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RESEARCH PAPER

Variational principle for the Kaup-Newell system

Yue Wu [a], Ji-Huan He [b,c*]

[a] *College of Economics and Management, Shanghai University of Political Science and Law, Qingpu Area, Shanghai, China*

[b] *School of Mathematics and Information Science, Henan Polytechnic University, Jiaozuo, China*

[c] *National Engineering Laboratory for Modern Silk, College of Textile and Clothing Engineering, Soochow University, Suzhou, China*

Abstract

The Kaup-Newell system describes the pulse propagation in optical fibers. This paper applies the semi-inverse method to construct the system's variational formulations, two trial-variational formulations with an unknown function are established, and a detailed derivation is given to determine the unknown function. Finally, the Kaup-Newell system with two variables is converted to a partial differential equation with only one variable.

Keywords: Variational theory; Calculus of variations; Lagrange multiplier; Schrödinger equation; potential function

1. Introduction

The variational theory plays a significant role in computational mechanics, it is the cornerstone of various numerical methods, e.g. the finite element method [1, 2], and various analytical methods, e.g., the Rayleigh-Ritz method [3, 4] the variational iteration method [5-7] and Hamiltonian-based frequency-amplitude formulation [8]. Variational theory is also useful tool to the analysis of dynamical properties of a nano-beam [9], thermal property of thermos-elasticity [10-12]and neural networks.

Recently the variational theory was extended to the fractal space, and the fractal variational principles have been attracted much attention due to their feasibility for the establishment of a real mathematical model for discontinuous problems. Now the variational theory becomes an effective tool to fractal soliton theory and nonlinear vibration systems.

In this paper, we want to search for a variational formulation for the Kaup-Newell system, which is a special case of the Schrödinger equation.

2. Kaup-Newell system

The Kaup-Newell system can be written in the form

$$\frac{\partial u}{\partial t}-\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+2u^2 v\right)=0 \tag{1}$$

$$\frac{\partial v}{\partial t}+\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}-2uv^2\right)=0 \tag{2}$$

* Corresponding author. Tel.: +86-139-1315-2427

E-mail address: Hejihuan@suda.edu.cn

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Eq.(1) and Eq.(2) can describe the pulse propagation in optical fibers. According to Eqs.(1) and (2), we can introduce two potential functions φ and ψ satisfying the following relationship:

$$\begin{cases} \dfrac{\partial \varphi}{\partial x} = u \\ \dfrac{\partial \varphi}{\partial t} = \dfrac{\partial u}{\partial x} + 2u^2 v \end{cases} \tag{3}$$

$$\begin{cases} \dfrac{\partial \psi}{\partial x} = v \\ \dfrac{\partial \psi}{\partial t} = -(\dfrac{\partial v}{\partial x} - 2uv^2) \end{cases} \tag{4}$$

In this paper we will apply the semi-inverse method to establish two variational principles, whose stationary conditions are Eqs.(1) &(4) and Eqs.(2) &(3) respectively.

3. Variational principle

The semi-inverse method is to construct a trial variational formulation in the form:

$$J(u,v,\varphi) = \iint \left\{ v\frac{\partial \varphi}{\partial t} + (\frac{\partial v}{\partial x} - 2uv^2)\frac{\partial \varphi}{\partial x} + F \right\} dtdx \tag{5}$$

where F is an unknown function of u and v and their derivatives. The stationary condition of Eq. (5) with respect to φ is

$$\frac{\partial L}{\partial \varphi} - \frac{\partial}{\partial t}(\frac{\partial L}{\partial \varphi_t}) - \frac{\partial}{\partial x}(\frac{\partial L}{\partial \varphi_x}) = 0 \tag{6}$$

where

$$L = v\frac{\partial \varphi}{\partial t} + (\frac{\partial v}{\partial x} - 2uv^2)\frac{\partial \varphi}{\partial x} + F \tag{7}$$

Eq.(6) leads to the following equation

$$-\frac{\partial v}{\partial t} - \frac{\partial}{\partial x}(\frac{\partial v}{\partial x} - 2uv^2) = 0 \tag{8}$$

This is Eq.(2). Now the stationary condition of Eq.(5) with respect to u and v are

$$-2v^2\frac{\partial \varphi}{\partial x} + \frac{\delta F}{\delta u} = 0 \tag{9}$$

$$\frac{\partial \varphi}{\partial t} - \frac{\partial^2 \varphi}{\partial x^2} - 4uv\frac{\partial \varphi}{\partial x} + \frac{\delta F}{\delta v} = 0 \tag{10}$$

where $\delta F / \delta u$ is the fractional derivative. In view of Eq.(3), Eqs.(9) and (10) becomes

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$$\frac{\delta F}{\delta u}=2v^2\frac{\partial \varphi}{\partial x}=2uv^2 \tag{11}$$

$$\frac{\delta F}{\delta v}=-\frac{\partial \varphi}{\partial t}+\frac{\partial^2 \varphi}{\partial x^2}+4uv\frac{\partial \varphi}{\partial x}=-\frac{\partial u}{\partial x}-2u^2v+\frac{\partial u}{\partial x}+4u^2v=2u^2v \tag{12}$$

From Eqs.(11) and (12), F can be identified as

$$F=u^2v^2 \tag{13}$$

Finally we obtain the following variational formulation

$$J(u,v,\varphi)=\iint\left\{v\frac{\partial \varphi}{\partial t}+(\frac{\partial v}{\partial x}-2uv^2)\frac{\partial \varphi}{\partial x}+u^2v^2\right\}dtdx \tag{14}$$

Alternatively, we can construct a trial variational formulation in the form

$$J(u,v,\psi)=\iint\left\{u\frac{\partial \psi}{\partial t}-(\frac{\partial u}{\partial x}+2u^2v)\frac{\partial \psi}{\partial x}+f\right\}dtdx \tag{15}$$

where f is an unknown function of u and v and their derivatives. The stationary condition of Eq.(5) with respect to ψ is

$$-\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}(\frac{\partial u}{\partial x}+2u^2v)=0 \tag{16}$$

This is Eq.(1). Now the stationary conditions of Eq.(16) with respect to u and v are

$$\frac{\partial \psi}{\partial t}+\frac{\partial^2 \psi}{\partial x^2}-4uv\frac{\partial \psi}{\partial x}+\frac{\delta f}{\delta u}=0 \tag{17}$$

$$-2u^2\frac{\partial \psi}{\partial x}+\frac{\delta f}{\delta v}=0 \tag{18}$$

In view of Eq.(3), we have

$$\frac{\delta f}{\delta u}=-\frac{\partial \psi}{\partial t}-\frac{\partial^2 \psi}{\partial x^2}+4uv\frac{\partial \psi}{\partial x}=\frac{\partial v}{\partial x}-2uv^2-\frac{\partial v}{\partial x}+4u^2v=2uv^2 \tag{19}$$

$$\frac{\delta f}{\delta v}=2u^2\frac{\partial \psi}{\partial x}=2u^2v \tag{20}$$

From Eqs.(19) and (20), f can be identified as

$$f=u^2v^2 \tag{21}$$

Finally we obtain the following variational formulation

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$$J(u,v,\psi)=\iint\left\{u\frac{\partial\psi}{\partial t}-(\frac{\partial u}{\partial x}+2u^2v)\frac{\partial\psi}{\partial x}+u^2v^2\right\}dtdx \tag{22}$$

4. Constrained variational principles

After integral by parts, the variational formulations given in Eq.(14) and Eq.(22) can be converted, respectively, into the following ones

$$J(u,v,\varphi)=\iint\left\{-\varphi\left[\frac{\partial v}{\partial t}+\frac{\partial}{\partial x}(\frac{\partial v}{\partial x}-2uv^2)\right]+u^2v^2\right\}dtdx \tag{23}$$

$$J(u,v,\psi)=\iint\left\{-\psi\left[\frac{\partial u}{\partial t}-\frac{\partial}{\partial x}(\frac{\partial u}{\partial x}+2u^2v)\right]+u^2v^2\right\}dtdx \tag{24}$$

It is obvious that φ in Eq.(23) and ψ in Eq.(24) are Lagrange multipliers, as a result, we obtain the following constrained variational formulations:

$$J(\varphi)=\iint u^2v^2dtdx \tag{25}$$

$$J(\psi)=\iint u^2v^2dtdx \tag{26}$$

Eq.(25) is subject to the constraint of Eq.(2), while Eq.(25) is under constraint of Eq.(1). In view of Eqs.(3) and (4), we have

$$\begin{cases}u=\varphi_x\\ v=\dfrac{\varphi_t-\varphi_{xx}}{2\varphi_x^2}\end{cases} \tag{27}$$

$$\begin{cases}v=\psi_x\\ u=\dfrac{\psi_t+\psi_{xx}}{2\psi_x^2}\end{cases} \tag{28}$$

Finally Eqs.(25) and (26) can be expressed, respectively, as

$$J(\varphi)=\iint\varphi_x^2(\frac{\varphi_t-\varphi_{xx}}{2\varphi_x^2})^2dtdx \tag{29}$$

$$J(\psi)=\iint\psi_x^2(\frac{\psi_t+\psi_{xx}}{2\psi_x^2})^2dtdx \tag{30}$$

The stationary conditions of Eqs.(29) and (30) are, respectively, as

$$-\frac{\partial}{\partial x}\left[2\varphi_x(\frac{\varphi_t-\varphi_{xx}}{2\varphi_x^2})^2-2\varphi_x^2(\frac{\varphi_t-\varphi_{xx}}{2\varphi_x^5})\right]-\frac{\partial^2}{\partial x^2}\left[2\varphi_x^2(\frac{\varphi_t-\varphi_{xx}}{2\varphi_x^2})\right]-\frac{\partial}{\partial t}\left[2\varphi_x^2(\frac{\varphi_t-\varphi_{xx}}{2\varphi_x^2})\right]=0 \tag{31}$$

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$$-\frac{\partial}{\partial x}\left[2\psi_x(\frac{\psi_t+\psi_{xx}}{2\psi_x^2})^2-2\psi_x^2(\frac{\psi_t+\psi_{xx}}{\psi_x^5})\right]+\frac{\partial^2}{\partial x^2}\left[2\psi_x^2(\frac{\psi_t+\psi_{xx}}{2\psi_x^2})\right]-\frac{\partial}{\partial t}\left[2\psi_x^2(\frac{\psi_t+\psi_{xx}}{2\psi_x^2})\right]=0 \quad (32)$$

or

$$\frac{\partial}{\partial t}\left[\varphi_x^2(\frac{\varphi_t-\varphi_{xx}}{2\varphi_x^2})\right]+\frac{\partial}{\partial x}\left[\varphi_x(\frac{\varphi_t-\varphi_{xx}}{2\varphi_x^2})^2-\varphi_x^2(\frac{\varphi_t-\varphi_{xx}}{2\varphi_x^5})\right]+\frac{\partial^2}{\partial x^2}\left[\varphi_x^2(\frac{\varphi_t-\varphi_{xx}}{2\varphi_x^2})\right]=0 \quad (33)$$

$$\frac{\partial}{\partial t}\left[\psi_x^2(\frac{\psi_t+\psi_{xx}}{2\psi_x^2})\right]+\frac{\partial}{\partial x}\left[\psi_x(\frac{\psi_t+\psi_{xx}}{2\psi_x^2})^2-\psi_x^2(\frac{\psi_t+\psi_{xx}}{\psi_x^5})\right]+\frac{\partial^2}{\partial x^2}\left[\psi_x^2(\frac{\psi_t+\psi_{xx}}{2\psi_x^2})\right]=0 \quad (34)$$

Eq.(33) or Eq.(34) is equivalent to the Kaup-Newell system of Eqs.(1) and (2).

5. Conclusions

In this paper, by the semi-inverse method, a family of variational principles are obtained, and the two-equation system can be converted into a partial differential equation as shown in Eq.(33) or Eq.(34). The strong coupling between *u* and *v* in the original system of Eqs.(1) and (2) is successfully decoupled, this makes the computation process extremely simple.

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