALPM is effective and extremely accurate in determining the analytical approximate solution for magnetohydrodynamic (MHD) boundary layer equations. Furthermore, the graphs of this novel solution demonstrate the validity, and usefulness of q-HALPM, and are consistent with the results of earlier studies.

Keywords: q-Homotopy analysis method; Laplace Transform; Padé approximation; MHD; Boundary layer**.**

1. Main text

Magnetohydrodynamics (MHD) is the study of the interaction of conductive fluids with magnetic phenomena (the flow of an electrically conductive fluid in the presence of a magnetic field) and has applications in various fields of technology and engineering, such as the MHD of generators, power generation, pumps, etc. The boundary velocity of the flow of a viscous fluid resulting from stretched boundaries is important in extrusion processes. The flow of the boundary layer in recent years has received great attention, and this stems from the many applications of engineering, technology, and metallurgical industries such as polymer extrusion, wire drawing, continuous casting, and others [1]. Most of the boundary layer MHD models can be reduced by converting them to systems of nonlinear ordinary differential equations (ODEs). Many researchers have solved the MHD boundary layer problem using numerical and analytical methods. Such as Sajid and Hayat [2] studied the magnetic hydrodynamic viscous flow problem due to the shrinkage of the plate in the case of two-dimensional shrinkage and axial contraction, and it was solved by the homotopy analytic method (HAM). Sajid et al [3] solved the MHD problem of the circulating flow of a viscous fluid over a shrinkage plate using the homotopy perturbation method, and the results

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for the contracted surface showed that the solutions are stable and convergent for MHD flows. Jasim [4] found a solution to the problem of the Jeffrey-Hammel flow for a non-Newtonian Casson fluid (JHFCF) by the perturbation iteration method (PIM) and Rang-Kata of the fourth order (RK 4), after converting the partial differential equations to a nonlinear ordinary differential equation using the similarity transformation. They compared the solutions with the variance parameter method (VPM) to verify the effectiveness of the PIM, and the effect of the parameters is explained graphically. Hammuch et al [5] investigated the MHD convection cooling problem of a low-temperature resistance plate moving downward in a viscous fluid. The equation resulting from the similarity transformation is solved by the homotopy perturbation method(HPM) to calculate the velocity and temperature. Jhankal [6] applied the HPM to solve nonlinear partial differential equations for the boundary layer of the MHD problem with a low-pressure gradient on a flat plate. Murad and Hassan [7] discovered a new method that combines HPM and Padé approximations, symbolized by (HPM- Padé), and applied it to the boundary value problem that results from the magnetic hydrodynamic viscous flow due to plate shrinkage. Rashidi and Ganji [8] found the analytical solution to the boundary layer problem of magnetic hydrodynamics of a two-dimensional viscous flow using HPM- Padé. We compared the results with the exact solution. Ali and Kishan [9] discussed the optimum homotopy analytic perturbation method (OHAM) for a Jeffrey MHD liquid flow on an extended surface in the presence of an electric field under the influence of thermal radiation and chemical interaction. Rashidi et al [10] used the homotopy perturbation method with two additional parameters to find approximate analytical solutions to the MHD boundary layer problems for the transfer of momentum and heat. He discussed the effect of various parameters on the distribution of velocity and temperature graphically. Daniel and Marinca [11] gave approximate solutions to the problems of the fixed MHD boundary layer of viscous flow and thermal radiation across an exponentially extended slab in a porous medium using OHAM and compare the solutions with the imaging method. It was concluded that this method has high accuracy and efficiency. Jabeen et al [11] developed the differential transformation method (DTM) and variation iteration method (VIM) along with Padé approximation (PA) to find analytical solutions for the static MHD dynamic magnetic layer flux; the results included velocity and temperature. Zhu and Zheng [12] analysed the MHD flux, for an incompressible non-Newtonian fluid at the stagnation point over theoretically extended plates, after converting the partial differential equations to non-linear ordinary equations with similarity and solving by using HAM, the solution was series-convergent, where they studied the effect of the magnetic parameter on the flow velocity. Majeed [13] a new iterative method (NIM) was used to analyze the metric flow problem of a two-dimensional incompressible viscous fluid under the influence of a magnetic field. The results proved the accuracy and efficiency of the method. Khan and Faraz [14] obtained a solution to the boundary layer problem by a modified Laplace method that combines Laplace transform and Padé approximation. Abbas et al [15] conducted a study of the effects of the Darcy-Furchheimer relationship with Joule heating, magnetic field, viscous dissipation, and thermal diffusion on mass concentration and convection through a third-order fluid flow on a plate extended within a porous medium. After converting PDEs to ODEs using a suitable similarity transformation and solving them numerically with the help of Matlab built-in numerical solution bvp4c. Parida and Padhy [16] explained the study of the effect of a magnetic field on the flow of a third-grade MHD fluid within a horizontal porous channel subject to Darcy's law impedance. The governing equations are numerically modeled by the finite difference method with Newton's method. Graphical analysis was used to examine the impact of different parameters, including the Reynolds number, elastic modulus, and Hartmann and Darcy numbers on flow velocity and temperature. Moreover, some researchers have solved the current problem in different ways such as Rashidi [17] deduced the solution to the boundary layer problem of hydrodynamic flow MHD. The fundamental equations are converted into coupled ordinary differential equations by a similarity transformation and solving it analytically by the DTM- Padé double method. Numerical comparison with (RK~4) and shooting method revealed that DTM-Pade is robust with graphs showing the effect of magnetic parameter on velocity. Fathizadeh et al [18] developed the HPM to solve the same problem; they concluded that the new method is powerful and effective by comparing it with HPM, exact solution, and shooting method. Awati et al [19] discovered a semi-numerical solution to the same problem by using the Dirichlet series method with the method of stretching variables, the convergence speed of this method is excellent compared to ADM, HAM, and HPM. Adel et al. [20] studied the different factors and the magnetic field on the movement of the nanofluid of the boundary layer on an extended plate, the basic equations were solved by using Chebyshev's method, the method proved its effectiveness and high accuracy in solving the system of ordinary equations numerically. Sarma et al [21, 22], the effects of radiation and chemical reaction are used to examine the steady two-dimensional Magneto hydrodynamic nanofluid flow across a stretched sheet. Although researchers try to improve and develop some analytical methods to solve MHD flow problems and the current problem (as mentioned in the above literature), some of these methods require great time and effort to get a solution to these problems, while others require a small parameter (the amount of disturbance). As in the homotopy perturbation method. In addition, there are many analytic methods and integrative transformations for solving linear/nonlinear differential equations that do not need a perturbation parameter, such as the q-homotopy analysis method (q-HAM) discovered by [22], and the Laplace transform discovered by Simon in 1942, and the Padé approximate method, which was discovered by Henri Padé in 1890. These reasons lead us to suggest a new analytical technique through which we can overcome numerical difficulties that appear in some previous methods. According to our limited knowledge and previous research, it has been shown that using approximate analytic methods along with integrative transformations may significantly reduce many of the difficulties and negative aspects of using each method independently. So, we proposed to combine the q-HAM with the Laplace transform and Padé approximation to obtain a new analytical technique, we will symbolize (q-HALPM). The aim of the current study is an analytical technique proposal used to find approximate analytical solutions to the boundary layer problem of magnetohydrodynamics. All results are achieved during the third iteration, which is a benefit of the new methodology. Moreover, the numerical results show the novel method's effectiveness and efficiency. By comparing with DTM- Padé, M-HPM, and the exact solution, we observe that the results agree well.

2. Governing equations

The MHD boundary layer flow of an incompressible viscous fluid is controlled by flat plate continuity and Navier-Stokes equations. Fluid from normal to the stretching sheet is electrically conductive under the influence of an applied magnetic field $B(x)$. Neglecting the induced magnetic field. The resulting boundary-layer equations are [18]

$$u_x + v_y = 0, \tag{1}$$

$$uu_x + vu_y = \upsilon u_{yy} + \frac{\sigma B^2(x)}{\rho} u \tag{2}$$

where u and v are the velocity components in the x and y directions respectively, υ is the kinematic viscosity, ρ is the fluid density and σ is the electrical conductivity of the fluid. In Equation (2), the external electric field and the polarization effects are negligible and

$$B(x) = B_0 x^{\frac{k-1}{2}} \tag{3}$$

the boundary conditions are

$u(x,0) = cx^k \quad , v(x,0) = 0,$ (4)
$u(x,y) \to 0, as\ y \to 0,$ (5)
using the similarity variables

$$t = \sqrt{\frac{c(k-1)}{2\upsilon}}\, x^{\frac{(k-1)}{2}} y \quad , u = cx^k f^{`}(t) \tag{6}$$

$$v = \sqrt{\frac{c\upsilon(k-1)}{2\upsilon}}\, x^{\frac{(k-1)}{2}} [f(t) + \frac{k+1}{k-1} t f^{`}(t)], \tag{7}$$

such that f be functioned to t .
The Equations (1-4) transform to

$$f^{```}(t) + f(t) f^{``}(t) - M f^{`}(t) - \beta f^{`}(t)^2 = 0 \tag{8}$$

$$f(0) = 0 \quad , f^{`}(0) = 1 , f^{`}(\infty) = 0 , \tag{9}$$

where

$$M = \frac{2\sigma B_0^{\ 2}}{\rho c(k-1)} \quad , \beta = \frac{2k}{k+1} \tag{10}$$

3. Analytical- approximation methods.

In this section, two analytical approximate methods, the q-homotopy analysis method, and the Padé -approximate method are illustrated as the following.

3.1. q-Homotopy Analysis Method.

Considering the following equation of the form [23]

$$N[f(t)] = 0 \tag{11}$$

where N is a nonlinear operator, t is an independent variable, and $f(t)$ is an unknown function, according to the zero-order deformation equation :

$$(1 - nq) L[\theta(t,q) - f_0(t,q)] - hqH(t)N[\theta(t,q)] = 0 \tag{12}$$

Or

$$H(t,q) = (1 - nq) L[\theta(t,q) - f_0(t,q)] - hqH(t)N[\theta(t,q)] = 0 \tag{13}$$

where $n \geq 1$, $q \in [0, \frac{1}{n}]$, is an embedding parameter, L is an auxiliary linear operator, $h \neq 0$ is a convergence-control parameter, $H \neq 0$ is an auxiliary function, $f_0(t,q)$ is the initial guess of $f(t)$ and $\theta(t,q)$ is the auxiliary function that should be satisfied in the function's initial conditions. The presence of h and $H(t)$ is necessary for the q-HAM solution series to converge. The solution sequence is generated when q changes from 0 to $\frac{1}{n}$, at $q = 0$ then

$$qH(t,0) = L[\theta(t,0) - f_0(t\, n, 0)] = 0 \tag{14}$$

and $q = \frac{1}{n}$, we have

$$H\left(t,\frac{1}{n}\right) = -\frac{1}{n}hH(t)N[\theta(t,q)] = 0 \quad (15)$$

Thus, $(1-nq)L[\theta(t,q) - f_0(t,q)] = hqH(t)N[\theta(t,q)]$

when $q = 0$ and $q = \frac{1}{n}$ the equations toward becomes

$$\theta(t,0) = f_0(t)\ ,\theta\left(t,\frac{1}{n}\right) = f(t) \quad (16)$$

As q decreases from 0 to $\frac{1}{n}$ the solution $\theta(t,q)$ changes from the initial guess $f_0(t)$ to the solution $f(t)$. The solution sequence for q can be obtained from Tyler's expansion of $\theta(t,q)$ as follows.

$$\theta(t,q) = f_0(\mathrm{t}) + \sum_{i=0}^{\infty} f_i(t)\ q^i \quad (17)$$

where $f_i(t) = \frac{1}{i!}\frac{d^i f(t,q)}{dq^i}\Big|_{q=0}$

if $q = \frac{1}{n}$, then

$$\theta\left(t,\frac{1}{n}\right) = f_0(t) + \sum_{i=0}^{\infty} f_i(t)(\tfrac{1}{n})^i \quad (18)$$

defining the vectors $\vec{f_j}(t) = (f_0(t), f_1(t), f_2(t), \ldots, f_i(t))$

derivatives of Eq. (17) i terms of q and set $q = 0$ after that dividing them by $i!$, obtain the i^{th} -order deformation equation

$$L[f_i(t) - k_i f_{i-1}(t)] = hR_i[\overrightarrow{f_{i-1}}(t)] \quad (19)$$

where $R_i[\overrightarrow{f_{i-1}}(t)] = \frac{1}{(i-1)!}\frac{d^{i-1}N[\theta(t,q)]}{dq^{i-1}}\Big|_{q=0}$

and $k_i = \begin{cases} 0, i \le 1 \\ n\ , i > 1 \end{cases}$

3.2. Padé approximate Method.

The function $f(x)$ define by the power series as follows;

$$f(x) = \sum_{i=0}^{\infty} c_i x^i. \quad (20)$$

The Padé approximate of $f(x)$ is a rational function and the notation of the Padé approximant can be written by the form

$$f(x) = \frac{P_p(x)}{P_r(x)} \quad (21)$$

such that, $P_r(x)$ is a polynomial of degree at most r and $P_p(x)$ is a polynomial of degree at most p, they defined as

$$P_p(x) = \sum_{j=0}^{p} a_n x^j \quad (22)$$

$$P_r(x) = \sum_{i=0}^{r} b_i x^i \quad (23)$$

substituting Equations (20, 22, and 23) into equation (21), we get

$$c_0x^0 + c_1x^1 + c_2x^2 + \cdots = \frac{a_0 + a_1x^1 + a_2x^2 + a_3x^3 + \cdots + a_lx^p}{b_0 + b_1x^1 + b_2x^2 + \cdots + b_mx^r} + O(x^{p+r+1}) \quad (24)$$

where $b_0 = 1$.

Rearranging Equation (24) to become in the form

$$(c_0x^0 + c_1x^1 + c_2x^2 + \cdots)(b_0 + b_1x^1 + b_2x^2 + \cdots + b_mx^r) = a_0 + a_1x^1 + a_2x^2 + a_3x^3 + \cdots + a_lx^p + O(x^{p+r+1}) \quad (25)$$

From Equation (25) obtain the following equations

$c_0 = a_0$

$c_0b_1 + c_1 = a_1$

$c_0b_2 + c_1b_1 + c_2 = a_2$

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$c_0b_r + c_1b_{r-1} + c_2b_{r-2} + \cdots + c_r = a_p$

and

$c_{p+1} + c_pb_1 + c_{p-1}b_2 + \cdots + c_{p-r+1}b_r = 0$

$c_{p+2} + c_{p+1}b_1 + c_pb_2 + \cdots + c_{p-r+2}b_r = 0$

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$c_{p+r} + c_{p+r-1}b_1 + c_{p+r-2}b_2 + \cdots + c_pb_r = 0.$

To obtain Padé approximation of the function $f(x)$, we solve the above equations to calculate the values of a_i *and* b_i and then substitute them into Equation(24). For more details, see Ref [24].

4. The q-HALP algorithm.

In this section, we discuss the basic idea of q- HALP depending on the q-Homotopy analysis method with Laplace transform and Padé approximation. From the q-homotopy analysis method, we get the following equation:

$$(1 - nq)L(v(y,q) - u_0(y)) - hqN[u(y)] = 0$$

Such as $v = \sum_{i=0}^{\infty} q^i v_i$

substituting v in the above equation obtain on a system of equations of power series in q and the solution of this system by using integral obtain on values v_i

the solution is $f = \lim_{q \to \frac{1}{n}} v = \sum_{i=0}^{\infty} v_i (\frac{1}{n})^i$, $n \geq 1$.

take Laplace transform of f

$$l(f(t)) = l(\sum_{i=0}^{\infty} v_i \left(\frac{1}{n}\right)^i, t, s)$$

Now, applying Padé - approximate for

$$\mathrm{P}_{\mathrm{r}}^{p}(l(f(s))) = \text{Pade}' \{l\left(f\left(\frac{1}{s}\right), [p,r]\right), 1 \le p, r \le p+r+1$$

Finally, taking the inverse Laplace to transform to obtain the approximate solution f as;

$$f = l^{-1}[\mathrm{P}_{\mathrm{r}}^{p}(l(f\left(\frac{1}{s}\right), s, t)] \ .$$

4.1- Application q-HALP on MHD Problem:

The application of a new technique q-HALPM for finding the analytical approximate solution for the MHD boundary–layer problem can be illustrated by the following.

Applying the q-HAM on Eq. (6), we have

$$(1-nq)[L(f(t,q)) - L(f_0(t,q))] - hq[f'''(t,q) - Mf'(t,q) - \beta f'(t,q)^2 + f(t,q)f''(t,q)] \tag{26}$$

such that $L = \frac{d^3}{dt^3}$ with the property $L(D) = 0$, where D is the constant.

Suppose $f(t) = \sum_{i=0}^{\infty} q^i f_i$ in Eq. (26) Constrict the zero order of the equation

$$L(\sum_{i=0}^{\infty} p^i f) - L(f_0(t,0)) = nq\left[L(\sum_{i=0}^{\infty} q^i f_i) - L(f_0(t,0))\right] + qh[\sum_{i=0}^{\infty} q^i f_i''' + (\sum_{i=0}^{\infty} p^i f_i)(\sum_{i=0}^{\infty} p^i f_i'') - M\sum_{i=0}^{\infty} p^i f_i' - \beta \sum_{i=0}^{\infty} p^i f_i'^2] \tag{27}$$

$$L(f_0) = 0 \; with \;\; f_0(0) = 0 , f_0'(0) = 1 , f_0''(0) = \alpha$$

then the i^{th} - order is

$$L(f_i) = x_i L(f_{i-1}) + hK_i(f_{i-1})$$

take L^{-1} to above equation, we have

$$f_i = x_i f_{i-1} + hL^{-1}(K_i(f_{i-1}))$$

with boundary conditions $f_i(0) = 0$, $f_i'(0) = 0$, $f_i''(0) = 0$ for $i \ge 1$.

as $x_i = \begin{cases} 0, i \le 1 \\ n, i > 1 \end{cases}$ and

$$K_i(f_{i-1}) = \frac{1}{(i-1)!} \frac{d^{i-1} f(t,q)}{dq^{i-1}}\bigg|_{q=0} = f_{i-1}''' - Mf_{i-1}' - \beta \sum_{j=0}^{i-1} f_j' f_{j-i-1}' + \sum_{j=0}^{i-1} f_j f_{j-i-1}''$$

hence the solutions are

$$f_i = x_i f_{i-1} + h \int\int\int (K_i(f_{i-1})) + D\frac{t^2}{2} + Ct + A.$$

The three integral constants D,C,A determine by boundary conditions.

The solution of q-HAM is

$$f_0 = t + \frac{1}{2}\alpha t^2 \ .$$

$$f_1 = h(\frac{-1}{6}(\beta + M)t^3 + \frac{1}{2}\alpha\left[\frac{1}{24}(4\beta + 2M - 2)t^4 + \frac{1}{60}(2\beta\alpha - \alpha)t^5\right]) \ .$$

$$f_2 = nh\left(\frac{-1}{6}(\beta+M)t^3 + \frac{1}{2}\alpha(\frac{1}{24}\,(4\beta+2M-2)\,t^4 + \frac{1}{60}\,(2\beta\alpha-\alpha)t^5)\right) + \frac{h}{120}[\,(12h\beta^2 + 3h\beta M + hM^2 - 2h\beta - 2hM + h\alpha^2 - 2h\beta\alpha^2)t^5 + (\frac{5}{3}h\alpha\beta^2 + \frac{5}{3}h\alpha\beta M + \frac{1}{6}h\alpha M^2 - 2h\alpha\beta - \frac{4}{3}h\alpha M + \frac{1}{2}h\alpha))t^6 + (\frac{10}{21}h\beta^2\alpha^2 + \frac{5}{21}h\beta\alpha^2 m - \frac{16}{21}h\beta\alpha^2 - \frac{4}{21}h\alpha^2 m + \frac{11}{42}h\alpha^2)t^7 + (\frac{21}{84}h\beta^2\alpha^3 - \frac{2}{21}h\beta\alpha^3 + \frac{11}{330}h\alpha^3)t^8]$$

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$$f_3 = n\left[nh\left(\frac{1}{6}(\beta+M)t^3 + \frac{1}{2}\alpha(\frac{1}{24}\,(4\beta+2M-2)\,t^4 + \frac{1}{60}\,(2\beta\alpha-\alpha)t^5)\right) + \frac{h}{120}[\,(12h\beta^2 + 3h\beta M + hM^2 - 2h\beta - 2hM + h\alpha^2 - 2h\beta\alpha^2)t^5 + (\frac{5}{3}h\alpha\beta^2 + \frac{5}{3}h\alpha\beta M + \frac{1}{6}h\alpha M^2 - 2h\alpha\beta - \frac{4}{3}h\alpha M + \frac{1}{2}h\alpha))t^6 + (\frac{10}{21}h\beta^2\alpha^2 + \frac{5}{21}h\beta\alpha^2 m - \frac{16}{21}h\beta\alpha^2 - \frac{4}{21}h\alpha^2 m + \frac{11}{42}h\alpha^2)t^7 + (\frac{21}{84}h\beta^2\alpha^3 - \frac{2}{21}h\beta\alpha^3 + \frac{11}{330}h\alpha^3)t^8]\right] + \frac{1}{6}(-\beta h^3 - 2\beta h^2 n - 2\beta h n^2 - Mh^3 - 3h^2Mn - 2hMn^2)t^3 - h(\frac{1}{967680}(80640\beta\alpha h^2 + 241920\beta\alpha hn + 161280\beta\alpha n^2 + 40320\alpha h^2 M + 120960\alpha hMn + 80640\alpha Mn^2 - 40320\alpha h^2 - 120960\alpha hn - 80640\alpha n^2)t^4 + \frac{1}{2419500}(40320\beta\alpha^2h^2 + 120960\beta\alpha^2 hn + 80640\beta\alpha^2n^2 - 80640\beta^2h^2 - 120960\beta^2 hn - 120960\beta h^2 M - 181440\beta hMn - 20160\alpha^2h^2 - 60480\alpha^2 hn - 40320\alpha^2n^2 - 40320M^2h^2 - 60480hM^2n + 80640\beta h^2 + 120960\beta hn + 80640h^2M + 120960hMn)t^5 + \frac{1}{4838400}(-134400\beta^2\alpha h^2 - 201600\beta^2\alpha nh^2 - 134400\beta\alpha h^2 M - 201600\beta\alpha hMn - 13440\alpha m^2h^2 - 20160\alpha hM^2n + 161280\beta\alpha h^2 + 241920\beta\alpha hn + 107520\alpha h^2 M + 161280\alpha hMn - 40320\alpha h^2 - 60480\alpha hn)t^6 + \frac{1}{846720}(-67200\beta^2\alpha^2h^2 - 1008600\beta^2\alpha^2h^2 - 33600\beta\alpha^2Mh^2 - 50400\beta\alpha^2h^2Mn + 16800\beta^3h^2 + 33600\beta^2Mh^2 + 107520\beta\alpha^2h^2 + 161280\beta\alpha^2h^2n + 18480\beta M^2h^2 + 26880\alpha^2Mh^2 + 40320n\alpha^2Mh^2 + 1680M^3h^2 - 26880h^2\beta^2 - 43680\beta Mh^2 - 1680M^2h^2 - 36690\alpha^2h^2 - 55440\alpha^2h^2n + 1344\beta h^2 + 1344Mh^2)t^7 + \frac{1}{13547520}(-13440\alpha^3\beta^2h^2 - 1596\alpha^3\beta^2h^2n + 26880\alpha\beta^3h^2 + 40320\alpha\beta^2Mh^2 + 14112\alpha\beta M^2h^2 + 21504\alpha^3\beta h^2 + 32256\alpha^3\beta h^2 n + 336\alpha M^3h^2 - 7392\alpha^3h^2 - 11088\alpha^3h^2n - 55776\alpha h^2\beta^2 - 65856\alpha\beta M - 13104\alpha M^2h^2 + 34608\alpha\beta h^2 + 26880\alpha Mh^2 - 5040\alpha h^2)t^8 + \frac{1}{20321280}(16800\beta^3\alpha^2h^2 + 16800\beta^2\alpha^2Mh^2 + 2352\beta\alpha^2M^2h^2 - 39760\beta^2\alpha^2h^2 - 30576\beta\alpha^2Mh^2 - 2184\alpha^2M^2h^2 + 30688\beta\alpha^2h^2 + 13608\alpha^2Mh^2 - 7224\alpha^2h^2)t^9 + \frac{1}{29030400}(4800\beta^3\alpha^3h^2 + 2400\beta^3\alpha^3Mh^2 - 12768\beta^2\alpha^3h^2 - 4368\beta\alpha^2Mh^2 + 11184\beta\alpha^3 + 1944\alpha^3h^2M - 3000\alpha^3h^2)t^{10} + \frac{1}{39916800}(600\beta^3\alpha^4h^2 - 1596\beta^2\alpha^4h^2 + 1398\beta\alpha^4h^2 - 375\alpha^4h^2)t^{11})$$

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substituting the functions f_i in the power series $\sum_{i=0}^{n} f_i\,(\frac{1}{n})^i$ to find the function f.
The solution of the Equation (6) by q-HAM at is $M = 10, \beta = 0.5$.

$$f(t) = -1.65t^2 + t - 2.838t^4 + 3.612t^3 - 0.3656e - 1t^7 - .2141t^6 + .5292t^5 - 0.2853e - 3t^{10} - 0.2584e - 2t^9 + 0.3536e - 2t^8$$

.

Now, take Laplace transform of f afterward us reception Padé approximate: we have

$$F(s)_{[2,1]} = \frac{(\frac{1}{s})^2}{1+(\frac{33}{10})\frac{1}{s}}\ .$$

Final, take inverse Laplace transform of $F(s)_{[2,1]}$ we get.

$$f(t) = \frac{10}{33} - \left(\frac{10}{33}\right) * exp\left(-\left(\frac{33}{10}\right)t\right)\ .$$

Obtain on value α from using the boundary condition $f'(+\infty) = 0$.
The results obtained from solving the problem are given in the below graphs.

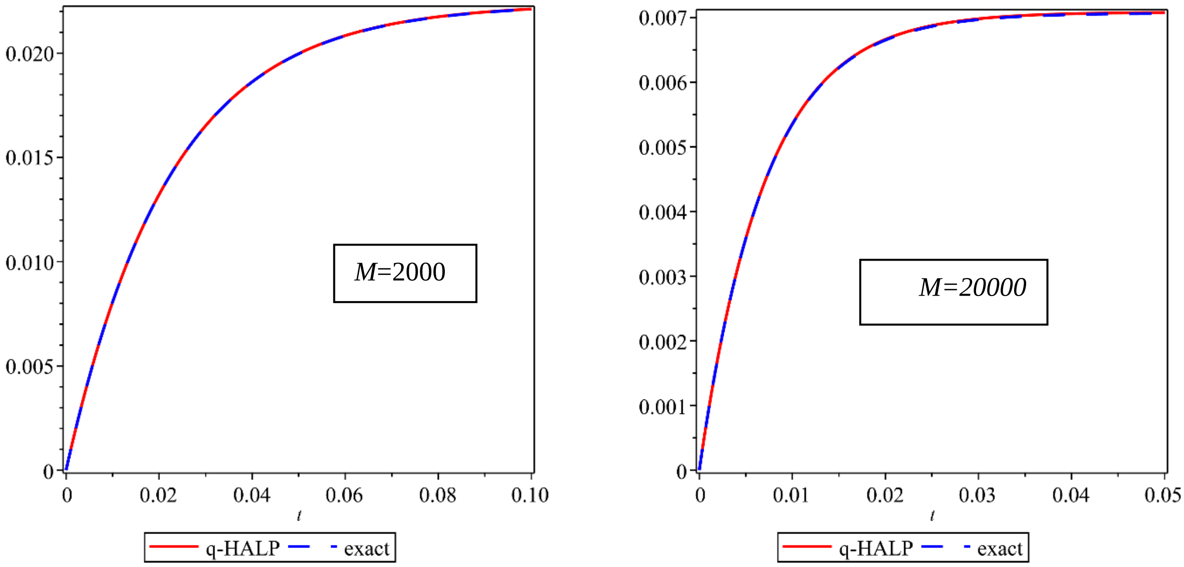

Fig-1: Analytical solution of $f(t)$ using q-HALP and exact solution at $\beta = 1$.

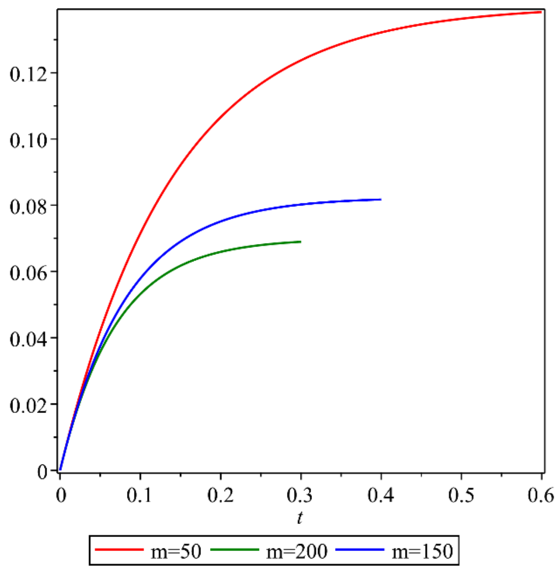

Fig-2: Effects magnetic parameter on an analytical solution using q-HALP at $\beta = 0.5$

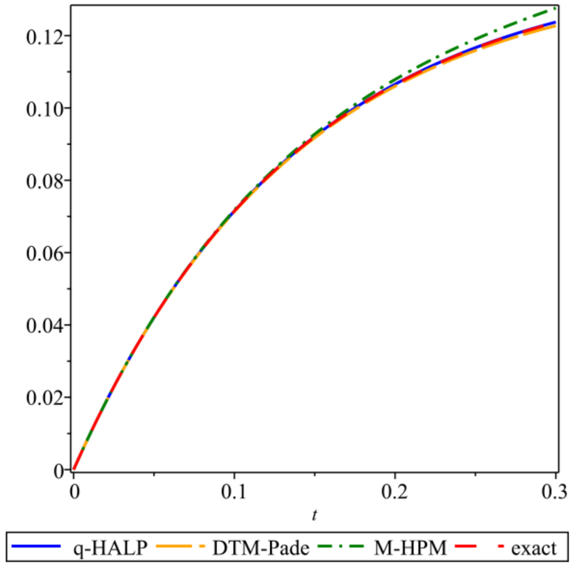

Fig-3: Compression between q-HALP, M-HPM and DTM- Padé with exact where $M = 50, \beta = 1.5$.

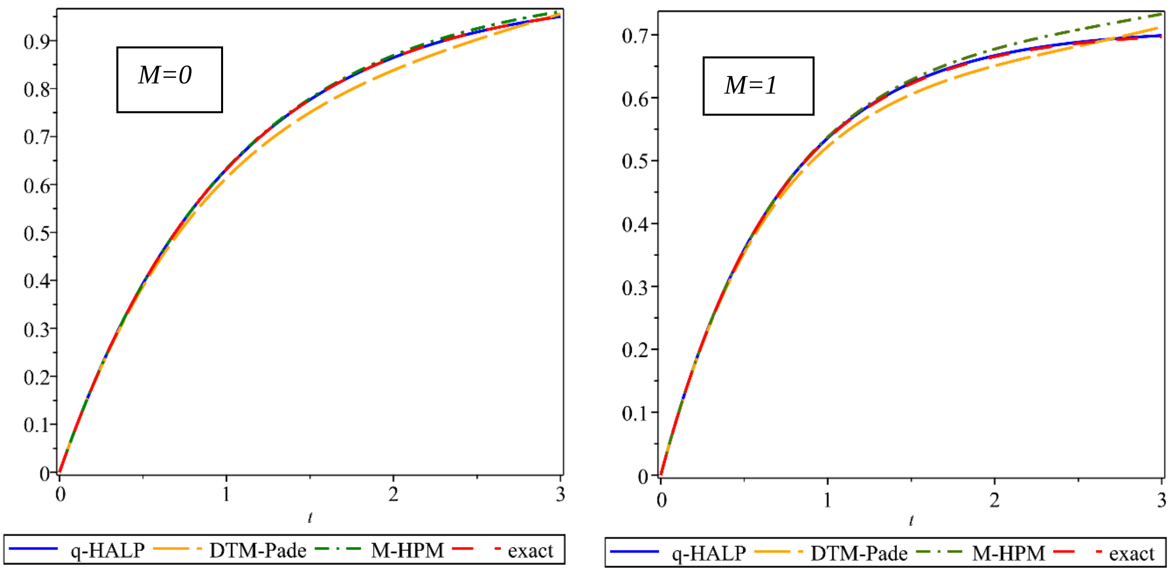

Fig-4: Compression between q- HALP , M-HPM and DTM- Padé with exact where $\beta = 1$

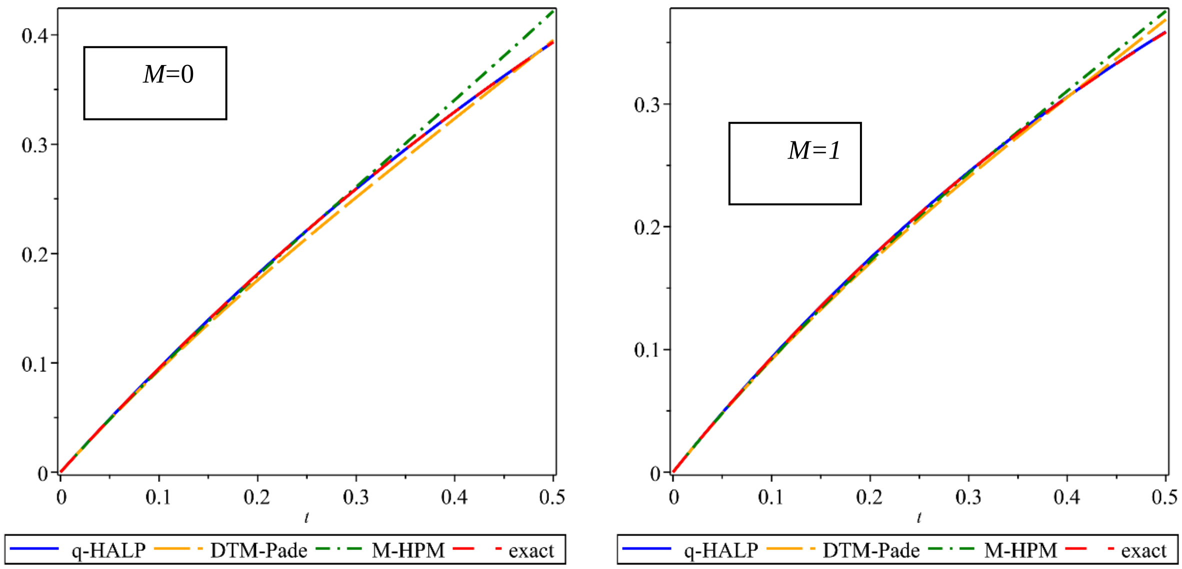

Fig-5: Compression between q- HALP , M-HPM and DTM- Pade´ with exact where $\beta = 5$

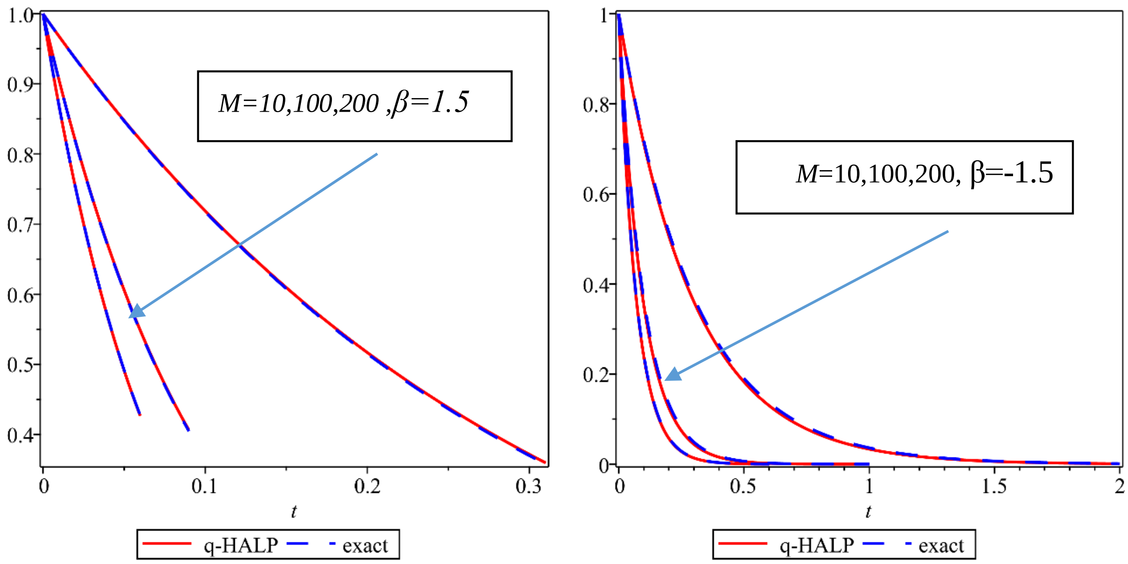

Fig-6: Compression of f' that resulting from q-HALP, and exact solution for differential value M .

Table-1: compression of the absolute errors of the analytical solution using the methods $q-HALPM$,$DTM-Pade'$, $M-HPM$ with an exact solution at $\beta=1.5$ and $t=0.3$.

M	Error measurement	q − HALP	DTM-Pade [17]	M-HPM [18]
50	L_2	3.48×10^{-6}	2.26×10^{-3}	1.87×10^{-4}
	L_∞	1.42×10^{-4}	0.08	0.01
100	L_2	2.63×10^{-8}	1.06×10^{-5}	5.29×10^{-5}
	L_∞	9.88×10^{-7}	3.98×10^{-4}	3.12×10^{-3}
500	L_2	1.95×10^{-9}	6.21×10^{-6}	3.51×10^{-5}
	L_∞	5.79×10^{-8}	1.84×10^{-5}	2.80×10^{-3}

Table-2: compression of the absolute errors of the analytical solution using the methods $q-HALPM$, DTM-Padé, $M-HPM$ with an exact solution at $M=10$ and $t=0.5$.

β	Error measurement	q − HALP	DTM-Pade [17]	M-HPM [18]
0	L_2	7.65×10^{-6}	7.50×10^{-4}	2.80×10^{-4}
	L_∞	1.96×10^{-4}	2.50×10^{-2}	9.82×10^{-3}
5	L_2	6.08×10^{-6}	2.35×10^{-3}	5.33×10^{-4}
	L_∞	1.56×10^{-4}	8.18×10^{-2}	2.04×10^{-2}
10	L_2	5.91×10^{-6}	2.54×10^{-3}	4.40×10^{-4}
	L_∞	1.51×10^{-4}	9.44×10^{-2}	1.92×10^{-2}

Where, L_2, and L_∞ are defined as;

$$\|E\|_{L_2}=\sqrt{h^2\sum_{i=1}^{n}\left|f_{approximate}-f_{exact}\right|^2}$$

$$\|E\|_{L_\infty}=max_{i=0..n}\left(\left|f_{approximate}-f_{exact}\right|\right)$$

5. Results and discussion

The new technique is successfully used to solve the equations of the magnetohydrodynamic boundary layer after converting it by the similarity transformation where the results are displayed graphically. To check the accuracy of our approximate solutions to q-HALPM, we did a comparative study between the solution of q-HALPM and DTM- Padé Ref [17] and M-HPM Ref [18]. Figure 1, represents the analytical solution of q- HALPM with the exact solution f for $\beta=1$ and at $M=2000$,20000 respectively, it is evident that the thickness of the boundary layer reduces at the magnetic parameter increases. Figure 2 explains how the flow velocity rate and heat transfer are affected by an increase of the magnetic parameter M at $\beta=0.5$, we notice that both the velocity rate and heat transfer decrease as the magnetic parameter M increases. Moreover, Figure 3, shows the comparison of the solution of the current method, DTM- Padé, M-HPM and the exact solution for values M=50 and β=1.5. Figures 4 and 5, illustrates a comparison between the solution of q-HALP, DTM- Padé , M-HPM and the exact solution for different values of M and β , it is clear that the solution for the current method is very close to the exact solution. Figure 6, displays the first derivative of the function $f(t)$ at $M=10,100,200$, we observed that there is an agreement between the results obtained by the new method with the exact solution for all values of t. Finally, from tables 1 and 2, we notice that the errors of the new technique q-HALP are smaller than other methods at $t = 0.3, 0.5$. Table 3 illustrated the values of convergence for the problem at $= 0.1$, $\beta = 0.5$ and different values of M. Through the comparison, it is noted that this method is convergent for all values of small or large M, unlike the homotopy perturbation method, which converges if the values of M are small.

6. A CONVERGENCE ANALYSIS of q-HALP

In this section, we will examine the analytical approximations derived by applying the new technique q-HALPM and will do so as follows:

Definition (6.1) : Assume that Z is Banach space, N is a nonlinear mapping defined by $N: Z \to R$ and R is the real number. Then, the sequence of the solutions of q-HALP can be written in the following form

$$F_{i+1} = N(F_i) \quad F_i = \sum_{j=0}^{i} f_j \ , i = 1,2,3,4, \dots$$

where, N satisfies Lipschitz condition such that for $\gamma \in R, 0 < \gamma < n$ we have

$$\|N(F_{i+1}) - N(F_i)\| \le \gamma \|F_{i+1} - F_i\| \ , 0 < \gamma < 1$$

Theorem 1. The series of the analytical approximate solution $f = \sum_{i=0}^{\infty} f_i$ resulted of q-HALP converges if the following condition satisfies $\|f_i - f_s\| \to 0 \ as \ s \to \infty, 0 < \gamma < 1$.

Proof:

$\|F_i - F_s\| = \|\sum_{j=0}^{i} f_j - \sum_{j=0}^{s} f_j\| = \|f_0 + \sum_{j=1}^{i} N(f_j) - f_0 - \sum_{j=1}^{s} N(f_j)\|$

$= \|N(\sum_{j=1}^{i} f_j) - N(\sum_{j=1}^{s} f_j)\| \ , F_{i+1} = N(F_i)$

$= \|N(\sum_{j=0}^{i-1} f_j) - N(\sum_{j=0}^{s-1} f_j)\|$

$= \|N(F_i) - N(F_s)\| \le \gamma \|F_i - F_s\|$

As N satisfies the Lipschitz condition. Let $i = s + 1$, then

$\|F_{s+1} - F_s\| \le \gamma \|F_s - F_{s-1}\| \le \gamma^2 \|F_{s-1} - F_{s-2}\| \le \dots \le \gamma^{s-1} \|F_1 - F_0\|$

hence,

$\|F_2 - F_1\| \le \gamma \|F_1 - F_0\|$

$\|F_3 - F_2\| \le \gamma^2 \|F_1 - F_0\|$

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$\|F_{s+1} - F_s\| \le \gamma^s \|F_1 - F_0\|$

By the triangle inequality, we have

$$\begin{aligned}\|F_i - F_s\| &= \|F_s - F_{s-1} - F_{s-2} - \dots - F_{s+1} - F_s\| \\ &\le \|F_s - F_{s-1}\| + \|F_{s-1} - F_{s-2}\| + \|F_{s-2} - F_{s-3}\| + \dots \|F_{s+1} - F_s\| \\ &\le (\gamma^{i-1} + \gamma^{i-2} + \dots + \gamma^s) \|F_1 - F_0\| \\ &\le \gamma^s (\gamma^{i-1-s} + \gamma^{i-2-s} + \dots + 1) \|F_1 - F_0\| \\ &\le \frac{\gamma^s}{1-\gamma} \|F_1 - F_0\|\end{aligned}$$

if $s \to \infty$, we have $\|F_i - F_s\| \to 0$ then F_i is a Cauchy sequence in Banach space Z.

From theorem1 and definition (6.1), to achieve convergence, the values of the parameter γ^i must be computed using the relationship shown below:

$$\gamma^i = \begin{cases} \frac{\|F_{i+1}\|}{\|F_1\|}, & \|F_1\| \neq 0 \\ 0, & \|F_1\| = 0 \end{cases} .$$

Table 3: The values of γ^i by using convergence condition at $t = 0.1$ and $\beta = 0.5$.

M	γ	γ^2	γ^3
1	3.47×10^{-4}	9.63×10^{-5}	4.78×10^{-6}
10	1.80×10^{-3}	4.98×10^{-4}	9.51×10^{-5}

7. Conclusions

1- A newly developed method called q-HALP is proposed to solve magnetic and dynamic boundary equations.
2- The effect of the magnetic parameter M on the velocity and heat transfer has been clarified. As M increases, the rate of velocity and heat transfer decreases.
3- The validity and efficiency of this method were proven by comparing the results with the results of the other methods used previously.
4- We conclude that q-HALP is an effective and highly accurate method for finding approximate analytical solutions to MHD problems. From the analysis of the results, we can conclude that q-HALPM is excellent with good convergence.
5- Expect that it can be used to treat a set of diverse complicated fluid flow problems.

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