Algebraic Structures and Their Applications Vol. 11 No. 1 (2024) pp 55-61.

Research Paper

ON HIGHER ORDER z-IDEALS AND z°-IDEALS IN COMMUTATIVE RINGS

ROSTAM MOHAMADIAN*

Abstract. A ring R is called radically z-covered (resp. radically z°-covered) if every $\sqrt{z}$-ideal (resp. $\sqrt{z^\circ}$-ideal) in R is a higher order z-ideal (resp. z°-ideal). In this article we show with a counter-example that a ring may not be radically z-covered (resp. radically z°-covered). Also a ring R is called z°-terminating if there is a positive integer n such that for every $m \geq n$, each $z^{\circ m}$-ideal is a $z^{\circ n}$-ideal. We show with a counter-example that a ring may not be z°-terminating. It is well known that whenever a ring homomorphism $\varphi : R \to S$ is strong (meaning that it is surjective and for every minimal prime ideal P of R, there is a minimal prime ideal Q of S such that $\varphi^{-1}[Q] = P$), and if R is a z°-terminating ring or radically z°-covered ring then so is S. We prove that a surjective ring homomorphism $\varphi : R \to S$ is strong if and only if $\ker(\varphi) \subseteq \mathrm{rad}(R)$.

DOI: 10.22034/as.2023.18637.1553

MSC(2010): Primary: 13A15, 13B30; Secondary: 13F10.

Keywords: Radically z-covered, Radically z°-covered, z^n-ideal, $z^{\circ n}$-ideal, z°-terminating.

Received: 22 June 2022, Accepted: 22 May 2023

*Corresponding author

1. Introduction

Throughout this paper R is a commutative ring with $1 \neq 0$. For any $a \in R$, we denote by $\mathcal{M}(a)$ (resp. $\mathcal{P}(a)$) the set of all maximal (resp. minimal prime) ideals of R containing a. An ideal I of a ring R is a z-ideal (resp. z°-ideal) if $\mathcal{M}(b) = \mathcal{M}(a)$ (resp. $\mathcal{P}(b) = \mathcal{P}(a)$) and $b \in I$, imply $a \in I$, for any $a, b \in R$. For each $a \in R$, $M(a)$ (resp. $P(a)$) is the intersection of all maximal (resp. minimal prime) ideals containing a. We use $\mathrm{Jac}(R)$ (resp. $\mathrm{rad}(R)$) instead of $M(0)$ (resp. $P(0)$). For a ring R the set of all minimal prime ideals of R is denoted by $\mathrm{Min}(R)$. It is well-known that every maximal (resp. minimal prime) ideal is a z-ideal (resp. z°-ideal).

We consider X to be a completely regular Hausdorff space and we denote by $C(X)$ the ring of all real-valued continuous functions on the space X. Concerning topological spaces and $C(X)$ the reader is referred to [8] and [9] respectively.

For more information about algebraic concepts see [2] and [11], z-ideals and z°-ideals in commutative rings see [12] and [4] and about z-ideals and z°-ideals in $C(X)$ see [3] and [5].

Let $n \in \mathbb{N}$. An ideal I of a ring R is a z^n-ideal (resp. $z^{\circ n}$-ideal) if $\mathcal{M}(a) = \mathcal{M}(b)$ (resp. $\mathcal{P}(a) = \mathcal{P}(b)$) and $a^n \in I$, imply $b^n \in I$, for any $a, b \in R$. The set of all z^n-ideals (resp. $z^{\circ n}$-ideals) of R denotes by $\mathcal{Z}^n(R)$ (resp. $\mathcal{Z}^{\circ n}(R)$). In particular $\mathcal{Z}(R)$ (resp. $\mathcal{Z}^{\circ}(R)$ denotes the set of all z-ideals (resp. z°-ideals) of R. For more information and details about z^n-ideals and $z^{\circ n}$-ideals, see [7], [14], respectively.

In Lemma 1 of [6] the z^n-ideals of a PID are characterized. In the next proposition we identify the z^n-ideals in $\mathbb{Z}$ by a preliminary method. Recall that maximal ideals of $\mathbb{Z}$ are exactly the principal ideals (p), for p a prime number. Thus if $a, b \in \mathbb{N}$ and $\mathcal{M}(a) = \mathcal{M}(b)$, then a and b are divisible by exactly the same prime numbers.

Proposition 1.1. *Let $n \in \mathbb{N}$. The ideal $I = (k)$ in $\mathbb{Z}$ is a z^n-ideal if and only if $k = p_1^{r_1} p_2^{r_2} ... p_t^{r_t}$ where $p_i^{,}s$ are distinct prime numbers and $1 \leq r_i \leq n$ for any $i = 1, \cdots, t$.*

Proof. ($\Leftarrow$) Suppose that $\mathcal{M}(a) = \mathcal{M}(b)$ and $a^n \in I$. Hence there exists $s \in \mathbb{Z}$ such that $a^n = p_1^{r_1} p_2^{r_2} ... p_t^{r_t} s$. Since $p_1 | a$ we infer that $p_1 | b$ and so $b = p_1 s_1$ for an $s_1 \in \mathbb{Z}$. Similarly, $p_2 | a$ and hence $p_2 | b$, therefore $b = p_2 s_2$, for an $s_2 \in \mathbb{Z}$. Now $p_2 | p_1 s_1$ and $(p_2, p_1) = 1$ implies that $p_2 | s_1$ and hence $s_1 = p_2 t_1$ for a $t_1 \in \mathbb{Z}$. This implies that $b = p_1 p_2 t_1$. Also $p_3 | a$ and so $p_3 | b$, hence there exists $s_3 \in \mathbb{Z}$ such that $b = p_3 s_3$. Now $p_3 | p_1 p_2 t_1$ and $(p_3, p_1 p_2) = 1$. Therefore $p_3 | t_1$ and so $t_1 = p_3 t_2$ for a $t_2 \in \mathbb{Z}$. It implies that $b = p_1 p_2 p_3 t_2$. By continuing this process there exists $s_0 \in \mathbb{Z}$ such that $b = p_1 p_2 ... p_t s_0$. Therefore $b^n = p_1^{r_1} p_2^{r_2} ... p_t^{r_t} u$ where $u = p_1^{n-r_1} p_2^{n-r_2} ... p_t^{n-r_t} s_0^n$. This consequence that $b^n \in I$ and we are done.
($\Rightarrow$) On the contrary and without loss of generality suppose that there exists $1 \leq i \leq t$ such that $r_i > n$ and $1 \leq r_j \leq n$ for any $j \neq i$. We consider $s \leq r_i$ such that $sn \geq r_i$.

We put $a = p_1 ... p_i ... p_t$ and $b = p_1 ... p_i^s ... p_t$. One can easily show that $\mathcal{M}(a) = \mathcal{M}(b)$ and $b^n = p_1^n ... p_i^{ns} ... p_t^n \in I$ while $a^n \notin I$ and it is a contradicts to assumption. □

We deduce the following result immediately. See also Corollary 1 of [6].

Corollary 1.2. *The ideal $I = (k)$ is a z-ideal in $\mathbb{Z}$ if and only if $k = p_1 p_2 ... p_t$ where $p_i^{,}s$ are distinct prime numbers.*

2. Radically z-covered and radically z°-covered

An ideal I of a ring R is said to be $\sqrt{z}$-ideal (resp. $\sqrt{z^\circ}$-ideal) if $\sqrt{I}$ is a z-ideal (resp. z°-ideal), see [5]. The set of all $\sqrt{z}$-ideals (resp. $\sqrt{z^\circ}$-ideals) of R is denoted by $\mathcal{Z}^{\mathrm{rad}}(R)$ (resp. $\mathcal{Z}^{\circ\mathrm{rad}}(R)$). Also an ideal I of a ring R is called higher order z-ideal (resp. z°-ideal) if there exist $n \in \mathbb{N}$ such that $I \in \mathcal{Z}^n(R)$ (resp. $I \in \mathcal{Z}^{\circ n}(R)$). A ring R is called radically z-covered (resp. radically z°-covered) if every $\sqrt{z}$-ideal (resp. $\sqrt{z^\circ}$-ideal) in R is a higher order z-ideal (resp. z°-ideal), see [7] and [14] for details.

It seems that an example of a non radically z-covered ring is essential which is not given in [7]. As a matter of fact we must show that there is an ideal I of a ring R such that $\sqrt{I}$ is a z-ideal but I is not a z^n-ideal for every $n \in \mathbb{N}$. See the following example for this purpose.

Example 2.1. Let F be a field and put $R = F[x_1, x_2, x_3, \cdots]$. Suppose that $I = (x_1, x_2^2, x_3^4, x_4^6, \cdots, x_{n+1}^{2n}, \cdots)$. It is clear that $\sqrt{I} = (x_1, x_2, x_3, \cdots)$ is a maximal ideal of R and hence it is a z-ideal of R, that is, $I \in \mathcal{Z}^{\mathrm{rad}}(R)$. One can easily see that $\mathcal{M}(x_{n+1}) = \mathcal{M}(x_{n+1}^2)$, for $n = 1, 2, \cdots$ and $(x_{n+1}^2)^n \in I$ while $(x_{n+1})^n \notin I$. This shows that I is not a z^n-ideal for any $n \in \mathbb{N}$ and consequently R is not radically z-covered.

Every z^n-ideal is a z^{n+1}-ideal, for any $n \in \mathbb{N}$, but the converse is not true, see Example 5 of [7].

Proposition 2.2. $\mathrm{rad}(R) = \mathrm{Jac}(R)$ *if and only if every $z^{\circ n}$-ideal is a z^n-ideal, for an $n \in \mathbb{N}$.*

Proof. ($\Leftarrow$) Similar to Proposition 1.3 in [13].

($\Rightarrow$) Suppose that $\mathcal{M}(a) = \mathcal{M}(b)$ and $a^n \in I$. We claim that $\mathcal{P}(a) = \mathcal{P}(b)$. To see this, let $P \in \mathcal{P}(a)$. Hence $a \in P$ and there is $c \notin P$ such that $ac \in \mathrm{rad}(R) = \mathrm{Jac}(R)$. Therefore $M(a) \cap M(c) = M(ac) \subseteq \mathrm{Jac}(R) = \mathrm{rad}(R) \subseteq P$. This implies that $M(a) \subseteq P$. Since $\mathcal{M}(a) = \mathcal{M}(b)$ we infer that $M(a) = M(b)$. Hence $M(b) \subseteq P$ and so $b \in P$. Thus $P \in \mathcal{P}(b)$, that is $\mathcal{P}(a) \subseteq \mathcal{P}(b)$. Similarly, $\mathcal{P}(b) \subseteq \mathcal{P}(a)$ and hence $\mathcal{P}(a) = \mathcal{P}(b)$. Since I is a $z^{\circ n}$-ideal we conclude that $b^n \in I$ and we are done. □

In $C(X)$ if $\sqrt{I}$ is a z°-ideal then so is I, see Proposition 3.4 in [5], therefore $C(X)$ is radically z°-covered.

It seems that an example of a non radically z°-covered ring is essential which is not given in [14]. The following example shows that a ring may not be radically z°-covered.

Example 2.3. Let F be a field and put $S = F[x_1, x_2, x_3, \cdots]$. Suppose that $I = (x_1^2, x_2^4, x_3^6, \cdots, x_n^{2n}, \cdots)$ and $J = (x_1, x_2^2, x_3^3, \cdots, x_n^n, \cdots)$. Now assume that $R = \frac{S}{I}$ and $K = \frac{J}{I}$. It is clear that $\sqrt{K} = \frac{(x_1, x_2, x_3, \cdots)}{I}$ is a minimal prime ideal of R and hence it is a z°-ideal of R, that is, $K \in \mathcal{Z}^{\circ\text{rad}}(R)$. We claim that K is not a $z^{\circ n}$-ideal for any $n \in \mathbb{N}$. To see this we observe that $\mathcal{P}(x_{n+1} + I) = \mathcal{P}(x_{n+1}^2 + I)$, for $n = 1, 2, \cdots$ and $(x_{n+1}^2 + I)^n \in K$ while $(x_{n+1} + I)^n \notin K$. This shows that K is not a $z^{\circ n}$-ideal for any $n \in \mathbb{N}$ and consequently R is not radically z°-covered.

3. z°-TERMINATING

Every $z^{\circ n}$-ideal is a $z^{\circ n+1}$-ideal, for any $n \in \mathbb{N}$. Hence we have the ascending chain $\mathcal{Z}^\circ(R) \subseteq \mathcal{Z}^{\circ 2}(R) \subseteq \mathcal{Z}^{\circ 3}(R) \subseteq \cdots$ of collections of ideals of R. We call it z°-tower of R. If there is a positive integer k such that $\mathcal{Z}^{\circ k}(R) = \mathcal{Z}^{\circ k+1}(R) = \cdots$ we say the z°-tower terminates.

Definition 3.1. ([14], Definition 4.2.8) A ring R is z°-terminating in case its z°-tower terminates.

In $C(X)$ we have $\mathcal{Z}^\circ(C(X)) = \mathcal{Z}^{\circ 2}(C(X)) = \cdots$, hence $C(X)$ is a z°-terminating ring. In $\mathbb{Z}$ for any $n \in \mathbb{N}$ we have $\mathcal{Z}^{\circ n}(\mathbb{Z}) = \{(0)\}$, so $\mathbb{Z}$ is z°-terminating.

The ring of integers is not z-terminating, see Example 5 of [7]. It seems that an example of a non z°-terminating ring is essential which is not given in [14]. The following example shows that a $z^{\circ n+1}$-ideal may not be a $z^{\circ n}$-ideal and consequence that a ring may not be z°-terminating.

Example 3.2. Let S be a reduced ring with subring $\mathbb{Z}$ and $P \neq (0)$ be a minimal prime ideal in S with $P \cap \mathbb{Z} = (0)$. By Lemma 3.6 in [5], $Q = xP[x] \subseteq S[x]$ is a minimal prime ideal in $R = \mathbb{Z} + xS[x]$ and hence it is a z°-ideal. Now we consider $Q_n = x^n P[x]$ with $1 \neq n \in \mathbb{N}$. Clearly, $\sqrt{Q_n} = Q$. We claim that $Q_{n+1} \in \mathcal{Z}^{\circ n+1}(R)$ but $Q_{n+1} \notin \mathcal{Z}^{\circ n}(R)$. For the former, suppose that $\mathcal{P}(f) = \mathcal{P}(g)$ and $f^{n+1} \in Q_{n+1}$. Hence $f \in \sqrt{Q_{n+1}} = Q$. Therefore $Q \in \mathcal{P}(f) = \mathcal{P}(g)$ implies that $g \in Q$. So there exists $h(x) \in P[x]$ such that $g(x) = xh(x)$. It implies that $g_0 = 0$, where g_0 is constant coefficient of g. Consequently, $(g(x))^{n+1} = x^{n+1} l(x)$ for an $l(x) \in P[x]$, that is, $g^{n+1} \in Q_{n+1}$. Next suppose that $0 \neq a \in P$. Put $f(x) = ax^2$ and $g(x) = ax$. Clearly, $\mathcal{P}(f) = \mathcal{P}(g)$. Now $(f(x))^n = x^{n+1} a^n x^{n-1} \in x^{n+1} P[x] = Q_{n+1}$ but $(g(x))^n = x^n a^n \notin x^{n+1} P[x] = Q_{n+1}$. This show that Q_{n+1} is not a $z^{\circ n}$-ideal.

Proposition 3.3. *([14], Theorem 4.2.11) Noetherian rings are radically z°-covered.*

If X is an infinite set then $C(X)$ is radically z°-covered ring which is not Noetherian. In Example 3.2 if S is a finitely generated $\mathbb{Z}$-module, then R is a Noetherian ring, see Proposition 2.1 in [10], so by the above proposition it is a radically z°-covered ring while is not z°-terminating.

It is well known that if $\varphi : R \to S$ is a surjective ring homomorphism then $\varphi(\mathrm{rad}(R)) \subseteq \mathrm{rad}(S)$. A ring homomorphism $\varphi : R \to S$ is strong if it is surjective and for every minimal prime ideal P of R, there is a minimal prime ideal Q of S such that $\varphi^{-1}[Q] = P$, see Definition 4.4.1 of [14].

Proposition 3.4. *Let $\varphi : R \to S$ is a strong homomorphism. Then*

(1) $\varphi(\mathrm{rad}(R)) = \mathrm{rad}(S)$.
(2) *if* $P \in \mathrm{Min}(R)$*, then* $\varphi[P] \in \mathrm{Min}(S)$.
(3) *if* $Q \in \mathrm{Min}(S)$*, then* $\varphi^{-1}[Q] \in \mathrm{Min}(R)$.

Proof. (1) It is clear.

(2) It is clear that $\varphi[P]$ is a proper prime ideal of S. We are to show that $\varphi[P] \in \mathrm{Min}(S)$. Let $y \in \varphi[P]$, hence there exists $x \in P$ such that $y = \varphi(x)$. Therefore there is $b \notin P$ such that $bx \in \mathrm{rad}(R)$. Now $\varphi(bx) = \varphi(b)\varphi(x) = \varphi(b)y \in \varphi(\mathrm{rad}(R)) = \mathrm{rad}(S)$. On the other hand $\varphi(b) \notin \varphi[P]$. Otherwise $\varphi(b) = \varphi(t)$ for a $t \in P$. Hence $b - t \in \ker(\varphi) \subseteq P$ implies that $b \in P$ which is not true. It implies that $\varphi[P]$ is a minimal prime ideal of S.

(3) Let $Q \in \mathrm{Min}(S)$ and $a \in \varphi^{-1}[Q]$. Hence $\varphi(a) \in Q$ and so there exists $y \notin Q$ such that $y\varphi(a) \in \mathrm{rad}(S)$. On the other hand, there is $x \in R$ such that $\varphi(x) = y$. Therefore $\varphi(ax) \in \mathrm{rad}(S) = \varphi(\mathrm{rad}(R))$. Thus $\varphi(ax) = \varphi(t)$ for a $t \in \mathrm{rad}(R)$. So $ax - t \in \ker(\varphi) \subseteq \mathrm{rad}(R)$ implies that $ax \in \mathrm{rad}(R)$. Furthermore since $\varphi(x) \notin Q$ we infer that $x \notin \varphi^{-1}[Q]$. It implies that $\varphi^{-1}[Q]$ is a minimal prime ideal of R. □

Proposition 3.5. *Let $\varphi : R \to S$ is a surjective ring homomorphism. Then the following statements are equivalent.*

(1) *φ is strong.*
(2) $\ker(\varphi) \subseteq \mathrm{rad}(R)$.
(3) *For any $a_1, a_2 \in R$, $\mathcal{P}(\varphi(a_2)) \subseteq \mathcal{P}(\varphi(a_1))$ implies that $\mathcal{P}(a_2) \subseteq \mathcal{P}(a_1)$.*

Proof. $(1 \Rightarrow 2)$ Let $P \in \mathrm{Min}(R)$, by hypothesis, there exists $Q \in \mathrm{Min}(S)$ such that $\varphi^{-1}[Q] = P$. Then $\ker(\varphi) \subseteq \varphi^{-1}[Q] = P$, and hence $\ker(\varphi) \subseteq \mathrm{rad}(R)$.

$(2 \Rightarrow 1)$ Let $P \in \mathrm{Min}(R)$. We will show that $P = \varphi^{-1}[\varphi[P]]$ and we conclude by Proposition 3.4. Let $a \in \varphi^{-1}[\varphi[P]]$, then $\varphi(a) \in \varphi[P]$, and so $\varphi(a) = \varphi(x)$ for some $x \in P$. It follows that $x - a \in \ker(\varphi) \subseteq P$, by hypothesis. Thus $a \in P$. The direct inclusion is clear.

$(2 \Rightarrow 3)$ Let $P \in \mathcal{P}(a_2)$, hence $a_2 \in P$. Therefore $\varphi(a_2) \in \varphi[P]$. By Proposition 3.4 we have $\varphi[P] \in \mathcal{P}(\varphi(a_2))$ and by hypothesis $\varphi[P] \in \mathcal{P}(\varphi(a_1))$, that is $\varphi(a_1) \in \varphi[P]$. Hence $\varphi(a_1) = \varphi(t)$ for a $t \in P$. This consequence $a_1 - t \in \ker(\varphi) \subseteq \mathrm{rad}(R) \subseteq P$ and so $a_1 \in P$, i.e., $P \in \mathcal{P}(a_1)$.

$(3 \Rightarrow 2)$ Suppose that $x \in \ker(\varphi)$, hence $\varphi(x) = 0$. Since $\mathcal{P}(\varphi(0)) \subseteq \mathcal{P}(\varphi(x))$ by hypothesis $\mathcal{P}(0) \subseteq \mathcal{P}(x)$. Therefore $P(x) \subseteq P(0) = \mathrm{rad}(R)$. It implies that $x \in \mathrm{rad}(R)$. □

Corollary 3.6. *([14], Lemma 4.4.6) Let $\varphi : R \to S$ be a strong homomorphism. If J is a $z^{\circ n}$-ideal of S, then $\varphi^{-1}[J]$ is a $z^{\circ n}$-ideal of R.*

Proposition 3.7. *([14], Proposition 4.4.7) Let $\varphi : R \to S$ is a strong homomorphism. If R is z°-terminating or radically z°-covered, then so is S.*

Remark 3.8. a) Let $\varphi : R \to S$ be a surjective ring homomorphism. If $\mathcal{Z}^{\mathrm{rad}}(R) = \mathcal{Z}(R)$ then $\mathcal{Z}^{\mathrm{rad}}(S) = \mathcal{Z}(S)$. Hence $\frac{C(X)}{I}$ is a radically z-covered ring, for every ideal I of $C(X)$.
b) Let $\varphi : R \to S$ is a strong homomorphism. If $\mathcal{Z}^{\circ \mathrm{rad}}(R) = \mathcal{Z}^{\circ}(R)$ then $\mathcal{Z}^{\circ \mathrm{rad}}(S) = \mathcal{Z}^{\circ}(S)$.
c) Let I be an ideal of R such that $I \subseteq \mathrm{rad}(R)$ and $\varphi : R \to \frac{R}{I}$ be a natural ring homomorphism. If R is z°-terminating (resp. radically z°-covered), then $\frac{R}{I}$ is z°-terminating (resp. radically z°-covered).
d) If $\mathrm{rad}(R)$ is contained in every higher order z°-ideal of R, then R is z°-terminating (resp. radically z°-covered) if and only if $\frac{R}{\mathrm{rad}(R)}$ has the same property.

4. Acknowledgments

The author would like to thank the referee for reading the article carefully and giving useful comments. Also I am grateful to the Research Council of Shahid Chamran University of Ahvaz financial support (GN:SCU.MM401.648).

References

[1] D. D. Anderson and J. Kintzinger, *Ideals in direct products of commutative rings*, Bull. Austr. Math. Soc., **77** (2008) 477-483.

[2] M. F. Atiyah and I. G. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesely, Reading Mass, 1969.

[3] F. Azarpanah, O. A. S. Karamzadeh and A. Rezaei Aliabad, *On z°-ideal in $C(X)$*, Fund. Math., **160** (1999) 15-25.

[4] F. Azarpanah, O. A. S. Karamzadeh and A. Rezaei Aliabad, *On ideals consisting entirely of zero divisors*, Comm. Algebra, **28** (2000) 1061-1073.

[5] F. Azarpanah and R. Mohamadian, *$\sqrt{z}$-ideals and $\sqrt{z^{\circ}}$-ideals in $C(X)$*, Acta. Math. Sinica. English Series, **23** (2007) 989-996.

[6] A. Benhissi and A. Maatallah, *A question about Higher order z-ideals in commutative rings*, Quaest. Math., **43** (2020) 1155-1157.

[7] T. Dube and O. Ighedo, *Higher order z-ideals in commutative rings*, Miskolc Math. Notes, **17** No. 1 (2016) 171-185.

[8] R. Engelking, *General Topology*, PWN-polish Science Publication, Warsaw, Poland, 1977.

[9] L. Gillman and M. Jerison, *Rings of Continuous Functions*, Springer-Verlag, Berlin, Germany, 1976.

[10] S. Hizem, *Chain conditions in rings of the form $A + xB[x]$ and $A + xI[x]$*, J. Commut. Algebra, (2009) 259-274.

[11] I. Kaplansky, *Commutative Rings*, Allyn and Bacon, Boston, MA, USA, 1970.

[12] G. Mason, *z-ideals and prime ideals*, J. Algebra, **26** (1973) 280-297.

[13] A. Rezaei Aliabad and R. Mohamadian, *On z-Ideals and z°-Ideals of Power Series Rings*, J. Math. Ext., **7** (2013) 93-108.

[14] Tlharesakgosi, B. *Topics on z-ideals of Commutative Rings*, University of South Africa, Pretoria, 2017.

Rostam Mohamadian

Department of Mathematics,

Faculty of Mathematical Sciences and Computer,

Shahid Chamran University of Ahvaz, Ahvaz, Iran.

mohamadian_r@scu.ac.ir