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Probability and Measurable Spaces on Modules Category

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Abstract. In this paper we show that the category of measurable spaces is closed under coproducts in the category of sets. For an arbitrary ring R, we define measurable and probability right R-modules and we prove that the categories of these new objects are closed under kernels, cokernels and pushouts in the category of right R-modules. We also show that the category of measurable right R-modules is closed under coproducts and products in the category of right R-modules. We end this paper by giving some results about stochastically independence in the category of probability right R-modules.

Keywords: Measurable space, Probability space, Ring, Module.

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1. Introduction

Independence is a fundamental notion in the probability theory. Two events are independent, or stochastically independent if the occurrence of one does not affect to the probability of occurrence of the other. When we are dealing with a collection of more than two events, or we are facing to a collection of the random variables, the stochastic independence of random variables is defined by the stochastic independence of σ-algebras generated by them. Hence we are

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dealing with a (finite or infinite) collection of measurable spaces. Combining these measurable spaces, a big measurable space is obtained. This is done by forming the notion product in the category of probability spaces and this is our motivation for studying the probability spaces from the categorical view.

The study of the probability spaces with an algebraic and categorical approach, has been done in the recent years by many authors [1, 2] and also the notion of stochastic independence by a categorical approach has been investigated by the authors in [7, 4, 5]. The main aim of this paper is to study the category of measurable (probability) spaces.

In Section 2, we recall some basic notions and we show that the category of measurable spaces is closed under coproducts in the category of sets (cf. Theorem 2.5). We notice that the existence of coproducts in the category of probability spaces is more complicated. We show that a coproduct in the category of measurable spaces is not necessarily a coproduct in category of probability spaces.

Let R be an arbitrary ring. In Section 3, we define measurable and probability right R-modules and we study the categories of these new objects. It is clear by our definition that these new categories are the subcategories of the category of right R-modules. As the category of right R-modules is abelian, it is natural to ask whether the category of measurable (probability) right R-modules is abelian as well. However, it is not clear whether these subcategories are preadditive.

We show that MeasMod-R, the category of measurable right R-modules, is closed under kernels, coproducts and products in the category of right R-modules.

As a conclusion, MeasMod-R is closed under equalizers. We also show that the categories MeasMod-R and Prob-R, the category of probability right R-modules are closed under cokernels and pushouts in the category of right R-modules. It is also proved that if $f_1 : (M, \sigma_M, P) \longrightarrow (M_1, \sigma_{M_1}, P_1)$ and $f_2 : (M, \sigma_M, P) \longrightarrow (M_2, \sigma_{M_2}, P_2)$ are measurable R-homomorphisms and if $f = (f_1, f_2) : (M, \sigma_M, P) \longrightarrow (M_1 \oplus M_2, \sigma(M_1 \oplus M_2), P_{M_1 \oplus M_2})$ is a probability R-homomorphism, then so are f_1 and f_2. Furthermore, f_1 and f_2 are stochastically independent. In view of [7], the category of probability spaces does not have universal products. However, given a family of probability right R-modules $\{(M_i, \sigma_{M_i}, P_i)| \ 1 \leq i \leq n\}$, it is shown that the probability right R-module $(\bigoplus_{i=1}^n M_i, \sigma(\bigoplus_{i=1}^n M_i), P_{\bigoplus_{i=1}^n M_i})$ is the product of $\{(M_i, \sigma_{M_i}, P_i)| \ 1 \leq i \leq n\}$ in Prob-R if and only if any family of probability R-homomorphisms $\{f_i : (M, \sigma_M, P) \longrightarrow (M_i, \sigma_{M_i})| \ 1 \leq i \leq n\}$ is stochastically independent.

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2. Preliminaries

In this section we recall some basic definitions and notions which are needed in this paper.

Definition 2.1. Let Ω be a nonempty set and $\mathcal{P}(\Omega)$ be the power set of Ω. A class of sets $\mathcal{F} \subseteq \mathcal{P}(\Omega)$ is called an *algebra* if (i) $\Omega \in \mathcal{F}$, (ii) $A \in \mathcal{F}$ implies that $A^c \in \mathcal{F}$ where $A^c = \Omega - A$ is the complement of A and (iii) $A, B \in \mathcal{F}$ implies that $A \cup B \in \mathcal{F}$. Furthermore, a class $\mathcal{F} \subseteq \mathcal{P}(\Omega)$ is called a *σ-algebra* if it is an algebra and if for each subclass $\{A_n | \ n \geq 1\} \subseteq \mathcal{F}$ we have $\bigcup_{n\geq 1} A_n \in \mathcal{F}$.

Definition 2.2. A *measurable space* (or Borel sapce) is an ordered pair $(\Omega, \mathcal{F})$ in which Ω is a set and $\mathcal{F}$ is a σ-algebra on Ω. Let $(\Omega_1, \sigma_{\Omega_1})$ and $(\Omega_2, \sigma_{\Omega_2})$ be measurable spaces. A *Morphism* $f : (\Omega_1, \sigma_{\Omega_1}) \longrightarrow (\Omega_2, \sigma_{\Omega_2})$ of measurable spaces is a function $f : \Omega_1 \longrightarrow \Omega_2$ such that $f^{-1}(X) \in \sigma_{\Omega_1}$ for every $X \in \sigma_{\Omega_2}$. For any morphisms $f : (\Omega_1, \sigma_{\Omega_1}) \longrightarrow (\Omega_2, \sigma_{\Omega_2})$ and $g : (\Omega_2, \sigma_{\Omega_2}) \longrightarrow (\Omega_3, \sigma_{\Omega_3})$ of measurable spaces, we define the composition of two morphisms, the usual composition of the functions $f : \Omega_1 \longrightarrow \Omega_2$ and $g : \Omega_2 \longrightarrow \Omega_3$. Moreover, for each measurable space $(\Omega, \mathcal{F})$, the identity morphism $1_{(\Omega,\mathcal{F})}$ is induced by the identity function 1_Ω. With these notations the class of all measurable spaces is a category.

Example 2.3. Let $\Omega = \{a, b, c, d\}$, $\mathcal{F}_1 = \{\Omega, \emptyset, \{a\}, \{b, c, d\}\}$ and let $\mathcal{F}_2$ be the set of all subsets of Ω. Then $f_1 : (\Omega, \mathcal{F}_1) \longrightarrow (\Omega, \mathcal{F}_2)$ defined by

$$f_1(\omega) = a \text{ for } \omega \in \Omega$$

is a morphism of measurable spaces. Also $f_2 : (\Omega, \mathcal{F}_2) \longrightarrow (\Omega, \mathcal{F}_1)$ defined by

$$f_2(\omega) = a \text{ if } \omega = a, b \text{ and } f_2(\omega) = c \text{ if } \omega = c, d$$

is a morphism of measurable spaces. However, the function $f_2 : (\Omega, \mathcal{F}_1) \longrightarrow (\Omega, \mathcal{F}_2)$ is not a morphism of measurable spaces from $(\Omega, \mathcal{F}_1)$ to $(\Omega, \mathcal{F}_2)$ since $f_2^{-1}(\{a\}) = \{a, b\} \notin \mathcal{F}_1$.

Definition 2.4. A *probability space* is a triple $(\Omega, \mathcal{F}, P)$ in which $(\Omega, \mathcal{F})$ is a measurable space equipped with a function $P : \mathcal{F} \longrightarrow [0, 1]$, called the *probability measure*, that satisfies the following conditions:

(i) P is *σ-additive*; meaning that if $\{A_n | \ n \geq 1\} \subseteq \mathcal{F}$ is a countable family of pairwise disjoint set, then $P(\bigcup_{n\geq 1} A_n) = \Sigma_{n\geq 1} P(A_n)$.

(ii) $P(\Omega) = 1$.

A *morphism* $\Phi : (\Omega_1, \mathcal{F}_1, P_1) \longrightarrow (\Omega_2, \mathcal{F}_2, P_2)$ between probability spaces $(\Omega_1, \mathcal{F}_1, P_1)$ and $(\Omega_2, \mathcal{F}_2, P_2)$ is a morphism $\Phi : (\Omega_1, \mathcal{F}_1) \longrightarrow (\Omega_2, \mathcal{F}_2)$ between the measurable spaces $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$ such that $P_2(X) = P_1(\Phi^{-1}(X))$ for every $X \in \mathcal{F}_2$. We consider the composition between the morphisms of probability spaces the same as in category of measurable space. Moreover, the identity morphism of each probability space $(\Omega, \mathcal{F}, P)$ is induced by $1_{(\Omega,\mathcal{F})}$. With these notations, the class of all probability spaces is a category.

We now give the main result of this section.

Theorem 2.5. *The category of measurable spaces is closed under coproducts in the category of sets.*

Proof. Given an arbitrary family $\{(\Omega_i, \sigma_i)\}_{i\in\Lambda}$ of measurable spaces, it follows from [3] that $\coprod_{i\in\Lambda} \Omega_i = \dot{\bigcup}_{i\in\Lambda}\Omega_i$ with the canonical injection $\iota_i : \Omega_i \longrightarrow \dot{\bigcup}_{i\in\Lambda}\Omega_i$ for each $i \in \Lambda$, where $\dot{\bigcup}_{i\in\Lambda}\Omega_i$ is disjoint union of $\{(\Omega_i, \sigma_i)\}_{i\in\Lambda}$. If we define

$$\sigma_{\dot{\bigcup}_{i\in\Lambda}\Omega_i} = \{X \subseteq \dot{\bigcup}_{i\in\Lambda}\Omega_i |\ \iota_i^{-1}(X) \in \sigma_i \text{ for all } i \in \Lambda\}$$

then we assert that $(\dot{\bigcup}_{i\in\Lambda}\Omega_i, \sigma_{\dot{\bigcup}_{i\in\Lambda}\Omega_i})$ is the coproduct of $\{(\Omega_i, \sigma_i)\}_{i\in\Lambda}$. It is easy to check that $\sigma_{\dot{\bigcup}_{i\in\Lambda}\Omega_i}$ is a σ-algebra of $\dot{\bigcup}_{i\in\Lambda}\Omega_i$. By the definition, $\iota_i : (\Omega_i, \sigma_i) \longrightarrow (\dot{\bigcup}_{i\in\Lambda}\Omega_i, \sigma_{\dot{\bigcup}_{i\in\Lambda}\Omega_i})$ is a morphism of measurable spaces for each $i \in \Lambda$. For every measurable space (Ω, σ) and every family of morphisms of measurable spaces $\{f_i : (\Omega_i, \sigma_i) \longrightarrow (\Omega, \sigma)\}$, there exists a unique function $f : \dot{\bigcup}_{i\in\Lambda}\Omega_i, \longrightarrow \Omega$ such that $f\iota_i = f_i$. For each $X \in \sigma$, and every $i \in \Lambda$ we have $\iota_i^{-1}(f^{-1}(X)) = f_i^{-1}(X) \in \sigma_i$ and so $f^{-1}(X) \in \sigma_{\dot{\bigcup}_{i\in\Lambda}\Omega_i}$ whence f is a morphism of measurable spaces. On the other hand, since $\dot{\bigcup}_{i\in\Lambda}\Omega_i$ is the coproduct of $\{\Omega_i\}_{i\in\Lambda}$ in the category of sets, f is unique as a morphism of measurable spaces. □

We notice that a coproduct in the category of measurable spaces is not necessarily a coproduct in the category of probability spaces.

Remark 2.6. If $\{(\Omega_i, \sigma_i, P_i)\}$ is a family of probability spaces, then the measurable space $(\dot{\bigcup}_{i\in\Lambda}\Omega_i, \sigma_{\dot{\bigcup}_{i\in\Lambda}\Omega_i})$ mentioned in Theorem 2.5 is not necessarily a probability space with a probability measure, denoted by $\coprod_{i\in\Lambda} P_i$. To be more precise, consider a probability space (Ω, σ, P) such that there exist $X, Y \in \sigma$ with $P(X) \neq P(Y)$. Now consider a family $\{(\Omega_i, \sigma_i, P_i)\}_{i\in\mathbb{N}}$ of probability spaces with $\Omega_i = \Omega$ and $\sigma_i = \sigma$ and $P_i = P$. If $(\dot{\bigcup}_{i\in\Lambda}\Omega_i, \sigma_{\dot{\bigcup}_{i\in\Lambda}\Omega_i}, \coprod_{i\in\Lambda} P_i)$ is a probability space with the canonical morphisms

$$\iota_i : (\Omega_i, \sigma_i, P_i) \longrightarrow (\dot{\bigcup}_{i\in\Lambda}\Omega_i, \sigma_{\dot{\bigcup}_{i\in\Lambda}\Omega_i}, \coprod_{i\in\Lambda} P_i)$$

for each $i \in \Lambda$, then $\coprod_{i\in\Lambda} P_i = P_i\iota_i^{-1}$ for each $i \in \Lambda$. We observe that $Z = (X, 1) \cup (Y, 2) \in \sigma_{\dot{\bigcup}_{i\in\Lambda}\Omega_i}$ so that $(\coprod_{i\in\Lambda} P_i)(Z) = P_1\iota_1^{-1}(Z) = P(X)$ and similarly $(\coprod_{i\in\Lambda} P_i)(Z) = P_2\iota_2^{-1}(Z) = P(Y)$ which makes a contradiction.

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3. Measurable and probability modules in modules category

Throughout this section R is an arbitrary ring and we denote by Mod-R the category of all right R-modules. For more details about Mod-R, we refer the reader to [6, 3]. We start this section with some new definitions.

Definition 3.1. Let M be a right R-module. A pair (M, σ_M) is called *measurable right R-module* if M is a right R-module equipped with a σ-algebra σ_M. Let (M, σ_M) and (N, σ_N) be measurable right R-modules. A right R-homomorphism $f : M \longrightarrow N$ is said to be a *measurable R-homomorphism* if $f^{-1}(X) \in \sigma_M$ for any $X \in \sigma_N$.

It is clear that $(\{0\}, \sigma_{\{0\}})$ is a measurable right R-module. Also for measurable right R-modules (M, σ_M) and (N, σ_N), the zero homomorphism $0 : M \longrightarrow N$ of right R-modules is a measurable R-homomorphism. More precisely, for every $X \in \sigma_N$, the set $0^{-1}(X)$ is either empty or M. It is clear that the identity homomorphism of a right R-module M is the identity morphism of measurable right R-module (M, σ_M) and we denote it by $1_{(M,\sigma_M)}$.

Definition 3.2. A measurable R-homomorphism $f : (M, \sigma_M) \longrightarrow (N, \sigma_N)$ is said to be *isomorphism* if there exists a measurable R-homomorphism $g : (N, \sigma_N) \longrightarrow (M, \sigma_M)$ such that $gf = 1_{(M,\sigma_M)}$ and $fg = 1_{(N,\sigma_N)}$.

In other words, f is an isomorphism if f is an isomorphism of right R-modules such that $f(X) \in \sigma_N$ for every $X \in \sigma_M$. We denote by MeasMod-R the category of all measurable right R-modules.

Example 3.3. Let k be a natural number and $\mathcal{B}(\mathbb{R}^k)$ be the Borel σ-algebra on $\mathbb{R}^k$. It is clear that $\mathbb{R}^k$ is a $\mathbb{Z}$-module and so $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$ is a measurable $\mathbb{Z}$-module. For any real number r, the linear function $f : \mathbb{R}^k \longrightarrow \mathbb{R}^k$ defined by $f(x) = rx$ is a $\mathbb{Z}$-homomorphism and further, $f : (\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k)) \longrightarrow (\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$ is a measurable $\mathbb{Z}$-homomorphism. Moreover, if r is nonzero, then f is an isomorphism with the inverse morphism $g : (\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k)) \longrightarrow (\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$ given by $g(x) = r^{-1}x$.

Definition 3.4. A triple (M, σ_M, P_M) is called a *probability right R-module* if (M, σ_M) is a measurable right R-module and P_M is a probability function on M. A *probability R-homomorphism* $f : (M, \sigma_M, P_M) \longrightarrow (N, \sigma_N, P_N)$, is a measurable R-homomorphism $f : (M, \sigma_M) \longrightarrow (N, \sigma_N)$ such that $P_N(X) = P_M(f^{-1}(X))$ for every $X \in \sigma_N$. We denote by Prob-R, the category of all probability right R-modules.

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Remark 3.5. If $0 : (M, \sigma_M, P_M) \longrightarrow (N, \sigma_N, P_N)$ is the zero morphism of probability spaces, it is clear that $P_N(X) = 1$ if $0 \in X$; otherwise $P_N(X) = 0$. Also, it is clear that the identity morphism of probability space (M, σ_M, P) is the identity R-homomorphism of M. We notice that if (M, σ_M, P_M) is a probability right R-module and N is a submodule of M such that $N \in \sigma_M$ with $P_M(N) \neq 1$, then the inclusion $\iota : (N, \sigma_N) \longrightarrow (M, \sigma_M)$ is a measurable R-homomorphism where $\sigma_N = \{N \cap X |\ X \in \sigma_M\}$; but it is not a morphism of probability modules under any probability function P_N on N. To be more precise, if $\iota : (N, \sigma_N, P_N) \longrightarrow (M, \sigma_M, P_M)$ is a morphism of probability right R-modules, then $1 = P_M(M) = P_N(N)$. On the other hand, $1 \neq P_M(N) = P_N(N)$ which is a contradiction.

Example 3.6. Let $\Omega = \mathbb{R}$, $\mathcal{F} = \mathcal{B}(\mathbb{R})$ and $P = \mu_F$, the Lebesgue-Stieltjes measure corresponding to a cumulative distribution function F, i.e., a function $F : \mathbb{R} \longrightarrow \mathbb{R}$ that is nondecreasing, right continuous and satisfies $F(-\infty) = 0$, $F(+\infty) = 1$. Then $(\mathbb{R}, \mathcal{F}, P)$ is a probability $\mathbb{Z}$-module. For any $r \in \mathbb{R}$, the linear function $f : (\mathbb{R}, \mathcal{B}(\mathbb{R}), P) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}), P')$, given by $f(x) = rx$ is a probability $\mathbb{Z}$-homomorphism $P'(X) = P(f^{-1}(X))$ for any $X \in \mathcal{B}(\mathbb{R})$.

The first result in this section shows that the category MeasMod-R has kernels.

Theorem 3.7. *The category* MeasMod-R *is closed under kernels in the category* Mod-R.

Proof. Assume that $f : (M, \sigma_M) \longrightarrow (N, \sigma_N)$ is a measurable R-homomorphism and $k : K \longrightarrow M$ is the kernel of R-homomorphism $f : M \longrightarrow N$. We know that $K = \mathrm{Ker} f$ and k is the inclusion map. If we consider the σ-algebra induced by K, namely $\sigma_K = \{K \cap X |\ X \in \sigma_M\}$, then clearly $k : (K, \sigma_K) \longrightarrow (M, \sigma_M)$ is a measurable R-homomorphism. We now assert that k is the kernel of f. Suppose that $g : (L, \sigma_L) \longrightarrow (M, \sigma_M)$ is a measurable R-homomorphism such that $fg = 0$. Then there exists a unique homomorphism $h : L \longrightarrow K$ of right R-modules such that $kh = g$. For every $X \cap K \in \sigma_K$ with $X \in \sigma_M$, we have $h^{-1}(X \cap K) = h^{-1}(k^{-1}(X)) = g^{-1}(X) \in \sigma_K$. The uniqueness of h in Mod-R implies that h is unique in MeasMod-R. □

Remark 3.8. It should be noticed that the category Prob-R is not closed under kernels in the category Mod-R. To be more precise, given a probability R-homomorphism $f : (M, \sigma_M, P) \longrightarrow (N, \sigma_N)$ such that $K = \mathrm{Ker} f \in \sigma_M$, $P_M(K) \neq 1$ and σ_K is any arbitrary σ-algebra of K. If the inclusion morphism $k : (K, \sigma_K) \longrightarrow (M, \sigma_M)$ is a measurable R-homomorphism, then in view of Remark 3.5, there does not exists any function P_K such that $k : (K, \sigma_K, P_K) \longrightarrow (M, \sigma_M, P)$ is a probability R-homomorphism.

Given a category $\mathcal{C}$ and two morphisms $\alpha, \beta : M \longrightarrow N$ in $\mathcal{C}$, we say that $k : K \longrightarrow M$ is an *equalizer* for α, β if $\alpha k = \beta k$, and if whenever $k' : K' \longrightarrow M$ is such that $\alpha k' = \beta k'$, there is a unique morphism $\gamma : K' \longrightarrow K$ such that $k\gamma = k'$. The *coequalizer* of α and β is defined dually.

Corollary 3.9. *The category* MeasMod-R *is closed under equalizers in the category* Mod-R*.*

Proof. Given two morphisms $\alpha, \beta : (M, \sigma_M) \longrightarrow (N, \sigma_N)$ in MeasMod-R, since Mod-R is additive, $\alpha - \beta$ is an R-homomorphism. Assume that $k : K \longrightarrow M$ is the kernel of $\alpha - \beta$. Using a similar proof of Theorem 3.7, we deduce that $k : (K, \sigma_K) \longrightarrow (M, \sigma_M)$ is the equalizer of α, β in MeasMod-R where $\sigma_K = \{K \cap X |\ X \in \sigma_M\}$. □

Definition 3.10. Let (M, σ_M) be a measurable right R-module and let K be a submodule of M. We define $\sigma_{M/K} = \{Y \subseteq M/K |\ \pi^{-1}(Y) \in \sigma_M\}$, where $\pi : M \longrightarrow M/K$ is the canonical epimorphism of right R-modules.

It is clear that $(M/K, \sigma_{M/K})$ is a measurable right R-module and the canonical epimorphism $\pi : (M, \sigma_M) \longrightarrow (M/K, \sigma_{M/K})$ is a measurable R-homomorphism.

Example 3.11. For $\Omega = \mathbb{R}$, assume that

$$\sigma_{\mathbb{R}} = \{A \subseteq \mathbb{R} |\ A \text{ is countable or } A^c \text{ is countable}\}.$$

It is easy to show that $\sigma_{\mathbb{R}}$ is a σ-algebra and so $(\mathbb{R}, \sigma_{\mathbb{R}})$ is a measurable $\mathbb{Z}$-module. We notice that $\mathbb{Q}$ is a submodule of $\mathbb{R}$. Then $\pi : \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Q}$ defined by $p(x) = x + \mathbb{Q}$ is a canonical epimorphism of $\mathbb{Z}$-modules and $\sigma_{\mathbb{R}/\mathbb{Q}} = \{\{x_i + \mathbb{Q} |\ i \in I, x_i \in \mathbb{R}\} |\ \{x_i |\ i \in I\}$ is countable or $\{x_i |\ i \in I\}^c$ is countable$\}$. It is easy to show that $\sigma_{\mathbb{R}/\mathbb{Q}}$ is a σ-algebra and so $(\mathbb{R}/\mathbb{Q}, \sigma_{\mathbb{R}/\mathbb{Q}})$ is a measurable $\mathbb{Z}$-module. It is clear by the definition of $\sigma_{\mathbb{R}/\mathbb{Q}}$ that π is a measurable $\mathbb{Z}$-homomorphism.

Proposition 3.12. *Assume that $f : (M, \sigma_M) \longrightarrow (N, \sigma_N)$ is a measurable R-homomorphism such that $A = \mathrm{Im} f$. Then $\sigma_N(A) \subseteq \sigma_M(A)$ where $\sigma_M(A) = \{X \subseteq A | f^{-1}(X) \in \sigma_M\}$ and $\sigma_N(A) = \{X \cap A | X \in \sigma_N\}$. Moreover if $f(\sigma_M) \subseteq \sigma_N$, then $\sigma_M(A) = \sigma_N(A)$*

Proof. For any $Y \cap A \in \sigma_N(A)$ in which $Y \in \sigma_N$, we have $f^{-1}(Y \cap A) = f^{-1}(Y) \in \sigma_M$; and hence $Y \cap A \in \sigma_M(A)$. In order to prove the second claim, if $Y \in \sigma_M(A)$, then $Y = f(f^{-1}(Y)) \in \sigma_N$. Therefore $Y = Y \cap A \in \sigma_N(A)$. □

Theorem 3.13. *The category* MeasMod-R *is closed under cokernels in the category* Mod-R*.*

Proof. Assume that $f : (M, \sigma_M) \longrightarrow (N, \sigma_N)$ is a measurable R-homomorphism. Considering, f as a homomorphism of right R-modules, assume that coker $f = p : N \longrightarrow C$, where $C = N/\mathrm{Im} f$ and p is the canonical epimorphism. In view of Definition 3.10, $p : (N, \sigma_N) \longrightarrow (C, \sigma_C)$ is a measurable R-homomorphism where $\sigma_C = \{Y \subseteq C | \; p^{-1}(Y) \in \sigma_N\}$. Now we show that $p : (N, \sigma_N) \longrightarrow (C, \sigma_C)$ is the cokernel of f. Assume that $g : (N, \sigma_N) \longrightarrow (L, \sigma_L)$ is a measurable R-homomorphism such that $gf = 0$. Then there exists a unique R-homomorphism $h : C \longrightarrow L$ such that $hp = g$. For every $X \in \sigma_L$, we have $p^{-1}(h^{-1}(X)) = g^{-1}(X) \in \sigma_N$. This implies that $h^{-1}(X) \in \sigma_C$; and hence $h : (C, \sigma_C) \longrightarrow (L, \sigma_L)$ is a measurable R-homomorphism. The uniqueness of h in Mod-R implies that h is unique in MeasMod-R. □

We have the following corollary about coequalizers in MeasMod-R.

Corollary 3.14. *The category* MeasMod-R *is closed under coequalizers in the category* Mod-R

Proof. Given two measurable R-homomorphisms $\alpha, \beta : (M, \sigma_M) \longrightarrow (N, \sigma_N)$, since Mod-$R$ is additive, $\alpha - \beta$ is an R-homomorphism. If we consider coker$(\alpha - \beta) = p : N \longrightarrow C$ where $C = N/\mathrm{Im}(\alpha - \beta)$, then using a similar proof of Theorem 3.13, the measurable R-homomorphism $p : (N, \sigma_N, C, \sigma_C)$ is a coequalizer of α, β in MeasMod-R where $\sigma_C = \{Y \subseteq C | \; p^{-1}(Y) \in \sigma_N\}$. □

Corollary 3.15. *The category* Prob-R *is closed under cokernels in the category* Mod-R.

Proof. Assume that $f : (M, \sigma_M, P_M) \longrightarrow (N, \sigma_N, P_N)$ is a probability R-homomorphism. In view of Theorem 3.13, assume that $p : (N, \sigma_N) \longrightarrow (C, \sigma_C)$ is the cokernel of the measurable R-homomorphism $f : (M, \sigma_M) \longrightarrow (N, \sigma_N)$. If we define $P_C(X) = P_N(p^{-1}(X))$ for all $X \in \sigma_C$, then it is clear that $p : (N, \sigma_N, P_N) \longrightarrow (C, \sigma_C, P_C)$ is a probability R-homomorphism. Given a probability R-homorphism $g : (N, \sigma_N, P_N) \longrightarrow (L, \sigma_L, P_L)$ with $gf = 0$, there exists a unique homomorphism $h : C \longrightarrow L$ of right R-modules such that $hp = g$. For every $X \in \sigma_L$, we have $P_C(h^{-1}(X)) = P_N(p^{-1}(h^{-1}(X)) = P_N(g^{-1}(X)) = P_L(X)$ so that $h : (C, \sigma_C, P_C) \longrightarrow (L, \sigma_L, P_L)$ is a probability R-homomorphism. The uniqueness of h follows from Theorem 3.13. □

Corollary 3.16. *The category* Prob-R *is closed under coequalizers in the category* Mod-R *.*

Proof. The proof is similar to the proof of Corollary 3.14 and 3.15. □

Definition 3.17. Given a category $\mathcal{C}$ and two morphisms $f_i : C \longrightarrow C_i$ of $\mathcal{C}$ for $i = 1, 2$, an object P of $\mathcal{C}$ together with two morphisms $g_i : C_i \longrightarrow P$ of $\mathcal{C}$ for $i = 1, 2$, is said to be the *pushout* of α_1 and α_2 if the following diagram is commutative

$$\begin{array}{ccc} C & \xrightarrow{f_1} & C_1 \\ {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle g_1} \\ C_2 & \xrightarrow[g_2]{} & P \end{array}$$

and if there exists another object P' together with two morphisms $g'_i : C_i \longrightarrow P'$ of $\mathcal{C}$ for $i = 1, 2$, then there exists a unique morphism $\gamma : P \longrightarrow P'$ of $\mathcal{C}$ such that $g'_i = \gamma g_i$ for $i = 1, 2$. The *pullback* of two morphism can be defined dually.

In the following result, we show that the category MeasMod-R has pushouts.

Proposition 3.18. *The category* MeasMod-R *is closed under pushouts in the category* Mod-R*.*

Proof. Assume that $f_1 : (M, \sigma_M) \longrightarrow (M_1, \sigma_{M_1})$ and $f_2 : (M, \sigma_M) \longrightarrow (M_2, \sigma_{M_2})$ are measurable R-homomorphisms. Since f_1 and f_2 are R-homomorphisms, there exists the following pushout diagram in Mod-R.

$$\begin{array}{ccc} M & \xrightarrow{f_1} & M_1 \\ {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle g_1} \\ M_2 & \xrightarrow[g_2]{} & D \end{array}$$

We define $\sigma_D = \{X \subseteq D$ such that $g_2^{-1}(X) \in \sigma_{M_1}$ and $g_1^{-1}(X) \in \sigma_{M_2}\}$; and hence g_1, g_2 are morphisms of measurable right R-modules. Assume that (E, σ_E) is a measurable right R-module and $h_i : (M_i, \sigma_{M_i}) \longrightarrow (E, \sigma_E)$, $i = 1, 2$ are morphisms in MeasMod-R such that $h_1 f_1 = h_2 f_2$. As the above diagram is pushout in Mod-R, there exists a unique R-homomorphism $h : D \longrightarrow E$ such that $hg_1 = h_1$ and $hg_2 = h_2$. For any $X \in \sigma_E$, we have $h_i^{-1}(X) \in \sigma_{M_i}$ for $i = 1, 2$. Thus $g_i^{-1}(h^{-1}(X)) \in \sigma_{M_i}$ for $i = 1, 2$ so that $h^{-1}(X) \in \sigma_D$. The uniqueness of h in Mod-R implies that h is unique in MeasMod-R. □

In view of Remark 3.8, since kernels are particular cases of pullbacks, Prob-R is not closed under pullbacks in Mod-R.

Corollary 3.19. *The category* Prob-R *is closed under pushouts in the category* Mod-R*.*

Proof. Assume that $f_1 : (M, \sigma_M, P) \longrightarrow (M_1, \sigma_{M_1}, P_1)$ and $f_2 : (M, \sigma_M, P) \longrightarrow (M_2, \sigma_{M_2}, P_2)$ are probability R-homomorphisms. Then $f_1 : (M, \sigma_M) \longrightarrow (M_1, \sigma_{M_1})$ and $f_2 : (M, \sigma_M) \longrightarrow (M_2, \sigma_{M_2})$ are measurable R-homomorphisms; and hence using Proposition 3.18, there exist the following pushout diagram in MeasMod-R

$$\begin{array}{ccc} M & \xrightarrow{f_1} & M_1 \\ {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle g_1} \\ M_2 & \xrightarrow[g_2]{} & D. \end{array}$$

Since f_1 and f_2 are morphisms in Prob-R, so are g_1 and g_2 and for any $X \in \sigma_D$, we have $P_D(X) = P_1(g_1^{-1}(X)) = P(f_1^{-1}(g_1^{-1}(X)) = P(f_2^{-1}(g_2^{-1}(X)) = P_2(g_2^{-1}(X))$ so that the diagram is commutative in Prob-R. Suppose that (E, σ_E, P_E) is a probability right R-module and $h_i : (M_i, \sigma_{M_i}, P_i) \longrightarrow (E, \sigma_E, P_E)$ for $i = 1, 2$ are probability R-homomorphisms such that $h_1 f_1 = h_2 f_2$. Using Proposition 3.18, there exists a unique measurable R-homomorphism $h : D \longrightarrow E$ such that $hg_1 = h_1$ and $hg_2 = h_2$. For any $X \in \sigma_E$, we have $P_E(X) = P_1(h_1^{-1}(X)) = P_1(g_1^{-1}(h^{-1}(X))) = P_D(h^{-1}(X))$ and so h is a probability R-homomorphism. The uniqueness of h in MeasMod-R implies that h is unique in Prob-R. □

Theorem 3.20. *The category* MeasMod-R *is closed under coproducts in the category* Mod-R.

Proof. Assume that $\{(M_i, \sigma_{M_i})\}_{i \in \Lambda}$ is a family of measurable right R-modules and let $\coprod_\Lambda M_i$ be the coproducts of M_i in Mod-R and $\iota_i : M_i \longrightarrow \coprod_\Lambda M_i$ be the canonical injection in Mod-R for each i. We define

$$\sigma_{\coprod_\Lambda M_i} = \{X \subseteq \coprod_\Lambda M_i |\ \iota_i^{-1}(X) \in \sigma_{M_i} \text{ for all } i \in \Lambda\}$$

and we show that $(\coprod_\Lambda M_i, \sigma_{\coprod_\Lambda M_i}) = \coprod_\Lambda (M_i, \sigma_{M_i})$. By the definition, $\iota_i : (M_i, \sigma_{M_i}) \longrightarrow (\coprod_\Lambda M_i, \sigma_{\coprod_\Lambda M_i})$ is a measurable R-homomorphism for each $i \in \Lambda$. Now, for any measurable right R-module (M, σ_M) and any family of measurable R-homomorphisms $\{f_i : (M_i, \sigma_{M_i}) \longrightarrow (M, \sigma_M) |\ i \in \Lambda\}$, there exists a unique R-homomorphism $f : \coprod_\Lambda M_i \longrightarrow M$ such that $f\iota_i = f_i$ for each $i \in \Lambda$. For any $X \in \sigma_M$ and any $i \in \Lambda$, we have $\iota_i^{-1}(f^{-1}(X)) = f_i^{-1}(X) \in \sigma_{M_i}$ and hence $f^{-1}(X) \in \sigma_{\coprod_\Lambda M_i}$ so that $f : (\coprod_\Lambda M_i, \sigma_{\coprod_\Lambda M_i}) \longrightarrow (M, \sigma_M)$ is a measurable R-homomorphism. The uniqueness of f in Mod-R implies that f is unique in MeasMod-R. □

We have the following result about products in MeasMod-R.

Proposition 3.21. *The category* MeasMod-R *is closed under products in the category* Mod-R.

Proof. Given a family $\{(M_i, \sigma_i)\}_{i\in\Lambda}$ of measurable right R-modules, let σ be the smallest σ-algebra generated by all subsets $\pi_i^{-1}(X)$ of $\prod_\Lambda M_i$, where $i \in \Lambda$, $X \in \sigma_i$, $\prod_\Lambda M_i$ is the product of M_i in Mod-R and $\pi_i : \prod_\Lambda M_i \longrightarrow M_i$ are the canonical projections in Mod-R. For any family $\{f_i : (M, \sigma') \longrightarrow (M_i, \sigma_i)\}_{i\in\Lambda}$ of measurable R-homomorphisms, we have a unique R-homomorphism $f = \prod_\Lambda f_i : M \longrightarrow \prod_\Lambda M_i$ such that $\pi_i f = f_i$ for each $i \in \Lambda$. Given $Y \in \sigma$, we may assume that $Y = \pi_i^{-1}(X)$ for some $i \in \Lambda$ and $X \in \sigma_i$. Then $f^{-1}(Y) = f^{-1}(\pi_i^{-1}(X)) = f_i^{-1}(X) \in \sigma'$; and hence $f : (M, \sigma') \longrightarrow (\prod_\Lambda M_i, \sigma)$ is a measurable R-homomorphism. The uniqueness of f in Mod-R implies that f is unique in MeasMod-R. □

Definition 3.22. Let $n \in \mathbb{N}$ and let $\{(M_i, \sigma_{M_i}) |\ 1 \le i \le n\}$ be a family of measurable right R-modules. A family $\{f_i : (M, \sigma_M, P) \longrightarrow (M_i, \sigma_{M_i}) |\ 1 \le i \le n\}$ of probability R-homomorphisms is said to be *stochastically independent* if for every $X_i \in \sigma_{M_i}$, $i = 1, \ldots, n$, we have the following equality

$$P(\bigcap_{i=1}^{n} f_i^{-1}(X_i)) = \prod_{i=1}^{n} P(f_i^{-1}(X_i)).$$

Example 3.23. Let (M_1, σ_1, P_1) and (M_2, σ_2, P_2) be probability right R-modules. For any $B_1 \in \sigma_1$ and $B_2 \in \sigma_2$, we set $\tilde{B_1} = B_1 \times M_2$ and $\tilde{B_2} = M_1 \times B_2$ and $\Sigma_i = \{\tilde{B_i} |\ B_i \in \sigma_i\}$ for $i = 1, 2$. Let σ be the σ-algebra generated by $\Sigma_1 \cup \Sigma_2$. For any $\tilde{B_1} \in \Sigma_1$ and $\tilde{B_2} \in \Sigma_2$, it is clear that $\tilde{B_1} \cap \tilde{B_2} = B_1 \times B_2$. We define probability function on σ with the property that $P(\tilde{B_1} \cap \tilde{B_2}) = P_1(B_1)P_2(B_2)$. Then $(M_1 \times M_2, \sigma, P)$ is a probability right R-module. By the construction of σ, the canonical projection $\pi_i : (M_1 \times M_2, \sigma) \longrightarrow (M_i, \sigma_i)$ is measurable for $i = 1, 2$. Furthermore, we have

$$\begin{aligned} P(\pi_1^{-1}(B_1) \cap \pi_2^{-1}(B_2)) &= P(\tilde{B_1} \cap \tilde{B_2}) \\ &= P(B_1 \times B_2) = P_1(B_1)P_2(B_2) \\ &= P(B_1 \times M_2)P(M_1 \times B_2) \\ &= P(\pi_1^{-1}(B_1))P(\pi_2^{-1}(B_2)). \end{aligned}$$

Thus, the canonical morphisms $\pi_i : (M_1 \times M_2, \sigma, P) \longrightarrow (M_i, \sigma_i, P_i)$ for $i = 1, 2$ are stochastically independent.

Let $n \in \mathbb{N}$. Given a family $\{(M_i, \sigma_{M_i}, P_i) |\ 1 \le i \le n\}$ of probability right R-modules, let $\sigma(\bigoplus_{i=1}^n M_i)$ be the least σ-algebra of $\bigoplus_{i=1}^n M_i$ such that the canonical projections $\pi_i : \bigoplus_{i=1}^n M_i \to M_i$ are measurable for $1 \le i \le n$. In view of [7], we can define a probability function $P_{\bigoplus_{i=1}^n M_i}$ on $\sigma(\bigoplus_{i=1}^n M_i)$ in such

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a way that for any $X_i \in \sigma_{M_i}$ with $1 \leq i \leq n$, we have $P_{\bigoplus_{i=1}^n M_i}(\bigoplus_{i=1}^n X_i) = \prod_{i=1}^n P_i(X_i)$.

The following lemma shows that Prob-R contains the canonical projections.

Lemma 3.24. *For $1 \leq i \leq n$, the canonical projections*

$$\pi_i : (\bigoplus_{i=1}^n M_i, \sigma(\bigoplus_{i=1}^n M_i), P_{\bigoplus_{i=1}^n M_i}) \longrightarrow (M_i, \sigma_{M_i}, P_i)$$

are morphisms in Prob-R*.*

Proof. It suffices to consider $n = 2$ and we prove the result for $i = 1$. The case $i = 2$ can be proved similarly. For any $X \in \sigma_{M_1}$, we have $\pi_1^{-1}(X) = X \oplus M_2$ and so $P_{M_1 \oplus M_2}(\pi_1^{-1}(X)) = P_{M_1 \oplus M_2}(X \oplus M_2) = P_1(X)$. □

Proposition 3.25. *Let M_1 and M_2 be right R-modules. Then $\sigma(M_1 \oplus M_2) \subseteq \sigma_{M_1 \oplus M_2}$ where $\sigma_{M_1 \oplus M_2}$ is as in Theorem 3.20. Moreover, if X is an element of $\sigma_{M_1 \oplus M_2}$ which is closed under operation $+$ and contains 0, then it contains an element $Y \in \sigma(M_1 \oplus M_2)$ such that $0 \in Y$.*

Proof. Assume that $\iota_i : M_1 \longrightarrow M_1 \oplus M_2$ are injections for $i = 1, 2$. It is enough to show that $\iota_i : (M_1, \sigma_{M_1}) \longrightarrow (M_1 \oplus M_2, \sigma(M_1 \oplus M_2))$ are measurable morphisms for $i = 1, 2$. We prove the assertion for $i = 1$ and for $i = 2$ is similar. Since $\sigma(M_1 \oplus M_2)$ is generated by all $X \oplus Y$ where $X \in \sigma_{M_1}$ and $Y \in \sigma_{M_2}$, it suffices to show that $\iota_1^{-1}(X \oplus Y) \in \sigma_{M_1}$ for any $X \in \sigma_{M_1}$ and $Y \in \sigma_{M_2}$. Clearly $\iota_1^{-1}(X \oplus M_2) = X \in \sigma_{M_1}$ and $\iota_1^{-1}(M_1 \oplus Y)$ is either M_1 or empty and so $\iota_1^{-1}(M_1 \oplus Y) \in \sigma_{M_1}$, too. Now $\iota_1^{-1}(X \oplus Y) = \iota_1^{-1}((X \oplus M_2) \cap (M_1 \oplus Y)) = \iota_1^{-1}(X \oplus M_2) \cap \iota_1^{-1}(M_1 \oplus Y) \in \sigma_{M_1}$. In order to prove the second claim, it follows from the definition that $\iota_i^{-1}(X) \in M_i$ for $i = 1, 2$; and hence $\pi_i^{-1}(\iota_i^{-1}(X)) \in \sigma(M_1 \oplus M_2)$ for $i = 1, 2$. Now, if we set $Y = \pi_1^{-1}(\iota_1^{-1}(X)) \cap \pi_2^{-1}(\iota_2^{-1}(X)) \in \sigma(M_1 \oplus M_2)$, then we deduce that $Y \subseteq X$ as X is closed under addition and clearly $0 \in Y$. □

Proposition 3.26. *If $f_1 : (M, \sigma_M) \longrightarrow (M_1, \sigma_{M_1})$ and $f_2 : (M, \sigma_M) \longrightarrow (M_2, \sigma_{M_2})$ are morphisms of measurable right R-modules, then so is $f = (f_1, f_2) : (M, \sigma_M) \longrightarrow (M_1 \oplus M_2, \sigma(M_1 \oplus M_2))$.*

Proof. It is enough to consider $X \oplus Y \in \sigma(M_1 \oplus M_2)$ where $X \in \sigma_{M_1}$ and $Y \in \sigma_{M_2}$. Then in this case, we have $f^{-1}(X \oplus Y) = f_1^{-1}(X) \cap f_2^{-1}(Y) \in \sigma_M$. □

We now give a result about the stochastically independent of two probability spaces.

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Theorem 3.27. *Let* $f_1 : (M, \sigma_M, P) \longrightarrow (M_1, \sigma_{M_1}, P_1)$ *and* $f_2 : (M, \sigma_M, P) \longrightarrow (M_2, \sigma_{M_2}, P_2)$ *be morphisms of measurable right* R*-modules. If the canonical morphism* $f = (f_1, f_2) : (M, \sigma_M, P) \longrightarrow (M_1 \oplus M_2, \sigma(M_1 \oplus M_2), P_{M_1 \oplus M_2})$ *is a probability* R*-homomorphism, then so are* f_1 *and* f_2*. Furthermore,* f_1 *and* f_2 *are stochastically independent.*

Proof. We observe that $f_i = \pi_i f$ for $i = 1, 2$ and hence it follows from the assumption and Lemma 3.24 that f_i are morphisms in Prob-R for $i = 1, 2$. To prove the second claim, using the first claim, for any $X \in \sigma_{M_1}$ and $Y \in \sigma_{M_2}$, we have the following equalities

$$P(f_1^{-1}(X) \cap f_2^{-1}(Y)) = P(f^{-1}(X \oplus Y))$$
$$= P_{M_1 \oplus M_2}(X \oplus Y) = P_1(X) P_2(Y) = P(f_1^{-1}(X)) P(f_2^{-1}(Y)).$$

□

In view of [7], the category of probability spaces does not have products. However, for finite products in Prob-R, we have the following proposition.

Proposition 3.28. *Let* $\{(M_i, \sigma_{M_i}, P_i) | \ 1 \leq i \leq n\}$ *be a family of probability right* R*-modules. Then* $(\bigoplus_{i=1}^n M_i, \sigma(\bigoplus_{i=1}^n M_i), P_{\bigoplus_{i=1}^n M_i})$ *is the product of* $\{(M_i, \sigma_{M_i}, P_i) | \ 1 \leq i \leq n\}$ *in* Prob-R *if and only if any family of probability* R*-homomorphisms* $\{f_i : (M, \sigma_M, P) \longrightarrow (M_i, \sigma_{M_i}) | \ 1 \leq i \leq n\}$ *is stochastically independent.*

Proof. It suffices to consider $n = 2$. Assume that f_1 and f_2 are stochastically independent. By the universal products in Mod-R, there exists a unique R-homomorphism $f = (f_1, f_2) : M \longrightarrow M_1 \oplus M_2$ such that $\pi_i f = f_i$ for $i = 1, 2$. According to Lemma 3.24, the canonical projections $\pi_i : (M_1 \oplus M_2), \sigma(M_1 \oplus M_2), P_{M_1 \oplus M_2})) \longrightarrow (M_i, \sigma_i)$ are probability R-homomorphisms. It suffices to show that $f : (M, \sigma, P) \longrightarrow (M_1 \oplus M_2, \sigma(M_1 \oplus M_2), P_{M_1 \oplus M_2})$ is a probability R-homomorphism. To do this, for any $X_i \in \sigma_i$ with $i = 1, 2$, we have

$$P(f^{-1}(X_1 \oplus X_2) = P(f_1^{-1}(X_1) \cap f_2^{-1}(X_2)) = P(f_1^{-1}(X_1)) P(f_2^{-1}(X_2))$$
$$= P_1(X_1) P_2(X_2) = P_{M_1 \oplus M_2}(X_1 \oplus X_2).$$

Conversely, if $\{f_i : (M, \sigma_M, P) \longrightarrow (M_i, \sigma_{M_i})$ for $i = 1, 2$ are probability R-homomorphisms, then there exists a probability R-homomorphism $f = (f_1, f_2) : (M, \sigma_M, P) \longrightarrow (M_1 \oplus M_2, \sigma(M_1 \oplus M_2), P_{M_1 \oplus M_2})$ and so it follows from Theorem 3.27 that f_1 and f_2 are stochastically independent. □

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