



A New Analytical Method to Improve Attitude Correction in Inertial Navigation Systems

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ABSTRACT: Inertial Navigation Systems suffer from accumulative errors due to their dead reckoning structure. Consequently, the navigation reset or realigning process of such systems is unavoidable. Realignment starts with estimating the navigation errors and correcting the navigation states. To correct the error of attitude states in the navigation reset process, different kinds of attitude correction method are used in the literature. This paper proposed an analytical attitude correction method that can calculate the error of Euler angles more precisely than the conventional method. In addition, this new approach preserves normality and orthogonality characteristics of the transformation matrix while the conventional method leads to losing both of these conditions. The proposed method expresses the error of Euler angles as functions of Euler angles and small rotation angles. The relation between the Euler angles error and the small rotation angles is nonlinear and mathematical calculation is performed to extract the explicit functions. Numerical simulations prove that the proposed method of attitude correction has the mentioned features and its performance is dominant over the conventional method.

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1- Introduction

Inertial Navigation Systems (INS) suffer from increasing navigation errors due to their dead-reckoning structure. Thus, in order to realign such systems, it is vital to correct the navigation states periodically [1]. Correction of navigation states is done by using external measurements including attitude, position, and velocity, called navigation aids. The above procedure is known as navigation error reset in the literature [2]. Realignment Inertial Navigation Systems usually starts with estimating navigation errors [3-5], and then the estimated errors are used to correct attitude with a conventional method. This paper introduces a new way for attitude correction with small rotation angles.

To correct the navigation errors, it is essential to focus on the source of these errors. Perturbing INS ential equations of attitude, position, and velocity provide INS navigation error equations [6]. INS error equations are usually linear because the derived perturbed equations include only first-order terms [7]. Some INS error modeling approaches for handling the significant amplitude attitude errors are reported in [8-12].

Phi angle and psi angle approaches are two different techniques for attitude error modeling. The first technique provides error equations in the true frame of navigation [13], and the latter defines the error equations in the estimated frame of navigation [14]. It is proven that both of these techniques are equivalent [13]. Navigation errors are modeled based on small rotation angles assumption. If the attitude error magni-

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tude diverges from the small rotation angle assumption, the transformation matrix estimated by these attitude errors will drop its normality and orthogonality characteristics. Consequently, efficient error detaching would not be reachable.

Relation of Euler angles error and small rotation angles are derived analytically in this paper. Found on this analytical relation, a new method to improve the realigning procedure in Inertial Navigation Systems is presented. Numerical simulation proved that the presented method does not affect normality and orthogonality characteristics of the matrix of transformation while the performance is dominant over the conventional algorithms.

The paper is organized in 5 sections: section 2 discusses the conventional method for correcting attitude in Inertial Navigation Systems. In section 3, a new algorithm for attitude correction in inertial navigation is introduced. Section 4 discusses the numerical simulation results, and finally, the conclusion is made in section 5.

2- Conventional Method for Correction of Attitude

Based on the Euler theorem, three sequential rotations result in a transformation from one coordinate system to another. Since the commutative rule is valid for small rotations, the order of rotations is not important, and it means the sequence of Euler angles can be changed. According to appendix A, a small rotation of frame \hat{N} (estimated frame of navigation) with respect to frame N (true frame of navigation) is provided by rotation tensor, and it is defined in the below equation:



$$\mathbf{R}^{\hat{N}N} = \mathbf{E} + \varepsilon \mathbf{R}^{\hat{N}N} \quad (1)$$

In equation (1), identity tensor is represented by \mathbf{E} , operator of perturbation is represented by ε , rotation tensor between N (true navigation frame) and \hat{N} (estimated navigation frame) are represented by $\mathbf{R}^{\hat{N}N}$, and $\varepsilon \mathbf{R}^{\hat{N}N}$ shows the perturbed representation of $\mathbf{R}^{\hat{N}N}$. It should be noted that by N , the North-East-Down (NED) navigation frame is intended. Appendix A shows that $\varepsilon \mathbf{R}^{\hat{N}N}$ is skew-symmetric, so the vector $\varepsilon \mathbf{R}^{\hat{N}N}$ is defined as follows:

$$[\varepsilon \mathbf{R}^{\hat{N}N}]^{\hat{N}} = [\varepsilon \mathbf{r}^{\hat{N}N}]^N = \begin{bmatrix} \varepsilon \phi \\ \varepsilon \theta \\ \varepsilon \psi \end{bmatrix} \quad (2)$$

It should be noted that yaw, roll, and pitch angle errors differ from vector elements $\varepsilon \phi, \varepsilon \theta$, and $\varepsilon \psi$, called small rotation angles. These Euler angles errors are defined as follows:

$$\begin{aligned} \Delta \psi &= \hat{\psi} - \psi \\ \Delta \theta &= \hat{\theta} - \theta \\ \Delta \phi &= \hat{\phi} - \phi \end{aligned} \quad (3)$$

where ψ, ϕ , and θ denote true Euler angles while $\hat{\psi}, \hat{\phi}$, and $\hat{\theta}$ represent estimated yaw, roll, and pitch angles. According to Appendix B, the small rotation angles dynamic model is defined as follows:

$$\begin{aligned} \varepsilon \dot{\phi} &= \omega_d \varepsilon \lambda - \frac{v_e}{R_c} \varepsilon h + \frac{1}{R_c} \varepsilon v_e + (\omega_d - \frac{v_e \tan \lambda}{R_c}) \varepsilon \theta + \frac{v_n}{R_c} \varepsilon \psi \\ &\quad - \cos \hat{\psi} \cos \hat{\theta} \delta \omega_x - (\cos \hat{\psi} \sin \hat{\theta} \sin \hat{\phi} - \sin \hat{\psi} \cos \hat{\phi}) \delta \omega_y \\ &\quad - (\cos \hat{\psi} \sin \hat{\theta} \cos \hat{\phi} + \sin \hat{\psi} \sin \hat{\phi}) \delta \omega_z \\ \varepsilon \dot{\theta} &= \frac{v_n}{R_c} \varepsilon h - \frac{1}{R_c} \varepsilon v_n + (\frac{v_e \tan \lambda}{R_c} - \omega_d) \varepsilon \phi + (\omega_n + \frac{v_e}{R_c}) \varepsilon \psi \\ &\quad - \sin \hat{\psi} \cos \hat{\theta} \delta \omega_x - (\sin \hat{\psi} \sin \hat{\theta} \sin \hat{\phi} + \cos \hat{\psi} \cos \hat{\phi}) \delta \omega_y \\ &\quad - (\sin \hat{\psi} \sin \hat{\theta} \cos \hat{\phi} - \cos \hat{\psi} \sin \hat{\phi}) \delta \omega_z \\ \varepsilon \dot{\psi} &= -(\omega_n + \frac{v_e \sec^2 \lambda}{R_c}) \varepsilon \lambda + \frac{v_e \tan \lambda}{R_c} \varepsilon h - \frac{\tan \lambda}{R_c} \varepsilon v_e - \frac{v_n}{R_c} \varepsilon \phi - \\ &\quad (\omega_n + \frac{v_e}{R_c}) \varepsilon \theta + \sin \hat{\theta} \delta \omega_x - \cos \hat{\theta} \sin \hat{\phi} \delta \omega_y - \cos \hat{\theta} \cos \hat{\phi} \delta \omega_z \end{aligned} \quad (4)$$

Although equation (4) reveals the relation between estimated Euler angles and small rotation angles, expression of $\Delta \phi, \Delta \theta, \Delta \psi$ in terms of $\varepsilon \phi, \varepsilon \theta, \varepsilon \psi$ is not developed. This relation is the subject of section 3.

In order to align Inertial Navigation Systems, the usual procedure starts with estimating small rotation angles by a

filter, which uses equation (4) as its dynamic model. Afterwards, employing equations (1-2) updates the rotation tensor $\mathbf{R}^{\hat{N}N}$, and finally attitude (Euler angles) correction will be done using the following equations [19]:

$$\begin{aligned} [\mathbf{T}]^{\hat{N}N} &= [\overline{\mathbf{R}}^{\hat{N}N}]^{\hat{N}} \\ [\mathbf{T}]^{\text{BN}} &= [\mathbf{T}]^{\text{BN}} [\mathbf{T}]^{\hat{N}N} \end{aligned} \quad (5)$$

$$\begin{cases} \theta = \arcsin(-t_{13}) \\ \phi = \arccos(t_{33} / \cos \theta) \text{sign}(t_{23}) \\ \psi = \arccos(t_{11} / \cos \theta) \text{sign}(t_{12}) \end{cases} \quad (6)$$

where, $[\mathbf{T}]^{\text{BN}} = [t_{ij}]$; $(i, j = 1, 2, 3)$.

Resetting the transformation matrix $[\mathbf{T}]^{\text{BN}}$ Employing equation (5) leads to numerical errors. Significantly, when the rotation angles deviate from the small rotation assumption, the transformation matrix loses its normality and orthogonality characteristics, and may eventually cause divergence of attitude propagation. A new analytical algorithm to correct the matrix of transformation without the mentioned drawbacks is introduced in the next part.

3-Proposed Analytical Algorithm for Correction of Attitude

This part specifies the relevance of Euler angles error and small rotation angles and a new technique for correcting attitude is obtained. This technique is advantageous to conventional algorithms. In the next section, a numerical comparison will prove this claim.

Expressing $\varepsilon \mathbf{R}^{\hat{N}N}$ in navigation frame is obtained by expanding equation (1):

$$\begin{aligned} [\varepsilon \mathbf{R}^{\hat{N}N}]^N &= [\mathbf{R}^{\hat{N}N}]^N - [\mathbf{E}]^N = \\ [\overline{\mathbf{T}}]^{\hat{N}N} - [\mathbf{E}]^N &= [\mathbf{T}]^{\text{NN}} - [\mathbf{E}]^N \\ &= [\mathbf{T}]^{\text{NB}} [\mathbf{T}]^{\text{BN}} - [\mathbf{E}]^N \end{aligned} \quad (7)$$

applying the transpose operator on equation (7) yields:

$$\begin{aligned} \overline{[\varepsilon \mathbf{R}^{\hat{N}N}]^N} &= \overline{[\mathbf{T}]^{\text{NB}} [\mathbf{T}]^{\text{BN}} - [\mathbf{E}]^N} \rightarrow \\ -[\varepsilon \mathbf{R}^{\hat{N}N}]^N &= [\overline{\mathbf{T}}]^{\text{BN}} [\overline{\mathbf{T}}]^{\text{NB}} - [\mathbf{E}]^N \rightarrow \\ [\varepsilon \mathbf{R}^{\hat{N}N}]^N &= [\mathbf{E}]^N - [\overline{\mathbf{T}}]^{\text{BN}} [\mathbf{T}]^{\text{BN}} \end{aligned} \quad (8)$$

Considering matrixes of transformation $[\mathbf{T}]^{\text{BN}}$ and $[\overline{\mathbf{T}}]^{\text{BN}}$ in close form, yields [16]:

$$[\mathbf{T}]^{\text{BN}} = \begin{bmatrix} \cos \psi \cos \theta & \sin \psi \cos \theta & -\sin \theta \\ \cos \psi \sin \theta \sin \phi - \sin \psi \cos \phi & \sin \psi \sin \theta \sin \phi + \cos \psi \cos \phi & \cos \theta \sin \phi \\ \cos \psi \sin \theta \cos \phi + \sin \psi \sin \phi & \sin \psi \sin \theta \cos \phi - \cos \psi \sin \phi & \cos \theta \cos \phi \end{bmatrix} \quad (9)$$

$$[T]^{BN} = \begin{bmatrix} \cos\psi \cos\hat{\theta} & \sin\psi \cos\hat{\theta} & -\sin\hat{\theta} \\ \cos\psi \sin\hat{\theta} \sin\hat{\phi} - \sin\psi \cos\hat{\phi} & \sin\psi \sin\hat{\theta} \sin\hat{\phi} + \cos\psi \cos\hat{\phi} & \cos\hat{\theta} \sin\hat{\phi} \\ \cos\psi \sin\hat{\theta} \cos\hat{\phi} + \sin\psi \sin\hat{\phi} & \sin\psi \sin\hat{\theta} \cos\hat{\phi} - \cos\psi \sin\hat{\phi} & \cos\hat{\theta} \cos\hat{\phi} \end{bmatrix} \quad (10)$$

Exerting equations (3), (9), and (10) into equation (8) and performing some algebraic works, outcomes are as follows:

$$[\varepsilon R^{NN}]^N = \begin{bmatrix} 0 & -\varepsilon\psi & \varepsilon\theta \\ \varepsilon\psi & 0 & -\varepsilon\phi \\ -\varepsilon\theta & \varepsilon\phi & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\Delta_3 & \Delta_2 \\ \Delta_1 & 0 & -\Delta_1 \\ -\Delta_2 & \Delta_1 & 0 \end{bmatrix} \quad (11)$$

$$\begin{aligned} \Delta_1 &= \Delta\theta \sin\hat{\psi} - \Delta\phi \cos\hat{\psi} \cos\hat{\theta} - \Delta\psi \Delta\theta \cos\hat{\psi} - \Delta\phi \Delta\psi \cos\hat{\theta} \sin\hat{\psi} \\ \Delta_2 &= -\Delta\theta \cos\hat{\psi} - \Delta\phi \cos\hat{\theta} \sin\hat{\psi} - \Delta\phi \Delta\theta \sin\hat{\psi} \sin\hat{\theta} \\ \Delta_3 &= \Delta\phi \sin\hat{\theta} - \Delta\psi - \Delta\phi \Delta\theta \cos^2\hat{\psi} \cos\hat{\theta} - \Delta\phi \Delta\psi \Delta\theta \cos\hat{\psi} \cos\hat{\theta} \sin\hat{\psi} \end{aligned}$$

Resorting Equation (11) results:

$$\begin{aligned} \varepsilon\phi &= \Delta\theta \sin\hat{\psi} - \Delta\phi \cos\hat{\psi} \cos\hat{\theta} - \Delta\psi \Delta\theta \cos\hat{\psi} - \Delta\phi \Delta\psi \cos\hat{\theta} \sin\hat{\psi} \\ \varepsilon\theta &= -\Delta\theta \cos\hat{\psi} - \Delta\phi \cos\hat{\theta} \sin\hat{\psi} - \Delta\phi \Delta\theta \sin\hat{\psi} \sin\hat{\theta} \\ \varepsilon\psi &= \Delta\phi \sin\hat{\theta} - \Delta\psi - \Delta\phi \Delta\theta \cos^2\hat{\psi} \cos\hat{\theta} - \Delta\phi \Delta\psi \Delta\theta \cos\hat{\psi} \cos\hat{\theta} \sin\hat{\psi} \end{aligned} \quad (12)$$

Equation (12) shows that it is possible to express small rotation angles as a function of error of Euler angles ($\Delta\phi$, $\Delta\theta$ and $\Delta\psi$) and estimated Euler angles ($\hat{\theta}$ and $\hat{\psi}$).

In integrated Inertial Navigation Systems, small rotation angles and Euler angles are estimated and calculated by filter and attitude propagation, respectively, so the error of Euler angles could be obtained by employing equation (12). Since equation (12) reveals the nonlinear relation of small rotation angles and Euler angles error, there is no way to obtain an explicit formula for calculating Euler angles error. In this situation, the authors suggest using explicit approximate equations to calculate Euler angles error, which is considered the main contribution of this study. In order to derive such explicit equations, higher-order terms (Δ^2 and Δ^3) are neglected, and only first-order terms are considered. The procedure to derive the explicit formula would be:

$$\begin{cases} \varepsilon\phi = \Delta\theta \sin\hat{\psi} - \Delta\phi \cos\hat{\psi} \cos\hat{\theta} \\ \varepsilon\theta = -\Delta\theta \cos\hat{\psi} - \Delta\phi \cos\hat{\theta} \sin\hat{\psi} \rightarrow \\ \varepsilon\psi = \Delta\phi \sin\hat{\theta} - \Delta\psi \\ \Delta\phi = -(\varepsilon\phi \cos\hat{\psi} + \varepsilon\theta \sin\hat{\psi}) \sec\hat{\theta} \\ \Delta\theta = \varepsilon\phi \sin\hat{\psi} - \varepsilon\theta \cos\hat{\psi} \\ \Delta\psi = -(\varepsilon\phi \cos\hat{\psi} + \varepsilon\theta \sin\hat{\psi}) \tan\hat{\theta} \end{cases} \quad (13)$$

If only third-order terms (Δ^3) are neglected, the explicit derived formula would be more accurate:

$$\begin{aligned} \varepsilon\phi &= \Delta\theta \sin\hat{\psi} - \Delta\phi \cos\hat{\psi} \cos\hat{\theta} - \Delta\psi (\Delta\theta \cos\hat{\psi} + \Delta\phi \cos\hat{\theta} \sin\hat{\psi}) \end{aligned} \quad (14)$$

$$\begin{aligned} \varepsilon\theta &= -(\Delta\theta \cos\hat{\psi} + \Delta\phi \cos\hat{\theta} \sin\hat{\psi}) - \Delta\phi \Delta\theta \sin\hat{\psi} \sin\hat{\theta} \end{aligned} \quad (15)$$

$$\varepsilon\psi = -\Delta\psi + \Delta\phi (\sin\hat{\theta} - \Delta\theta \cos^2\hat{\psi} \cos\hat{\theta}) \quad (16)$$

The simplified form of equation (16) would be:

$$\Delta\psi = \Delta\phi (\sin\hat{\theta} - \Delta\theta \cos^2\hat{\psi} \cos\hat{\theta}) - \varepsilon\psi \quad (17)$$

Exerting equation (17) into equation (14) and eliminating the third-order (Δ^3) terms yields:

$$\begin{aligned} \varepsilon\phi &= +\Delta\theta (\sin\hat{\psi} + \cos\hat{\psi} \varepsilon\psi) + \Delta\phi (\cos\hat{\theta} \sin\hat{\psi} \varepsilon\psi - \cos\hat{\psi} \cos\hat{\theta}) \\ &\quad - \Delta\phi^2 \sin\hat{\theta} \cos\hat{\theta} \sin\hat{\psi} - \Delta\phi \Delta\theta \sin\hat{\theta} \cos\hat{\psi} \end{aligned} \quad (18)$$

Another appearance of equation (15) would be:

$$\Delta\theta = \frac{-\varepsilon\theta - \Delta\phi \cos\hat{\theta} \sin\hat{\psi}}{\cos\hat{\psi} + \Delta\phi \sin\hat{\psi} \sin\hat{\theta}} \quad (19)$$

Exerting equation (19) into equation (18), eliminating third-order (Δ^3) terms, and performing some algebraic works yields:

$$a(\Delta\phi)^2 + b(\Delta\phi) + c = 0 \quad (20)$$

$$\begin{aligned} a &= \sin\hat{\theta} \cos\hat{\theta} \sin\hat{\psi} (\sin\hat{\psi} \varepsilon\psi - \cos\hat{\psi}) \\ b &= -\cos\hat{\theta} + \sin\hat{\theta} (\cos\hat{\psi} \varepsilon\theta - \sin\hat{\psi} \varepsilon\phi) \\ c &= -\sin\hat{\psi} \varepsilon\theta - \cos\hat{\psi} (\varepsilon\phi + \varepsilon\theta \varepsilon\psi) \end{aligned} \quad (21)$$

Equation (20) has two solutions:

$$\Delta\phi = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad (22)$$

$$\Delta\phi = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \quad (23)$$

Equations (3, 17, 19, and 23) show that Euler angles can be corrected by removing attitude errors as follows:

$$\begin{aligned} \psi &= \hat{\psi} - \Delta\psi \\ \theta &= \hat{\theta} - \Delta\theta \\ \phi &= \hat{\phi} - \Delta\phi \end{aligned} \quad (24)$$

Using equation (24) for attitude correction leads to preserving the normality and orthogonality characteristics of the matrix of transformation. It should be noted that the derived formulas are used for attitude resetting, not for attitude propagation which may be done by the Rodrigues formulas, quaternions or etc.

4- Simulation

In order to investigate the performance of the proposed attitude correction algorithms, some simulations are performed. As a result, the dominant performance of the introduced algorithm in equation (20) over the other algorithm based on equation (5) is proved.

In the simulation scenario, 50 uniformly distributed random groups of Euler angle errors ($\Delta\varphi, \Delta\theta, \Delta\psi$) and random Euler angles ((φ, θ, ψ)) are considered. These samples and their range are reported in Table 1 and equation (25), respectively. In the next step, small angles of rotation ($(\varepsilon\varphi, \varepsilon\theta, \varepsilon\psi)$) are determined using equation (12). It may also be noted that an estimation filter calculates these small rotation angles in real applications.

$$\begin{cases} \phi \in [-180 \ 180]^\circ \\ \theta \in [-90 \ 90]^\circ \\ \psi \in [-180 \ 180]^\circ \end{cases}, \begin{cases} \hat{\phi} = \phi + \Delta\phi \\ \hat{\theta} = \theta + \Delta\theta \\ \hat{\psi} = \psi + \Delta\psi \end{cases}, \begin{cases} \Delta\phi \in [-30 \ 30]^\circ \\ \Delta\theta \in [-30 \ 30]^\circ \\ \Delta\psi \in [-30 \ 30]^\circ \end{cases} \quad (25)$$

In Table 2, the results of three attitude correction methods are reported. In the first method, which is the typical attitude correction [15-18], the matrix of transformation $[T]^{BN}$ is determined employing equation (5). According to appendix C, normality and orthogonality conditions are applied to $[T]^{BN}$, then Euler angles are corrected using equation (6). Finally, the conventional method error would be:

$$\begin{aligned} [T]^{BN} &\equiv [t_{ij}] \\ \hat{\Delta}\theta &= \arcsin(-t_{13}) - \theta \\ \hat{\Delta}\phi &= \arccos(t_{33}/\cos\theta)\text{sign}(t_{23}) - \phi \\ \hat{\Delta}\psi &= \arccos(t_{11}/\cos\theta)\text{sign}(t_{12}) - \psi \end{aligned} \quad (26)$$

The second method is based on the first-order approximation of rotation angles error, and Columns 5 to 7 of Table 2 belong to this algorithm. Using this algorithm, computing $\Delta\varphi, \Delta\theta, \Delta\psi$ by equation (13) leads to the correction of attitude angles through equation (24). Finally, first-order method error would be:

$$\begin{aligned} \hat{\Delta}\phi &= (\hat{\phi} - \Delta\phi) - \phi \\ \hat{\Delta}\theta &= (\hat{\theta} - \Delta\theta) - \theta \\ \hat{\Delta}\psi &= (\hat{\psi} - \Delta\psi) - \psi \end{aligned} \quad (27)$$

The third method is based on the second-order approximation of rotation angles error, introduced in equations (17), (19), and (23). Columns 8 to 9 of Table 2 report the results of this algorithm. This table reveals that attitude correction using the lowest error among other discussed methods belong to the second-order approximation technique. Equation (27) calculates the method error.

In order to compare the performance of the algorithms, some error definitions are used in Table 3. These errors are defined as:

$$e_1 = \left\| \begin{bmatrix} \hat{\Delta}\phi & \hat{\Delta}\theta & \hat{\Delta}\psi \end{bmatrix} \right\|_2 \quad (28)$$

$$e_2 = \left\| I - [\bar{T}]^{BN} [T]^{BN} \right\|_{\text{Frobenius}} \quad (29)$$

$$e_3 = \left| 1 - [\bar{T}]_1^{BN} [T]_1^{BN} \right| + \left| 1 - [\bar{T}]_2^{BN} [T]_2^{BN} \right| + \left| 1 - [\bar{T}]_3^{BN} [T]_3^{BN} \right| \quad (30)$$

where e_1 represents error magnitude in corrected attitude angles, e_2 is orthogonality index, and e_3 evaluates normality characteristics. The term $[T]_i^{BN}$ ($i = 1, 2, 3$) represents the i^{th} column of the corrected matrix of transformation $[T]^{BN}$ in equation (30).

Table 3, is constructed by the data given in Table 1 and Table 2. Attitude correction errors of the algorithms (e_1) are illustrated in the following figures:

Figures (1) and (2) reveal that the least numerical error belongs to the method with second-order approximated formula. Additionally, based on Table 3, the first-order and the second-order approximation formulas do not induce normality and orthogonality errors (i.e. e_2 and e_3) to the matrix of transformation.

Table 1. Random scenarios Description

Scenario Number	true Euler angles (in deg)			computed Euler angles (in deg)			small angles of rotation (in deg)		
	ϕ	θ	ψ	$\hat{\phi}$	$\hat{\theta}$	$\hat{\psi}$	$\varepsilon\phi$	$\varepsilon\theta$	$\varepsilon\psi$
1	-146.040	-14.061	29.510	-150.470	-14.238	19.991	3.630	1.597	10.463
2	-117.820	-37.852	-57.367	-123.740	-33.899	-59.382	-0.547	-6.100	5.422
3	-23.880	-10.134	59.545	-21.552	-16.270	53.050	-6.419	1.842	6.017
4	159.230	10.623	-82.694	168.420	14.207	-78.405	-4.811	8.124	-1.932
5	160.380	19.175	98.570	170.120	14.550	89.763	-2.913	-9.129	11.136
6	64.762	-4.387	97.450	59.027	-6.784	95.789	-3.135	5.441	2.356
7	-9.909	-83.719	-41.908	-8.800	-74.202	-45.241	-6.550	-6.585	1.830
8	-13.597	-16.365	123.980	-13.119	-6.541	130.500	8.372	6.000	-6.951
9	-8.386	-21.436	52.392	-15.252	-31.139	60.630	-4.174	10.195	-4.697
10	-157.630	7.653	-10.179	-165.440	15.445	-7.018	5.999	-8.514	-4.189
11	173.720	-39.119	-22.277	166.370	-44.044	-20.102	6.820	2.427	2.541
12	120.650	58.349	16.112	121.540	52.423	22.779	-2.161	5.276	-5.793
13	-57.638	39.553	-15.124	-49.754	40.093	-17.100	-6.014	1.523	7.053
14	-79.195	-76.653	99.653	-80.796	-69.359	97.241	7.069	1.284	3.832
15	-7.373	17.383	-93.124	-1.016	26.077	-85.153	-8.432	5.314	-5.111
16	-47.914	1.324	117.460	-52.124	2.020	111.990	-1.317	4.158	5.370
17	-154.530	-73.875	-154.520	-155.670	-81.408	-156.330	3.078	-7.003	3.082
18	-151.580	-63.664	-163.990	-148.210	-64.533	-159.180	1.640	-0.245	-7.806
19	46.072	12.702	71.330	53.580	15.949	78.885	0.869	-7.767	-5.571
20	-154.610	-63.842	-11.629	-151.180	-67.682	-7.344	-0.492	3.957	-7.352
21	77.008	5.527	54.567	76.965	5.287	54.663	-0.171	0.174	-0.100
22	-137.060	-19.708	156.080	-130.280	-16.010	150.890	7.394	0.219	3.066
23	93.203	-50.700	168.310	93.404	-52.641	169.990	-0.274	-1.933	-1.843
24	87.247	26.944	-1.559	93.766	18.736	-5.520	-5.940	8.805	6.845
25	-114.790	49.004	-39.528	-107.260	57.107	-47.341	-8.413	-1.668	13.410
26	-165.800	64.837	178.930	-165.580	74.790	179.820	0.243	9.902	-0.697
27	174.540	-76.501	133.720	173.170	-75.215	142.170	0.684	1.214	-7.112
28	-174.110	39.289	-58.052	-173.180	47.742	-60.570	-7.481	-3.510	2.894
29	-68.388	-0.616	-9.349	-68.117	9.024	-0.332	-1.823	-9.593	-8.987
30	-26.477	-7.860	176.350	-19.822	-13.995	170.610	6.037	-7.023	4.784
31	117.810	5.295	81.136	112.080	6.334	81.374	1.930	5.485	-0.907
32	19.800	23.124	-33.552	9.820	16.508	-41.000	11.402	-0.838	4.052
33	-121.770	-2.050	-29.298	-129.370	-11.691	-25.961	11.181	5.218	-2.882
34	-167.930	84.702	161.860	-173.080	91.956	161.740	2.417	6.999	-5.130
35	46.172	55.819	119.670	54.260	53.414	109.710	0.533	-5.161	16.280

36	47.240	-21.569	64.566	56.874	-20.141	59.431	-2.829	-8.536	1.464
37	65.320	-38.547	125.190	57.000	-28.808	122.350	3.466	10.785	6.962
38	-148.780	55.420	-89.390	-144.000	61.432	-88.765	-6.104	2.580	3.564
39	-53.564	-11.055	-128.280	-60.648	-9.307	-128.710	-5.627	-4.274	1.460
40	-120.670	24.955	145.110	-115.010	33.587	146.430	8.838	4.214	1.110
41	-85.806	-6.013	66.476	-89.420	-11.038	67.861	-3.197	5.210	-0.698
42	105.460	68.569	147.070	105.810	62.146	155.580	-3.347	-5.890	-8.290
43	58.701	73.546	45.468	65.053	78.347	43.252	2.568	-4.651	8.427
44	151.200	-73.871	35.923	143.580	-73.330	26.995	2.001	0.357	16.007
45	-91.614	55.684	-42.909	-94.051	59.937	-35.220	-2.014	-4.243	-9.751
46	95.673	-43.992	-89.887	98.079	-40.286	-98.890	-3.680	2.285	7.422
47	-41.559	84.952	88.313	-35.437	92.501	83.517	7.518	-1.381	10.860
48	23.008	-72.809	-13.919	20.109	-72.175	-20.171	0.714	-0.910	8.984
49	5.647	40.360	-66.308	9.209	34.450	-60.495	4.169	5.290	-3.813
50	63.928	53.609	-160.210	72.096	45.440	-151.290	8.020	-5.036	-2.292

Table 2. Error comparison of three different attitude correction algorithms

Sc.	<u>Conventional algorithm</u>			<u>1st order approximation algorithm</u>			<u>2nd order approximation algorithm</u>		
No.	$ \hat{\Delta}\phi^\circ $	$ \hat{\Delta}\theta^\circ $	$ \hat{\Delta}\psi^\circ $	$ \hat{\Delta}\phi^\circ $	$ \hat{\Delta}\theta^\circ $	$ \hat{\Delta}\psi^\circ $	$ \hat{\Delta}\phi^\circ $	$ \hat{\Delta}\theta^\circ $	$ \hat{\Delta}\psi^\circ $
1	0.394	0.249	0.139	0.352	0.083	10.524	0.098	0.008	0.112
2	0.081	0.316	0.141	0.063	0.375	5.355	0.146	0.108	0.067
3	0.251	0.175	0.043	0.157	0.101	5.799	0.026	0.042	0.081
4	0.198	0.224	0.430	0.019	0.504	2.030	0.010	0.176	0.123
5	0.561	1.135	0.458	0.289	1.751	11.180	0.078	0.026	0.097
6	0.036	0.073	0.129	0.035	0.173	2.343	0.009	0.027	0.021
7	4.897	0.444	2.039	1.343	0.229	3.557	0.393	0.083	0.798
8	0.090	0.299	0.312	0.402	0.439	6.627	0.065	0.114	0.349
9	0.272	0.721	0.539	1.123	1.065	4.107	0.165	0.135	0.306
10	0.504	0.331	0.376	0.551	0.076	5.093	0.224	0.148	0.030
11	0.150	0.207	0.158	0.401	0.302	3.212	0.180	0.202	0.124
12	0.428	0.277	0.620	0.971	0.225	6.732	0.061	0.020	0.174
13	0.142	0.330	0.138	0.215	0.227	7.192	0.156	0.233	0.071
14	1.027	0.189	0.776	0.515	0.119	3.429	0.008	0.040	0.072
15	0.305	0.462	0.727	0.331	0.742	5.031	0.054	0.325	0.095
16	0.095	0.154	0.009	0.141	0.361	5.314	0.010	0.010	0.043
17	0.211	0.090	0.792	1.186	0.117	1.758	0.057	0.032	0.115
18	0.001	0.123	0.094	0.002	0.057	7.847	0.033	0.034	0.053

19	0.068	0.434	0.159	0.244	0.898	5.425	0.070	0.040	0.044
20	0.648	0.048	0.542	0.806	0.022	6.706	0.056	0.042	0.035
21	0.000	0.000	0.000	0.000	0.000	0.100	0.000	0.000	0.000
22	0.004	0.221	0.076	0.176	0.091	3.369	0.132	0.047	0.028
23	0.047	0.008	0.040	0.092	0.010	1.769	0.000	0.001	0.005
24	0.355	0.088	0.906	0.618	0.015	6.254	0.157	0.085	0.024
25	0.533	1.222	1.217	0.712	0.786	14.731	0.219	0.090	0.637
26	0.287	0.105	2.625	0.589	0.050	0.113	0.006	0.050	0.000
27	0.168	0.066	0.173	0.570	0.093	7.680	0.040	0.004	0.026
28	0.037	0.282	0.382	0.010	0.214	3.200	0.019	0.050	0.277
29	0.709	0.105	0.274	1.518	0.037	8.736	0.012	0.046	0.032
30	0.198	0.121	0.018	0.664	0.191	3.966	0.131	0.091	0.024
31	0.011	0.027	0.089	0.013	0.046	0.872	0.002	0.026	0.040
32	0.162	0.474	0.105	0.431	0.232	4.735	0.378	0.002	0.063
33	0.208	0.057	0.433	0.327	0.055	1.729	0.097	0.135	0.080
34	2.037	0.181	5.284	2.163	0.149	2.856	0.100	0.043	0.008
35	0.006	0.791	0.810	0.365	1.166	16.743	0.349	0.516	0.469
36	0.241	0.090	0.313	0.274	0.477	1.911	0.208	0.162	0.229
37	0.014	1.055	0.604	0.038	1.040	6.829	0.203	0.305	0.335
38	0.269	0.028	0.145	0.893	0.034	4.354	0.009	0.169	0.003
39	0.164	0.024	0.201	0.138	0.031	1.553	0.100	0.026	0.181
40	0.095	0.230	0.342	0.380	0.234	2.024	0.052	0.102	0.241
41	0.021	0.078	0.156	0.075	0.100	0.679	0.025	0.007	0.044
42	0.644	0.239	0.509	1.663	0.323	9.676	0.100	0.030	0.192
43	0.558	0.282	0.590	0.165	0.348	8.600	0.283	0.033	0.320
44	1.152	0.193	0.438	0.843	0.050	15.423	0.579	0.053	0.315
45	0.354	0.255	0.253	0.836	0.374	9.074	0.031	0.019	0.039
46	0.281	0.100	0.167	0.191	0.283	7.571	0.013	0.081	0.014
47	10.077	0.223	7.228	18.114	0.076	7.185	0.042	0.047	0.094
48	0.237	0.019	0.361	0.314	0.026	9.312	0.022	0.018	0.058
49	0.057	0.140	0.178	0.469	0.323	4.063	0.061	0.080	0.123
50	0.278	0.339	0.752	1.591	0.100	4.234	0.533	0.078	0.594
average	0.591	0.267	0.666	0.868	0.296	5.572	0.116	0.085	0.146

Table 3. Performance comparison of three different Euler angles correction methods

Sc. No.	<u>Conventional method</u>			<u>1st order approximation</u>			<u>2nd order approximation method</u>		
	e_1°	e_2	e_3	e_1°	e_2	e_3	e_1°	e_2	e_3
1	0.486	0.002	0.001	10.530	0.000	0.000	0.149	0.000	0.000
2	0.356	0.000	0.000	5.369	0.000	0.000	0.193	0.000	0.000
3	0.309	0.001	0.000	5.802	0.000	0.000	0.095	0.000	0.000
4	0.524	0.001	0.001	2.092	0.000	0.000	0.216	0.000	0.000
5	1.347	0.004	0.003	11.320	0.000	0.000	0.128	0.000	0.000
6	0.153	0.000	0.000	2.350	0.000	0.000	0.035	0.000	0.000
7	5.323	0.001	0.001	3.809	0.000	0.000	0.893	0.000	0.000
8	0.441	0.002	0.002	6.653	0.000	0.000	0.373	0.000	0.000
9	0.941	0.002	0.001	4.389	0.000	0.000	0.373	0.000	0.000
10	0.710	0.002	0.001	5.123	0.000	0.000	0.270	0.000	0.000
11	0.300	0.000	0.000	3.251	0.000	0.000	0.298	0.000	0.000
12	0.803	0.000	0.000	6.806	0.000	0.000	0.186	0.000	0.000
13	0.385	0.001	0.001	7.199	0.000	0.000	0.289	0.000	0.000
14	1.301	0.000	0.000	3.469	0.000	0.000	0.083	0.000	0.000
15	0.914	0.002	0.001	5.096	0.000	0.000	0.343	0.000	0.000
16	0.181	0.000	0.000	5.328	0.000	0.000	0.045	0.000	0.000
17	0.825	0.000	0.000	2.124	0.000	0.000	0.132	0.000	0.000
18	0.155	0.000	0.000	7.847	0.000	0.000	0.071	0.000	0.000
19	0.468	0.001	0.001	5.504	0.000	0.000	0.092	0.000	0.000
20	0.846	0.000	0.000	6.754	0.000	0.000	0.078	0.000	0.000
21	0.000	0.000	0.000	0.100	0.000	0.000	0.000	0.000	0.000
22	0.234	0.000	0.000	3.375	0.000	0.000	0.143	0.000	0.000
23	0.062	0.000	0.000	1.771	0.000	0.000	0.005	0.000	0.000
24	0.977	0.002	0.002	6.284	0.000	0.000	0.180	0.000	0.000
25	1.804	0.006	0.004	14.769	0.000	0.000	0.680	0.000	0.000
26	2.643	0.001	0.001	0.601	0.000	0.000	0.051	0.000	0.000
27	0.250	0.000	0.000	7.702	0.000	0.000	0.048	0.000	0.000
28	0.476	0.001	0.000	3.207	0.000	0.000	0.282	0.000	0.000
29	0.767	0.003	0.002	8.867	0.000	0.000	0.057	0.000	0.000
30	0.233	0.001	0.001	4.026	0.000	0.000	0.161	0.000	0.000
31	0.093	0.000	0.000	0.873	0.000	0.000	0.048	0.000	0.000
32	0.512	0.002	0.001	4.761	0.000	0.000	0.383	0.000	0.000
33	0.483	0.002	0.002	1.761	0.000	0.000	0.185	0.000	0.000
34	5.666	0.001	0.000	3.585	0.000	0.000	0.109	0.000	0.000
35	1.132	0.008	0.006	16.788	0.000	0.000	0.780	0.000	0.000

36	0.405	0.001	0.000	1.989	0.000	0.000	0.349	0.000	0.000
37	1.215	0.003	0.002	6.908	0.000	0.000	0.496	0.000	0.000
38	0.307	0.000	0.000	4.444	0.000	0.000	0.170	0.000	0.000
39	0.261	0.000	0.000	1.560	0.000	0.000	0.208	0.000	0.000
40	0.423	0.001	0.001	2.073	0.000	0.000	0.267	0.000	0.000
41	0.175	0.000	0.000	0.690	0.000	0.000	0.051	0.000	0.000
42	0.855	0.001	0.001	9.823	0.000	0.000	0.218	0.000	0.000
43	0.859	0.001	0.001	8.608	0.000	0.000	0.429	0.000	0.000
44	1.248	0.006	0.005	15.446	0.000	0.000	0.661	0.000	0.000
45	0.504	0.001	0.001	9.120	0.000	0.000	0.054	0.000	0.000
46	0.342	0.001	0.000	7.579	0.000	0.000	0.084	0.000	0.000
47	12.404	0.003	0.002	19.487	0.000	0.000	0.113	0.000	0.000
48	0.432	0.001	0.000	9.317	0.000	0.000	0.064	0.000	0.000
49	0.233	0.000	0.000	4.103	0.000	0.000	0.159	0.000	0.000
50	0.870	0.001	0.001	4.524	0.000	0.000	0.802	0.000	0.000
average	1.053	0.001	0.001	5.899	0.000	0.000	0.232	0.000	0.000
Maximum	12.404	0.008	0.006	19.487	0.000	0.000	0.893	0.000	0.000

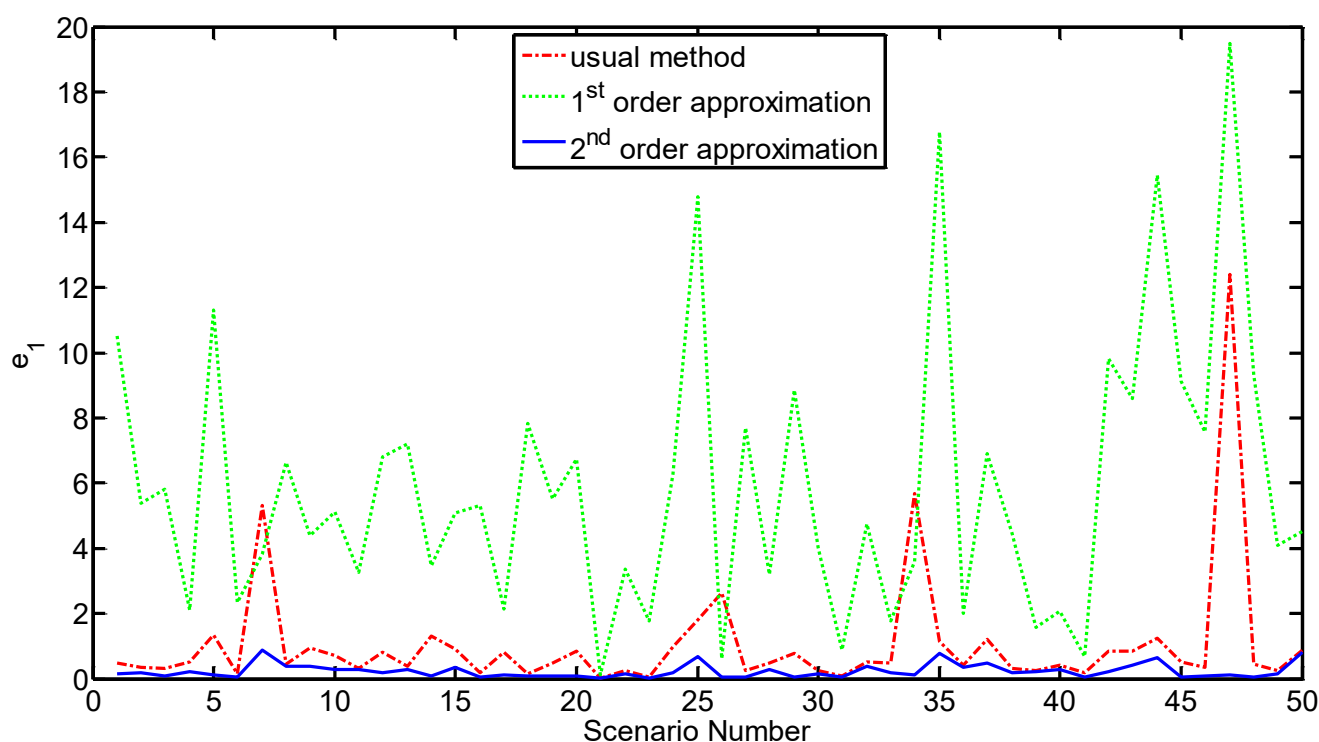


Fig. 1. Error amplitude Comparison in three different attitude resetting methods

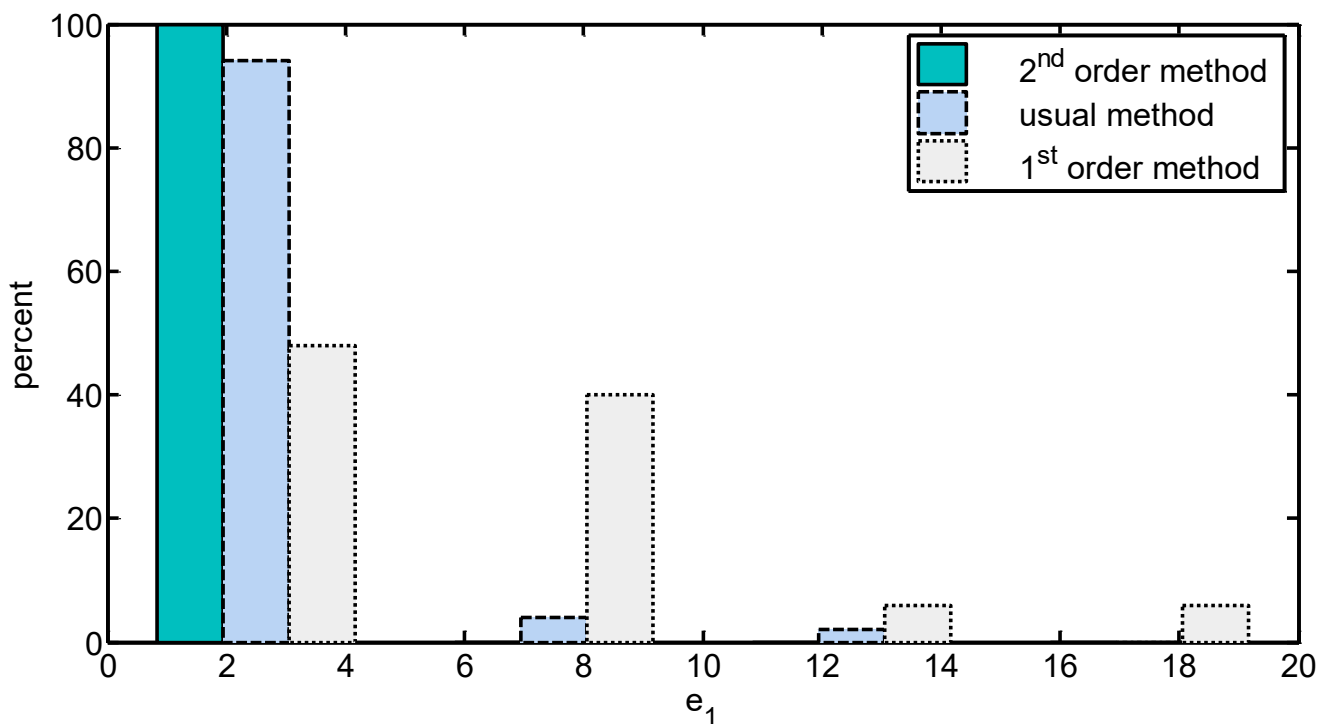


Fig. 2. Error histogram in three different attitude resetting methods

5- Conclusion

Euler angles error was obtained in terms of small rotation angles and vice versa. These equations revealed that small angles of rotation and Euler angles error relation are nonlinear. Thus, a first-order approximation was made to remove the nonlinearity relation between small angles of rotation and Euler angles error. An analytical attitude correction algorithm was found on this approximation. Numerical simulations proved that the proposed method for attitude correction is more accurate than the conventional method. Another advantage of the proposed method appears when a significant error of attitude occurs. Unlike the conventional method, the new algorithm preserves normality and orthogonality characteristics of the matrix of transformation in this situation.

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