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Difference Labeling and Decomposition

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Abstract. A difference labeling of a graph G is an injective function $f : V(G) \to N \cup \{0\}$ together with the weight function f^* on $E(G)$ given by $f^*(uv) = |f(u) - f(v)|$ for every edge uv in G. The collection of subgraphs induced by the edges of the same weight is a decomposition of G and is called the *common weight decomposition* of G induced by f. Let ψ_f denote the collection of all the paths taken from each member of the common weight decomposition induced by f. A difference labeling f of G is said to be a *graphoidal difference labeling* if ψ_f is an acyclic graphoidal decomposition of G. This paper initiates a study on this concepts.

Keywords: Decomposition, Graphoidal difference labeling.

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1. Introduction

By a graph $G = (V, E)$ we mean a non-trivial, finite, connected and undirected graph without loops or multiple edges. For terms not defined here,

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we refer to [7]. Throughout the paper the order and size of G are denoted by m and n respectively.

A decomposition of a graph G is a collection of its subgraphs such that every edge of G lies in exactly one member of the collection. Various types of decompositions have been introduced and studied by imposing conditions on the members of the decomposition. For instance, Harary introduced the notion of path decomposition [8] which demands each member of a decomposition to be a path. Following Harary, several variations of decomposition have been introduced and extensively studied. Unrestricted path decompositions [9], geodesic path decompositions[5] and simple path decompositions[2] are some variations of decomposition. In this direction Acharya and Sampathkumar[1] introduced the concept of graphoidal decomposition of a graph. A *graphoidal decomposition* of a graph G is a decomposition ψ of G all of whose members are paths or cycles such that every vertex of G is an internal vertex of at most one member of ψ. A graphoidal decomposition wherein no member is a cycle is called an *acyclic graphoidal decomposition* which was introduced by Arumugam and Suresh Suseela [4]. The minimum cardinality of an acyclic graphoidal decomposition of a graph G is called the *acyclic graphoidal decomposition number* and is denoted by $\eta_a(G)$.

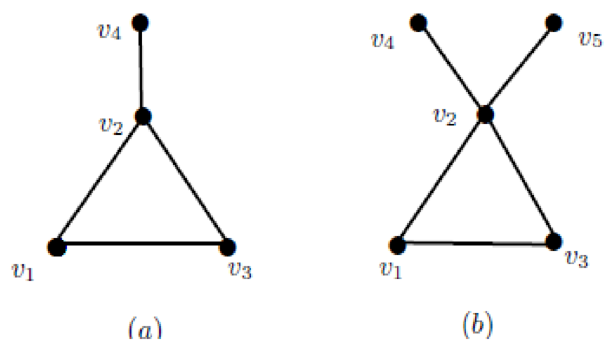

FIGURE 1. An example of difference labeling.

The concept of difference labeling was introduced by Bloom and Ruiz [6]. A difference labeling of a graph G is an injective function $f : V(G) \to N \cup \{0\}$ together with the weight function f^* on $E(G)$ given by $f^*(uv) = |f(u) - f(v)|$ for every edge uv in G. Let us denote the weight $f^*(uv)$ of the edge uv by $w_f(uv)$. Certainly, the collection of subgraphs induced by the edges with the same weight is a decomposition of G; this is called the *common weight decomposition* of G induced by f. For example, for the graph G given in Figure 1(a), consider the difference labeling $f : V(G) \to N \cup \{0\}$ defined by $f(v_i) = \deg(v_i)$, for all $i = 1, 2, 3, 4$. Then, the common weight decomposition ψ_f of G induced by f is given by $\psi_f = \{(v_1, v_2, v_3), (v_1, v_3), (v_2, v_4)\}$. Now, consider the graph

G given in Figure 1(b) together with a difference labeling f on $V(G)$ defined by $f(v_i) = \sum_{w \in V(G)} d(v_i, w)$, where $d(v_i, w)$ denotes the distance between vertices v_i and w. Then, the collection ψ_f associated with this labeling is given by $\psi_f = \{(v_1, v_2, v_3), (v_1, v_3), (v_4, v_2, v_5)\}$.

Several graph theoretic concepts have been emerged by interrelating different areas in graph theory. For example, the notion of graphoidal labeling is derived by combining the major areas decompositions and labelings of graphs(For details on graphoidal labeling one can refer to [1], [3], [11] and [12]). A similar study has been carried out in [10] where the notion of acyclic graphoidal decomposition is linked with difference labeling. In this direction of research, this paper introduces the concept of graphoidal difference labeling.

We need the following theorems which provide the η_a-value for trees and complete bipartite graphs.

Definition 1.1. Let ψ be a collection of internally disjoint paths in G. A vertex of G is said to ba an *interior vertex* of ψ if it is an internal vertex of a path in ψ. Any vertex which is not an interior vertex of ψ is said to be an *exterior vertex* of ψ.

Theorem 1.2. [4] *If there exists a acyclic graphoidal decomposition ψ of a graph G such that every vertex of G with degree at least two is interior to ψ, then ψ is a minimum acyclic graphoidal decomposition of G.*

Theorem 1.3. [4] *For a tree T, $\eta_a(T) = n - 1$, where n is the number of pendant vertices in T.*

Theorem 1.4. [4] *For a complete bipartite graph $K_{r,s}$,*

(i) $\eta_a(K_{1,1}) = 1$, $\eta_a(K_{1,s}) = s - 1$, *for all* $s \geq 2$.
(ii) $\eta_a(K_{2,2}) = 2$, $\eta_a(K_{2,s}) = s - 1$, *for all* $s \geq 3$.
(iii) $\eta_a(K_{r,s}) = rs - r - s$, *if* $r, s > 2$.

2. Difference graphoidal labeling

Bloom and Ruiz [6] proved that each member of a common weight decomposition of G induced by a difference labeling of G is a linear forest. That is, if ψ_f denotes the collection of all the paths taken from each member of the common weight decomposition induced by f, then ψ_f is an acyclic path decomposition of G. However this acyclic path decomposition ψ_f do not need to be an acyclic graphoidal decomposition of G.

For example, for the graph given in Figure 2, consider the difference labeling $f : V(G) \to N \cup \{0\}$ defined by $f(v_i) = 2i$, for all $i = 1, 2, ..., 6$, $f(v_7) = 1$, $f(v_8) = 7$, $f(v_9) = 15$ and $f(v_{10}) = 20$. Then, the collection ψ_f is given by $\psi_f = \{(v_1, v_2, v_3, v_4, v_5, v_6), (v_7, v_2, v_8, v_5), (v_5, v_9, v_{10})\}$. As, the vertex v_2 is an internal vertex of the paths $(v_1, v_2, v_3, v_4, v_5, v_6)$ and (v_7, v_2, v_8, v_5), ψ_f is

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not an acyclic graphoidal decomposition of G. Motivated by this observation we introduce the concept of graphoidal difference labeling of a graph which is defined as follows.

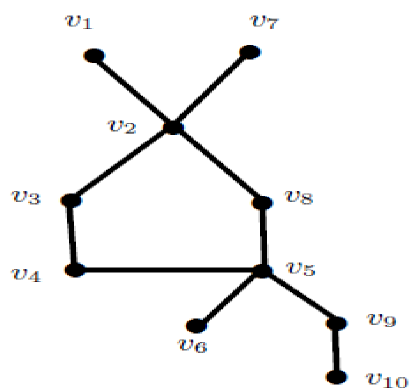

FIGURE 2. A graph G.

Definition 2.1. A difference labeling f of a graph G is said to be a *graphoidal difference labeling*($\mathcal{GDL}$) if ψ_f is an acyclic graphoidal decomposition of G and if G admits such a labeling f, then G is called a *difference label graphoidal graph.* When f is a $\mathcal{GDL}$, the collection ψ_f is called the *difference label graphoidal decomposition*($\mathcal{DLGD}$) induced by f.

Example 2.2. *A graph G together with a difference labeling f on $V(G)$ defined by $f(v_i) = i$, for all $i = 1, 2, ..., 6$ and $f(v_7) = 8$ is given in Figure 3(a). The collection ψ_f associated with this labeling is given by $\psi_f = \{(v_1, v_2, v_3, v_4), (v_5, v_2),$ $(v_2, v_6), (v_3, v_7)\}$. Certainly ψ_f is an acyclic graphoidal decomposition of G. So,*

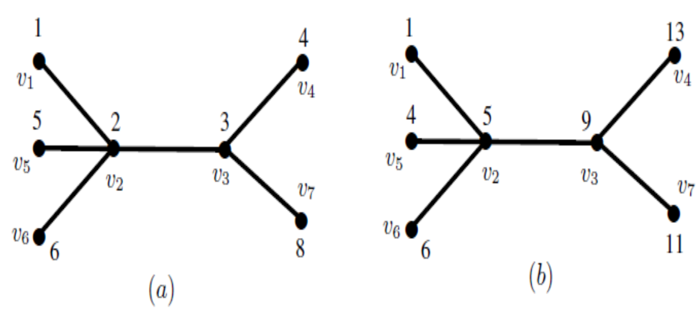

FIGURE 3. (a): An example of $\mathcal{GDL}$. (b): A difference labeling that is not a $\mathcal{GDL}$.

the difference labeling f of G is a $\mathcal{GDL}$ of G. Now, for the same graph G, consider the difference labeling f_1 defined by $f_1(v_i) = 4i - 3$, for all $i = 1, 2, 3, 4$,

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$f(v_5) = 4$, $f(v_6) = 6$ and $f(v_7) = 11$(see Figure 3(b)). It is certain that the collection $\psi_{f_1} = \{(v_1, v_2, v_3, v_4), (v_5, v_2, v_6), (v_3, v_7)\}$ is not an acyclic graphoidal decomposition of G as the vertex v_2 is an internal vertex in two paths in ψ_{f_1}. Hence the difference labeling f_1 of G is not a $\mathcal{GDL}$ of G.

Theorem 2.3. *Every graph admits a $\mathcal{GDL}$.*

Proof. Let G be a graph on n vertices with $V(G) = \{v_1, v_2, ..., v_n\}$. Define $f : V(G) \to N \cup \{0\}$ by $f(v_i) = 2^i$, for all $i = 1, 2, ..., n$. We prove that $\psi_f(G) = E(G)$. For this it is enough to prove that different edges receive different weights under the weight function f^* on $E(G)$. Suppose $e = v_i v_j$ and $e' = v_k v_l$ are edges of G such that $w_f(e) = w_f(e')$. Assume without loss of generality that $i > j$ and $k > l$. So, $w_f(e) = w_f(e')$ implies that

$$2^i - 2^j = 2^k - 2^l \qquad \cdots\cdots\cdots \quad (1)$$

Then $2^j(2^{i-j} - 1) = 2^l(2^{k-l} - 1)$. As 2^l divides $2^l(2^{k-l} - 1)$, it follows that 2^l divides $2^j(2^{i-j} - 1)$. This implies that 2^l divides 2^j as $(2^l, 2^{i-j} - 1) = 1$. In a similar way, we can prove that 2^j divides 2^l. Hence $j = l$. Therefore, the equation (1) now implies that $i = k$ and thus $e = e'$. So, different edges receive different weights which in turn implies that $\psi_f = E(G)$, which is obviously an acyclic graphoidal decomposition of G. Therefore f is a $\mathcal{GDL}$ of G. □

As proved in Theorem 2.3, every graph has at least one $\mathcal{GDL}$. Indeed, a graph can have infinitely many graphoidal difference labelings as shown below.

Theorem 2.4. *Every graph admits infinitely many graphoidal difference labelings.*

Proof. Suppose f is a $\mathcal{GDL}$ of a graph G of size m (Note that the existence of a $\mathcal{GDL}$ is guaranteed in view of Theorem 2.3). Consider the difference labelings f_1, f_2 and f_3 of G that are defined as follows.

(i) For a positive integer k, define $f_1(u) = f(u) + k$, for all $u \in V(G)$.
(ii) Define $f_2(u) = f(u) - m$, for all $u \in V(G)$, where $m = Min\{f(x) : x \in V(G)\}$.
(iii) Define $f_3(u) = M - f(u)$, for all $u \in V(G)$, where $M = Max\{f(w) : w \in V(G)\}$.

Certainly, for any edge $e = uv$, we have $|f_1(u) - f_1(v)| = |f(u) - f(v)|$ and so $\psi_f = \psi_{f_1}$. As f is a $\mathcal{GDL}$, ψ_f is an acyclic graphoidal decomposition of G and so is ψ_{f_1}. Therefore f_1 is a $\mathcal{GDL}$. Note that f_1 is different from f. In a similar way, one can prove that both f_2 and f_3 are graphoidal difference labelings of G distinct from f and f_1. Thus infinitely many $\mathcal{GDL}$ can be constructed from a $\mathcal{GDL}$ of G. □

3. The parameter $\eta_{\mathbf{d}}$

We have observed in Theorem 2.4 that there are infinitely many $\mathcal{GDL}s$ for a graph. But one can note that all the $\mathcal{GDL}s$ for a graph G provided in the proof of Theorem 2.4 give raise to the same $\mathcal{DLGD}$. This is not the case always. Indeed a graph may admit many graphoidal difference labelings such that the respective $\mathcal{DLGD}$s induced by them are of different cardinalities. For example,

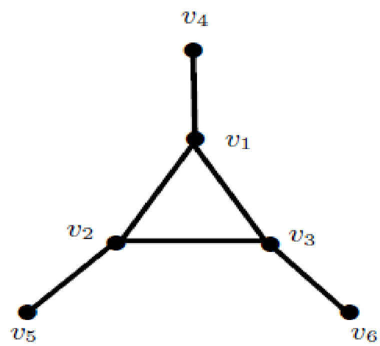

Figure 4. A graph with two $\mathcal{DLGD}s$ of different cardinalities.

for the graph G of Figure 4, consider the difference labelings f_1 and f_2 defined by $f_1(v_1) = 2,\ f_1(v_2) = 5,\ f_1(v_3) = 3,\ f_1(v_4) = 1,\ f_1(v_5) = 8,\ f_1(v_6) = 4$ and $f_2(v_1) = 4,\ f_2(v_2) = 7,\ f_2(v_3) = 6,\ f_2(v_4) = 2,\ f_2(v_5) = 15,\ f_2(v_6) = 5$. Then

$$\psi_{f_1} = \{(v_4, v_1, v_3, v_6), (v_1, v_2, v_5), (v_2, v_3)\} \text{ and}$$
$$\psi_{f_2} = \{(v_4, v_1, v_3), (v_6, v_3, v_2), (v_1, v_2), (v_2, v_5)\}$$

Clearly, both ψ_{f_1} and ψ_{f_2} are distinct acyclic graphoidal decompositions and so f_1 and f_2 are $\mathcal{GDL}$s. Note that $|\psi_{f_1}| \neq |\psi_{f_2}|$.

While it is possible for a graph to have more than one $\mathcal{DLGD}s$ of different cardinalities, it would be interesting to study the $\mathcal{DLGD}$ of minimum cardinality for a graph G. Motivated by this we define the notion of difference label graphoidal decomposition number of a graph.

Definition 3.1. The *difference label graphoidal decomposition number* $\eta_d(G)$ of a graph G is defined to be the minimum cardinality of ψ_f where the minimum is taken over all possible graphoidal difference labeling f of G. That is,

$$\eta_d(G) = Min\{|\psi_f| : f \text{ is a } \mathcal{GDL} \text{ of } G\}$$

where $|\psi_f|$ denotes the cardinality of ψ_f.

Example 3.2. (i). *Consider the graph G given in Figure 5. Define f by $f(v_1) = 1,\ f(v_2) = 3,\ f(v_3) = 5,\ f(v_4) = 7,\ f(v_5) = 9$ and $f(v_6) = 2$. Then $\psi_f = \{(v_1, v_2, v_3, v_4, v_5), (v_1, v_6, v_2), (v_3, v_5)\}$ is a $\mathcal{DLGD}$ of G.*

Hence $\eta_d(G) \leq 3$. Since, any $\mathcal{DLGD}$ is also an acyclic graphoidal decomposition and an acyclic graphoidal decomposition must have at least 3 paths it follows that $\eta_d(G) \geq 3$. Thus $\eta_d(G) = 3$.

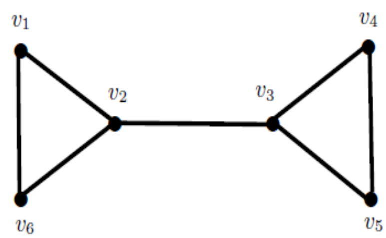

FIGURE 5. A graph G with $\eta_d(G) = 3$.

(ii). *For a path $P_n = (v_1, v_2, ..., v_n)$ on n vertices, the difference labeling f defined by $f(v_i) = i$, for each $i \in \{1, 2, ..., n\}$ is a $\mathcal{GDL}$ with $\psi_f = \{(v_1, v_2, ..., v_n)\}$ so that $\eta_d(P_n) = 1$.*

(iii). *For a cycle $C_n = (v_1, v_2, ..., v_n, v_1)$, if f is defined by $f(v_i) = i$, for each $i \in \{1, 2, ..., n\}$, then $\psi_f = \{(v_1, v_2, ..., v_n), (v_1, v_n)\}$ which is obviously an acyclic graphoidal decomposition of C_n so that $\eta_d(C_n) \leq 2$. Further at least two paths are required in order to cover the edges of C_n so that $\eta_d(C_n) = 2$.*

Let us now proceed to obtain a bound for η_d in terms of order and size which will be more helpful in dertermining η_d for several families of graphs. To start with, we prove the following lemmas.

Lemma 3.3. *For a $\mathcal{GDL}$ f of a graph G, let t_{ψ_f} denote the number of exterior vertices to ψ_f and let $t = \min_f t_{\psi_f}$. Then $\eta_d(G) = m - n + t$.*

Proof. Let f be any $\mathcal{GDL}$ of G and let ψ_f be the $\mathcal{DLGD}$ of G induced by f. Then

$$\begin{aligned} m &= \sum_{P \in \psi_f} |E(P)| \\ &= \sum_{P \in \psi_f} (i(P) + 1)\,, \textit{where } i(P) \textit{ is the number of internal} \\ &\qquad\qquad\qquad\qquad \textit{vertices of } P. \\ &= \sum_{P \in \psi_f} i(P) + |\psi_f| \end{aligned}$$

Therefore, $|\psi_f| = m - \sum_{P \in \psi_f} i(P)$. Now, since ψ_f is an acyclic graphoidal decomposition of G, every vertex of G is either an exterior vertex to ψ_f or an internal vertex of exactly one path in ψ_f and so $n = t_{\psi_f} + \sum_{P\in\psi_f} i(P)$. Hence $|\psi_f| = m - n + t_{\psi_f}$. Thus $\eta_d(G) = \min_f \{|\psi_f|\} = (m-n) + \min_f \{t_{\psi_f}\} = m - n + t$. □

Lemma 3.4. *Let G be a graph and let f be any $\mathcal{GDL}$ of G. Then the vertices with maximum and minimum labels with respect to f are exterior to ψ_f.*

Proof. Let u be the vertex of G with minimum label under f. Suppose u is an internal vertex of a path P in ψ_f. Let x and y be the vertices on P that are adjacent to the vertex u. Therefore $|f(u) - f(x)| = |f(u) - f(y)|$. As $f(u)$ is minimum, $f(x) > f(u)$ and $f(y) > f(u)$. So, the above equation implies that $f(x) - f(u) = f(y) - f(u)$ and so $f(x) = f(y)$, a contradiction. Hence the vertex u cannot be an internal vertex of any path in ψ_f. That is, u is exterior to ψ_f. In a similar way, we can prove that v is also exterior to ψ_f. □

As a consequence of the above two lemmas, we now obtain a bound for η_d in terms of order and size of G.

Theorem 3.5. *For any graph G, we have $\eta_d(G) \geq m - n + 2$. Further, the equality holds if and only if there exists a $\mathcal{DLGD}$ ψ_f induced by a difference labeling f such that all the vertices other than the vertices with maximum and minimum labels under f are interior to ψ_f.*

Proof. For any $\mathcal{GDL}$ f of G, by Lemma 3.4, at least two vertices of G would be exterior to ψ_f so that $t_{\psi_f} \geq 2$. Hence $t \geq 2$ and so Lemma 3.3 implies that $\eta_d(G) \geq m - n + 2$. Now, suppose $\eta_d(G) = m - n + 2$. By Lemma 3.3, $t = 2$. That is, there is a difference labeling f of G such that exactly two vertices of G are exterior to ψ_f. Now, in view of Lemma 3.4, those two exterior vertices are the vertices with minimum and maximum labels under f. Conversely, if there is a difference labeling f with the given property, then $t \leq 2$ so that $\eta_d(G) = m - n + t \leq m - n + 2$. The other inequality is always true and thus $\eta_d(G) = m - n + 2$. □

Corollary 3.6. *A $\mathcal{DLGD}$ ψ_f of a graph G with the property that all the vertices other than the vertices with maximum and minimum labels under f are interior to ψ_f, is a minimum $\mathcal{DLGD}$.*

Proof. Follows from Theorem 3.5. □

The Corollary 3.6 will be very useful to determine the value of η_d for several graphs. For example, η_d for the Petersen graph is determined with the aid of the corollary.

Example 3.7. *For the Petersen graph G, $\eta_d(G) = 7$.*

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Proof. Let the vertices of the Petersen graph G be labeled as in Figure 6. Then the labeling f on G defined by $f(v_i) = i$, for all i, is a $\mathcal{GDL}$ as $\psi_f = \{(v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}), (v_5, v_1), (v_1, v_9), (v_2, v_7), (v_6, v_{10}), (v_3, v_{10}), (v_4, v_8)\}$ is an acyclic graphoidal decomposition of G. Certainly, v_1 and v_{10} are the vertices with minimum and maximum labels and they are the only vertices exterior to ψ_f. Therefore, by Corollary 3.6, ψ_f is a minimum $\mathcal{DLGD}$ of G so that $\eta_d(G) = |\psi_f| = 7$. □

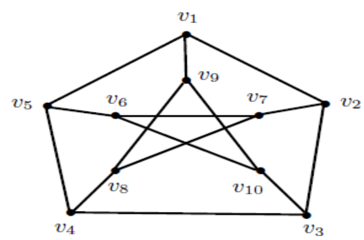

FIGURE 6. The Petersen graph.

Let us now proceed to determine the value of η_d for some common classes of graphs. For this purpose we prove the following lemma.

Lemma 3.8. *For any graph G, we have $\eta_d(G) \leq m - l + 1$, where l is the length of a longest path. Further, the bound is sharp.*

Proof. Let $V(G) = \{v_1, v_2, ..., v_n\}$. Let P be a longest path in G, say $P = (v_1, v_2, ..., v_{l+1})$. Now, define a difference labeling f on G by

$$f(v_i) = \begin{cases} i & if\ 1 \leq i \leq l+1 \\ 2^i & elsewhere \end{cases}$$

As discussed in the proof of Theorem 2.3, $\left|2^i - 2^j\right| \neq \left|2^k - 2^l\right|$ when i, j, k and l are distinct integers greater than $l + 1$. That is, no two edges of G lying outside P have the same weight. Further, if v_i is a vertex lying outside P

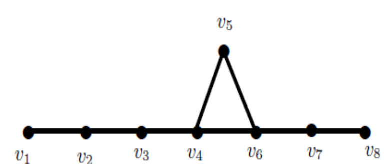

FIGURE 7. A graph G with $\eta_d(G) = m - l + 1$.

adjacent to a vertex v_j that lies on P, then $i > l + 1$ and $1 < j < l + 1$. So, $w_f(v_iv_j) = |f(v_i) - f(v_j)| = \left|2^i - j\right| = 2^i - j$. Certainly, $2^i - j$ is neither 1 nor

$|2^r - 2^s|$, for any $r, s > l+1$. Hence $\psi_f = \{P\} \cup (E(G) - E(P))$, which is an acyclic graphoidal decomposition of G and hence ψ_f is a $\mathcal{DLGD}$ of G. Thus $\eta_d(G) \leq |\psi_f| = m - l + 1$. For the sharpness of the bound, consider the graph G given in Figure 7. Here, $m = 8$ and $l = 7$. Define f by $f(v_i) = i$ for all i. Then $\psi_f = \{(v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8), (v_4, v_6)\}$ is a minimum $\mathcal{DLGD}$ of G. Hence $\eta_d(G) = m - l + 1 = 2$. □

With the aid of Lemma 3.8 we determine η_d for Hamiltonian graphs and in particular for complete graphs and wheels.

Theorem 3.9. *For a Hamiltonian graph G, $\eta_d(G) = m - n + 2$.*

Proof. As G is Hamiltonian, the length l of a detour path is $n-1$ and so by Lemma 3.8 we have $\eta_d(G) \leq m - n + 2$. The other inequality is always true as seen in Theorem 3.5. Thus $\eta_d(G) = m - n + 2$. □

Corollary 3.10. (i) *For a complete graph K_n, where $n \geq 2$, $\eta_d(K_n) = \frac{n^2-3n+4}{2}$.*

(ii) *For a wheel W_n, where $n \geq 4$, $\eta_d(W_n) = n$.*

Proof. As K_n and W_n are Hamiltonian, the result follows from Theorem 3.9. □

In the following theorems, we determine the value of η_d for trees and complete bipartite graphs.

Theorem 3.11. *For a tree T, $\eta_d(T) = n-1$, where n is the number of pendant vertices in T.*

Proof. It has been proved in [10] that every minimum acyclic graphoidal decomposition of a tree T can be realized as a $\mathcal{DLGD}$ ψ_f by a suitable difference labeling f of T so that $\eta_d(T) = \eta_a(T)$. Now Theorem 1.3 completes the proof. □

Theorem 3.12. *Let r and s be positive integers with $r \leq s$. Then*

$$\eta_d(K_{r,s}) = \begin{cases} s-1 & if\ r = 1 \\ r(s-1) - r + 2 & if\ r \geq 2 \text{ and } s \leq rC_2 + 2 \\ r(s-1) - rC_2 & if\ r \geq 2 \text{ and } s > rC_2 + 2 \end{cases}$$

where $rC_2 = \frac{r(r-1)}{2}$.

Proof. Let $X = \{x_1, x_2, ..., x_r\}$ and $Y = \{y_1, y_2, ..., y_s\}$ be the bipartition of $K_{r,s}$. If $r = 1$, then $K_{1,s}$ is the star with s pendant vertices and so it follows from Theorem 3.11 that $\eta_d(K_{1,s}) = s - 1$. Assume $r \geq 2$.

Case 1. $s \leq rC_2 + 2$.

Define f on $V(K_{r,s})$ by $f(x_1) = 4$, $f(x_2) = 10$, $f(y_1) = 1$, $f(y_2) = 7$, $f(y_3) = 13$,

$$f(y_{i+3}) = 3f(y_{i+2}) - 8 \ : \ i = 1, 2, 3, ..., r-2$$
$$f(x_{i+2}) = 2f(y_{i+2}) - 4 \ : \ i = 1, 2, 3, ..., r-2 \ and$$
$$f(y_t) = \frac{f(x_{j+1}) + f(x_{i+j+1})}{2} \ : \ 1 \le j \le r-2, 1 \le i \le r-j-1,$$

where $t = jr - \frac{(j-1)(j+2)}{2} + 1 + i$. Now, it is not difficult to verify that the collection ψ_f induced by f is given by $\psi_f = \{P_{(1)}, Q_{(1)}, Q_{(2)}, ..., Q_{(r-2)}\} \cup \{R^j_{(i)} : 1 \le j \le r-2 \text{ and } 1 \le i \le r-j-1\} \cup X$, where

$$P_{(1)} = (y_1, x_1, y_2, x_2, y_3)$$
$$Q_{(i)} = (x_1, y_{i+2}, x_{i+2}, y_{i+3})$$
$$R^j_{(i)} = (x_{j+1}, y_t, x_{(i+j+1)}) \ and$$

X is the set of all edges of $K_{r,s}$ not covered by the above paths in ψ_f, is $\mathcal{DLGD}$ of $K_{r,s}$. Clearly, all the vertices of $K_{r,s}$ other than the vertices y_1 and y_{r+1} with minimum and maximum labels respectively, are interior to ψ_f so that by Corollary 3.6, ψ_f is a minimum $\mathcal{DLGD}$ of $K_{r,s}$. Hence, by Theorem 3.5, $\eta_d(K_{r,s}) = rs - (r+s) + 2$.

Case 2. $s > rC_2 + 2$.

In this case, we define a difference labeling f_1 on $V(K_{r,s})$ using the labeling f defined in the Case 1 as follows. Let

$$f_1(x_i) = f(x_i) \ : \ i = 1, 2, 3, ..., r$$
$$f_1(y_i) = f(y_i) \ : \ i = 1, 2, 3, ..., rC_2 + 2 \ \ and$$
$$f_1(y_i) = f(y_{r+1}) + i \ \ : \ i = rC_2 + 3, rC_2 + 4, ..., s$$

Then $\psi_{f_1} = \psi_f \cup X_1$, where X_1 is the set of all edges not covered by the paths in ψ_f, is a $\mathcal{DLGD}$ of $K_{r,s}$ with $|\psi_{f_1}| = |\psi_f| + |X_1| = (1+(r-2)+\frac{(r-2)(r-1)}{2})+(rs-(4+3(r-2)+2\frac{(r-2)(r-1)}{2})) = (r-1+\frac{(r-2)(r-1)}{2})+(rs-(4+3r-6+r^2-3r+2)) = \frac{r(r-1)}{2}+(rs-r^2) = \frac{2rs-r^2-r}{2} = r(s-1)-rC_2$. Hence $\eta_d(K_{r,s}) \le r(s-1)-rC_2$. Further, let g be any $\mathcal{GDL}$ of $K_{r,s}$. Then, for a vertex $y_i \in Y$ that is interior to ψ_g, there corresponds a pair of vertices in X and hence at most rC_2 vertices of $K_{r,s}$ belonging to Y can be interior to ψ_g which implies that at least $s - rC_2$ vertices of $K_{r,s}$ are exterior to ψ_g so that $t \ge s - rC_2$. Hence, by Lemma 3.3, we have $\eta_d(K_{r,s}) \ge rs - (r+s) + s - rC_2 = r(s-1) - rC_2$. □

The $\mathcal{GDL}$ f for $K_{r,s}$ defined in the above theorem and the respective $\mathcal{DLGD}$ ψ_f are illustrated in the following example.

Example 3.13. *For the complete bipartite graph $K_{4,8}$ with bipartition $X = \{x_1, x_2, x_3, x_4\}$ and $Y = \{y_i : 1 \le i \le 8\}$, the $\mathcal{GDL}$ f defined in the proof of Theorem 3.12 is given by $f(x_1) = 4$, $f(x_2) = 10$, $f(x_3) = 22$, $f(x_4) = 58$, $f(y_1) = 1$, $f(y_2) = 7$, $f(y_3) = 13$, $f(y_4) = 31$, $f(y_5) = 85$, $f(y_6) = 16$,*

$f(y_7) = 34$ and $f(y_8) = 40$. Further, the $\mathcal{DLGD}$ ψ_f of $K_{4,6}$ is given by $\psi_f = \{P_{(1)}, Q_{(1)}, Q_{(2)}, R^1_{(1)}, R^1_{(2)}, R^2_{(1)}\} \cup X$, where

$$P_{(1)} = (y_1, x_1, y_2, x_2, y_3)$$
$$Q_{(1)} = (x_1, y_3, x_3, y_4)$$
$$Q_{(2)} = (x_1, y_4, x_4, y_5)$$
$$R^1_{(1)} = (x_2, y_6, x_3)$$
$$R^1_{(2)} = (x_2, y_7, x_4)$$
$$R^2_{(1)} = (x_3, y_8, x_4)$$

and X is the set of all edges of $K_{4,6}$ not covered by the above paths. Further, for the complete bipartite graph $K_{4,12}$ with bipartition $X = \{x_1, x_2, x_3, x_4\}$ and $Y = \{y_i : 1 \leq i \leq 12\}$, the $\mathcal{GDL}$ f_1 is given by $f_1(x_i) = f(x_i)$, for all $i \in \{1, 2, 3, 4\}$ and for each $i \in \{1, 2, ..., 8\}$, $f_1(y_i) = f(y_i)$, $f_1(y_9) = 94$, $f_1(y_{10}) = 95$, $f_1(y_{11}) = 96$ and $f_1(y_{12}) = 97$. The $\mathcal{DLGD}$ ψ_{f_1} of $K_{4,12}$ is given by $\psi_{f_1} = \psi_f \cup X_1$, where X_1 is the set of all edges of $K_{4,12}$ not covered by the paths in ψ_f.

4. Relation between $\eta_{\mathbf{d}}$ and $\eta_{\mathbf{a}}$

In this section, we discuss the relationship of η_d with the acyclic graphoidal decomposition number η_a. Certainly, $\eta_a(G) \leq \eta_d(G)$ as every difference label graphoidal decomposition of G is an acyclic path decomposition of G. The following theorem shows that the absolute difference between the parameters η_d and η_a can be made arbitrarily large.

Theorem 4.1. *For a given positive integer k, there exists a graph G such that $\eta_d(G) - \eta_a(G) = k$.*

Proof. Consider a path P on $5k$ vertices, say $P = (v_1, v_2, ..., v_{5k})$. Introduce k vertices namely w_1, w_2,..., w_k. Join the vertex w_i to the vertices v_{5i-3} and v_{5i-1} for each $i \in \{1, 2, ..., k\}$. Let G be the resultant graph(see Figure 8). We prove that $\eta_d(G) = 2k + 1$ and $\eta_a(G) = k + 1$. It is clear that $\psi_a = \{P\} \cup \{(v_{5i-3}, w_i, v_{5i-1}) : 1 \leq i \leq k\}$ is an acyclic graphoidal decomposition of G such that every vertex of G with degree at least two is interior to ψ_a so that by Theorem 1.2, ψ_a is a minimum acyclic graphoidal decomposition of G. Hence $\eta_a(G) = |\psi_a| = k + 1$.

We next prove that $\eta_d(G) = 2k + 1$. For this consider the difference labeling f defined by $f(v_i) = i$, for all $i = 1, 2, 3, ..., 5k$ and $f(w_j) = 5k + j$, for each $j = 1, 2, 3, ..., k$. Now, for each $i \in \{1, 2, ..., 5k - 1\}$, the weight of the edge $v_i v_{i+1}$ is given by $w_f(v_i v_{i+1}) = |f(v_i) - f(v_{i+1})| = |i - (i + 1)| = 1$. Further, for each $i \in \{1, 2, ..., k\}$, the weight of the edge $v_{5i-3} w_i$ is given by $w_f(v_{5i-3} w_i) = |f(v_{5i-3}) - f(w_i)| = |(5i - 3) - (5k + i)| = |4i - 3 - 5k|$ and the weight of the edge $w_i v_{5i-1}$ is $w_f(w_i v_{5i-1}) = |f(w_i) - f(v_{5i-1})| = |(5k +$

$i) - (5i-1)| = |5k-4i+1|$. We need to verify that $w_f(v_{5i-3}w_i) \neq w_f(w_i v_{5i-1})$, for all $i = 1, 2, ..., k$. Suppose $w_f(v_{5i-3}w_i) = w_f(w_i v_{5i-1})$, for at least one i. Then $|4i-3-5k| = |5k-4i+1|$. If both $4i-3-5k$ and $5k-4i+1$ are positive, then $4i-3-5k = 5k-4i+1$ so that $5i-2 = 5k+i$ which implies that $f(v_{5i-2}) = f(w_i)$, a contradiction. Similarly, one can prove the other cases. Therefore, $w_f(v_{5i-3}w_i) \neq w_f(w_i v_{5i-1})$, for all $i = 1, 2, ..., k$. Also, $w_f(v_{5i-3}w_i) \neq 1$ and $w_f(w_i v_{5i-1}) \neq 1$. Hence $\psi_f = \{P\} \cup \{(v_{5i-3}, w_i) : 1 \leq i \leq k\} \cup \{(v_{5i-1}, w_i) : 1 \leq i \leq k\}$, which is certainly an acyclic graphoidal decomposition of G so that f is $\mathcal{GDL}$ of G and hence $\eta_d(G) \leq |\psi_f| = 2k+1$.

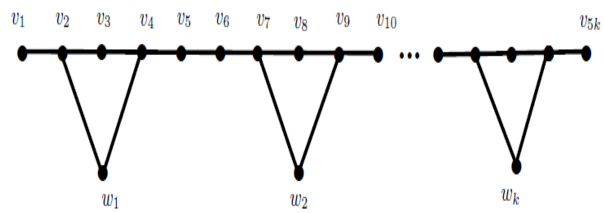

Figure 8. A graph G with $\eta_d(G) - \eta_a(G) = k$.

Further, for any $\mathcal{DLGD}$ ψ_g of G, at least one vertex lying on the cycle $C_{(i)} = (v_{5i-3}, v_{5i-2}, v_{5i-1}, w_i, v_{5i-3})$, for all $i = 1, 2, ..., k$, is exterior to ψ_g. On the contrary, suppose all the vertices of the cycle $C_{(j)} = (v_{5j-3}, v_{5j-2}, v_{5j-1}, w_j, v_{5j-3})$, for some $j \in \{1, 2, ..., k\}$, are interior to ψ_g. Then the path $(v_{5j-3}, w_j, v_{5j-1})$ would be a section of the path in ψ_g having w_j as an internal vertex. So $|g(v_{5j-3}) - g(w_j)| = |g(v_{5j-1}) - g(w_j)|$. Now, if for instance $g(v_{5j-3}) < g(v_{5j-2}) < g(v_{5j-1})$ and $g(v_{5j-3}) < g(w_j) < g(v_{5j-1})$, then $g(w_j) = \frac{g(v_{5j-1})+g(v_{5j-3})}{2} = g(v_{5j-2})$, a contradiction. In a similar way, the remaining cases can be discussed. Hence, for any $\mathcal{DLGD}$ ψ_g of G, the two pendant vertices of G and at least k vertices of degree greater than one of G are exterior to ψ_g and so $t \geq k+2$. By Lemma 3.3, $\eta_d(G) \geq (7k-1) - 6k + (k+2) = 2k+1$. Therefore $\eta_d(G) = 2k+1$. Hence $\eta_d(G) - \eta_a(G) = k$. □

Even if the above theorem asserts that the absolute difference between the parameters η_d and η_a can be made arbitrarily large, they do not assume arbitrary values. That is, given positive integers a and b with $a \leq b$, it is not always possible to find a graph G for which $\eta_a(G) = a$ and $\eta_d(G) = b$. For example, when $\eta_a(G) = 1$, G is a path to which the value of $\eta_d(G)$ is also 1. In this connection we pose the following conjecture.

Conjecture 4.2. *For any graph G, $\eta_d(G) \leq 2\eta_a(G) - 1$.*

We find some classes of graphs that support the Conjecture 4.2. For example, the common classes of graphs such as complete graphs, complete bipartite graphs, wheels and trees support the Conjecture 4.2 as shown below. For the value of η_a to these common class graphs, one may refer to [4].

Theorem 4.3. *Complete bipartite graphs, complete graphs, wheels and trees support the Conjecture 4.2.*

Proof. For a tree T, by Theorem 3.11 that $\eta_d(T) = n-1$, where n is the number of pendant vertices of T. By Theorem 1.3, $\eta_a(T) = n-1$ and so $\eta_d(T) = \eta_a(T)$. Now, for a wheel W_n on n vertices, $\eta_a(W_n) = n-2$ and by Corollary 3.10, $\eta_d(W_n) = n$. Hence $\eta_d(W_n) \leq 2\eta_a(W_n) - 1$. For a complete graph K_n, $\eta_a(K_n) = \frac{n^2-3n}{2}$ and by Corollary 3.10, $\eta_d(K_n) = \frac{n^2-3n+4}{2}$. Hence $\eta_d(K_n) \leq 2\eta_a(K_n) - 1$. Again one can verify from Theorem 3.12 and Theorem 1.4 that complete bipartite graphs support the Conjecture 4.2. $\square$

We conclude this section with the following realization theorem in connection with the Conjecture 4.2

Theorem 4.4. *For any two positive integers a and b with $1 < a \leq b \leq 2a-1$, there exists a graph G for which $\eta_a(G) = a$ and $\eta_d(G) = b$.*

Proof. Suppose a and b are two positive integers with $1 < a \leq b \leq 2a-1$. We construct a graph G with $\eta_a(G) = a$ and $\eta_d(G) = b$ as follows. Let $b = a + r$, where $0 \leq r \leq a-1$. Consider a path $P = (v_1, v_2, ..., v_{2a+r})$ on $2a+r$ vertices. Introduce $a-1$ vertices namely $w_1, w_2, \ldots, w_{a-1}$. Now, join w_i to the vertices v_{3i-1} and v_{3i+1}, for all $i = 1, 2, ..., r$ and for each $i \in \{r+1, r+2, ..., a-1\}$, join w_i to the vertices v_{2i+r} and v_{2i+r+1}. Let G be the resultant graph. For $a = 6$ and $b = 8$, the graph G is given in the Figure 9. We prove that $\eta_a(G) = a$ and $\eta_d(G) = b$. It is clear that $\psi_1 = \{P\} \cup \{(v_{3i-1}, w_i, v_{3i+1}) : 1 \leq i \leq r\} \cup \{(v_{2i+r}, w_i, v_{2i+r+1}) : r+1 \leq i \leq a-1\}$ is an acyclic graphoidal decomposition of G such that every vertex of G with degree at least two is interior to ψ_a so that by Theorem 1.2, ψ_1 is a minimum acyclic graphoidal decomposition of G and hence $\eta_a(G) = |\psi_1| = 1 + r + ((a-1) - r) = a$. We now prove that $\eta_d(G) = b$. For this consider the difference labeling f defined by

$$\begin{aligned} f(v_i) &= 2i &&: i = 1, 2, ..., 2a+r \\ f(w_i) &= 2(2a+r)+i &&: i = 1, 2, ..., r \textit{ and} \\ f(w_i) &= 2r+4i+1 &&: i = r+1, r+2, ..., a-1. \end{aligned}$$

Now, for each $i \in \{1, 2, ..., 2a+r-1\}$, the weight of the edge v_iv_{i+1} is given by $w_f(v_iv_{i+1}) = |f(v_i) - f(v_{i+1})| = |2i - 2(i+1)| = 2$. Further, for each $i \in \{r+1, r+2, ..., a-1\}$, the weight of the edge $v_{2i+r}w_i$ is given by $w_f(v_{2i+r}w_i) = |f(v_{2i+r}) - f(w_i)| = |2(2i+r) - (2r+4i+1)| = |-1| = 1$ and the weight of the edge w_iv_{2i+r+1} is given by $w_f(w_iv_{2i+r+1}) = |f(w_i) - f(v_{2i+r+1})| = |2r+4i+1-2(2i+r+1)| = |-1| = 1$. Hence $w_f(v_{2i+r}w_i) =$

$w_f(w_i v_{2i+r+1})$, for all $i \in \{r+1, r+2, ..., a-1\}$. Now, the weight of the edge $v_{3i-1}w_i$, where $1 \leq i \leq r$, is given by $w_f(v_{3i-1}w_i) = |f(v_{3i-1}) - f(w_i)| = |2(3i-1) - (2(2a+r)+i)| = |5i - 4a - 2r - 2|$ and the weight of the edge $w_i v_{3i+1}$, where $1 \leq i \leq r$, is $w_f(w_i, v_{3i+1}) = |f(w_i) - f(v_{3i+1})| = |(2(2a+r)+i) - 2(3i+1)| = |4a+2r-5i-2|$. Also, it can be proved that $w_f(v_{3i-1}w_i) \neq w_f(w_i v_{3i+1})$, for all $i = 1, 2, ..., r$. Hence, $\psi_f = \{P\} \cup \{(v_{3i-1}, w_i) : 1 \leq i \leq r\} \cup \{(w_i, v_{3i+1}) : 1 \leq i \leq r\} \cup \{(v_{2i+r}, w_i, v_{2i+r+1}) : r+1 \leq i \leq a-1\}$, which is certainly an acyclic graphoidal decomposition of G so that f is $\mathcal{GDL}$ of G and hence $\eta_d(G) \leq |\psi_f| = 1 + r + r + ((a-1) - r) = a + r = b$.

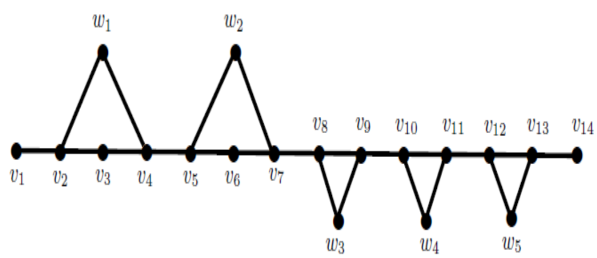

FIGURE 9. A graph G with $\eta_a(G) = 6$ and $\eta_d(G) = 8$.

Further, let ψ_g be an arbitrary $\mathcal{DLGD}$ of G. Suppose all the vertices of the cycle $C_{(j)} = (v_{3j-1}, v_{3j}, v_{3j+1}, w_j, v_{3j-1})$, for some $j \in \{1, 2, ..., r\}$, are interior to ψ_g. Then the path $(v_{3j-1}, w_j, v_{3j+1})$ would be a section of the path in ψ_g with w_j as an internal vertex. So $|g(w_j) - g(v_{3j-1})| = |g(v_{3j+1}) - g(w_j)|$. Now, if $g(v_{3j-1}) < g(v_{3j}) < g(v_{3i+1})$ and $g(v_{3j-1}) < g(w_j) < g(v_{3i+1})$ then $g(w_j) = \frac{g(v_{3j-1})+g(v_{3i+1})}{2} = g(v_{3j})$, a contradiction. In a similar way, the remaining cases can be discussed. So, what we have proved is that at least one vertex lying on the cycle $C_{(i)} = (v_{3i-1}, v_{3i}, v_{3i+1}, w_j, v_{3i-1})$, for all $i = 1, 2, ..., r$, is exterior to ψ_g. Hence, for any $\mathcal{DLGD}$ ψ_g, the two pendant vertices of G and at least r vertices of degree greater than one of G are exterior to ψ_g and so $t \geq r+2$. By Lemma 3.3 that $\eta_d(G) \geq (4a+r-3) - (3a+r-1) + (r+2) = a + r = b$. Hence $\eta_d(G) = b$. □

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