Algebraic Structures and Their Applications Vol. 11 No. 2 (2024) pp 131-150.

Research Paper

A STUDY ON CONSTACYCLIC CODES OVER THE RING $\mathbb{Z}_4 + u\mathbb{Z}_4 + u^2\mathbb{Z}_4$

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Abstract. This paper studies λ-constacyclic codes and skew λ-constacyclic codes over the finite commutative non-chain ring $R = \mathbb{Z}_4 + u\mathbb{Z}_4 + u^2\mathbb{Z}_4$ with $u^3 = 0$ for $\lambda = (1 + 2u + 2u^2)$ and $(3 + 2u + 2u^2)$. We introduce distinct Gray maps and show that the Gray images of λ-constacyclic codes are cyclic, quasi-cyclic, and permutation equivalent to quasi-cyclic codes over $\mathbb{Z}_4$. It is also shown that the Gray images of skew λ-constacyclic codes are quasi-cyclic codes of length $2n$ and index 2 over $\mathbb{Z}_4$. Moreover, the structure of λ-constacyclic codes of odd length n over the ring R is determined and give some suitable examples.

1. Introduction

In the beginning of coding theory, the study of linear codes was within the confines of vector spaces over finite fields. After the landmark paper of Hammon et al. [9], in which certain good non-linear binary codes are constructed from cyclic codes over $\mathbb{Z}_4$ via the Gray map, there has been a paradigm shift in the studies of codes towards finite rings. Since then, many researchers

DOI: 10.22034/as.2023.3145

MSC(2010): Primary: 94B05, 94B15, 94B60.

Keywords: Constacyclic code, Cyclic code, Gray map, Quasi-cyclic code, Skew constacyclic code.

Received: 17 April 2021, Accepted: 22 July 2023.

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are interested in codes over finite rings because of their new role in algebraic coding theory and a wide range of applications in various fields. Cyclic codes are a significant class of linear codes over finite rings and have been studied by many authors in various rings [1, 2, 8, 13, 15, 19]. For instance, Özen et al. [13] studied cyclic codes over the ring $\mathbb{Z}_4 + u\mathbb{Z}_4 + u^2\mathbb{Z}_4$ with $u^3 = 0$ and obtained their generators and minimal spanning sets. By considering the Gray map, they obtained many new linear codes over $\mathbb{Z}_4$.

Constacyclic codes are a well-known generalization of cyclic codes. Much research on constacyclic codes over various rings has been done as it can be effectively implemented by shift constant. In [16], Qian et al. studied the constacyclic codes over the ring $\mathbb{F}_2 + u\mathbb{F}_2$ where $u^2 = 0$ and showed that the Gray image of $(1+u)$-constacyclic code of length n is distance invariant cyclic codes of length $2n$. Later on, many researchers have been studying constacyclic codes over other finite rings like $\mathbb{Z}_4$ and its extensions to get optimal codes. In [21], Yildiz and Aydin discussed linear codes and cyclic codes over $\mathbb{Z}_4 + u\mathbb{Z}_4,\ u^2 = 0$ and many new linear codes over $\mathbb{Z}_4$ were obtained. Later, Yu et al. [22] studied codes on the same ring and proved that $\mathbb{Z}_4$-image of a $(1+u)$-constacyclic code of length n is a cyclic code over $\mathbb{Z}_4$ of length $4n$. In fact, there is a vast literature on constacyclic codes over various finite rings, we refer to [3, 4, 5, 6, 10, 12, 14, 17], along with their references.

Recently, Islam and Prakash [11] considered the ring $\mathbb{Z}_4 + u\mathbb{Z}_4 + v\mathbb{Z}_4$, where $u^2 = v^2 = uv = vu = 0$ of order 64 and determined the generator polynomials and minimal spanning set for cyclic codes over the ring. Further, the authors proved that the Gray images of $(1+2u)$-constacyclic codes are cyclic, quasi-cyclic and permutation equivalent to a quasi-cyclic code over $\mathbb{Z}_4$. In [7], Dertli and Cengellenmis introduced the ring $\mathbb{Z}_4 + u\mathbb{Z}_4 + v\mathbb{Z}_4,\ u^2 = u, v^2 = v, uv = vu = 0$ and studied the Gray images of cyclic, constacyclic, quasi-cyclic and their skew codes over the ring. Moreover, they determined the cyclic DNA and skew cyclic DNA codes over the ring.

Indeed, Islam et al. [10] discussed the λ-constacyclic and skew λ-constacyclic codes over the ring $\mathbb{Z}_4[u]/\langle u^k\rangle$, where $u^k = 0$ with $\lambda = (1+2u^{k-1})$ and $(3+2u^{k-1})$. The authors have shown that the Gray images of λ-constacyclic and skew λ-constacyclic codes over the ring are cyclic, quasi-cyclic, permutation equivalent to a quasi-cyclic code over $\mathbb{Z}_4$. Further, they obtained the generators of the λ-constacyclic codes over the ring.

Being motivated by the above-mentioned works, we consider the commutative ring $R = \mathbb{Z}_4 + u\mathbb{Z}_4 + u^2\mathbb{Z}_4$, where $u^3 = 0$, as a particular case of [10], by taking different units $\lambda = (1+2u+2u^2)$ and $(3+2u+2u^2)$ and study the algebraic properties of the ring. In this paper, we introduce new Gray maps and study their images of λ-constacyclic codes over $\mathbb{Z}_4$ with $\lambda = (1+2u+2u^2)$ and $(3+2u+2u^2)$. The intention of this article is to establish relations among the known linear codes like cyclic, quasi-cyclic or permutation equivalent to quasi-cyclic code over $\mathbb{Z}_4$

via the newly introduced Gray maps obtained as $\mathbb{Z}_4$ -images of λ-constacyclic codes over the ring R. The presentation of this paper is organized as follows. In Section 2, we discuss some preliminary concepts of the ring R. Some new Gray maps are introduced in Section 3, and we investigate the properties of the Gray images of λ-constacyclic codes with $\lambda = (1 + 2u + 2u^2)$ and $(3 + 2u + 2u^2)$, respectively. In Section 4, we discuss skew constacyclic codes over R and obtain that some particulars $\mathbb{Z}_4$-images are quasi-cyclic codes. Furthermore, in Section 5, we determine the algebraic structures of the λ-constacyclic codes over the ring R with some suitable examples and study some results on λ-constacyclic codes with Nechaev permutation and other permutations. Section 6 concludes the paper.

2. Preliminaries

In [13], Özen et al. considered the commutative ring $R = \mathbb{Z}_4 + u\mathbb{Z}_4 + u^2\mathbb{Z}_4$ with $u^3 = 0$ and studied the cyclic codes over R. Clearly, R is isomorphic to $\mathbb{Z}_4[u]/\langle u^3\rangle$ and it has characteristic 4 and order 64. Any element x of R can be written as $x = a + ub + u^2c$, where $a, b, c \in \mathbb{Z}_4$ and x is a unit in R if only if a is a unit in $\mathbb{Z}_4$. There are 32 units and 32 non-units in R. The set of units $U = \{1, 3, 1 + 2u, 1 + 2u^2, 1 + 2u + 2u^2, 3 + 2u, 3 + 2u^2, 3 + 2u + 2u^2\}$ satisfies $\lambda^2 = 1$ for all $\lambda \in U$. The units $(1 + 2u + 2u^2)$ and $(3 + 2u + 2u^2)$ are used in the studies of this paper. The ring R has 13 ideals given by $\{\langle 0\rangle, \langle 2\rangle, \langle u\rangle, \langle 2u\rangle, \langle u^2\rangle, \langle 2u^2\rangle, \langle 2+u^2\rangle, \langle 2u+u^2\rangle, \langle 2+u\rangle, \langle 2, u\rangle, \langle 2, u^2\rangle, \langle 2, 2u^2\rangle, R\}$. It is a local ring with unique maximal ideal ring $\langle 2, u\rangle$. Also, R is not a chain ring as the ideals $\langle u^2\rangle$ and $\langle 2u\rangle$ are not comparable under the set inclusion.

We recall that a linear code C of length n over R is an R-submodule of R^n and elements of the code are called codewords. A linear code C of length n over R is said to be a cyclic code if it is invariant under the cyclic shift operator $\sigma, i.e., \sigma(C) = C$, where $\sigma(c_0, c_1, \ldots, c_{n-1}) = (c_{n-1}, c_0, \ldots, c_{n-2})$ for all $(c_0, c_1, \ldots, c_{n-1}) \in C$. Let λ be a unit in R. A linear code C of length n over R is said to be a λ-constacyclic code if it is invariant under the constacyclic shift operator τ_λ, *i.e.*, $\tau_\lambda(C) = C$, where $\tau_\lambda(c_0, c_1, \ldots, c_{n-1}) = (\lambda c_{n-1}, c_0, \ldots, c_{n-2})$ for all $(c_0, c_1, \ldots, c_{n-1}) \in C$. Moreover, a λ-constacyclic code of length n over R can be identified as an ideal of the quotient ring $R_{n,\lambda} = R[x]/\langle x^n - \lambda\rangle$ by the correspondence

$$c = (c_0, c_1, \ldots, c_{n-1}) \to c(x) = c_0 + c_1x + \cdots + c_{n-1}x^{n-1}(\mathrm{mod}\langle x^n - \lambda\rangle).$$

Definition 2.1. [11] Let σ be the cyclic shift operator and $n = ml$. Then, the quasi-cyclic shift operator ρ_l is defined by

$$\rho_l(c^1|c^2|\cdots|c^l) = (\sigma(c^1)|\sigma(c^2)|\cdots|\sigma(c^l)|),$$

where $c^i \in \mathbb{Z}_4^m$ for $i = 1, 2, \ldots, l$. A linear code C of length n over $\mathbb{Z}_4$ is said to be a quasi-cyclic code of index l if and only if $\rho_l(C) = C$.

3. Gray maps and $\mathbb{Z}_4$-images of λ-constacyclic codes

In the present section, we introduce new Gray maps and discuss some relations between the Gray images of λ-constacyclic codes with $\lambda = (1 + 2u + 2u^2)$ and $(3 + 2u + 2u^2)$ and some well-known linear codes over $\mathbb{Z}_4$. It is divided into two subsections and discussed below.

3.1. $(1 + 2u + 2u^2)$**-constacyclic codes over** R **and their** $\mathbb{Z}_4$**-images.** In this section, we consider three different Gray maps on the ring R and show that the Gray images of $(1+2u+2u^2)$-constacyclic codes are cyclic, quasi-cyclic and permutation equivalent to quasi-cyclic codes over $\mathbb{Z}_4$.

We first take a Gray map ψ_1 from R to $\mathbb{Z}_4^2$ as

$$\psi_1 : R \to \mathbb{Z}_4^2,$$

defined by

$$\psi_1(a + ub + u^2c) = (b + 2c, 2a + b + 2c) \quad \forall\ a, b, c \in \mathbb{Z}_4.$$

Clearly, ψ_1 is a $\mathbb{Z}_4$-linear map but not bijective. This map can be extended to R^n componentwise as follows:

$$\psi_1 : R^n \to \mathbb{Z}_4^{2n},$$

$$\begin{aligned} \psi_1(r_0, r_1, \ldots, r_{n-1}) = (&b_0 + 2c_0, b_1 + 2c_1, \ldots, b_{n-1} + 2c_{n-1}, 2a_0 + b_0 + 2c_0, \\ &2a_1 + b_1 + 2c_1, \ldots, 2a_{n-1} + b_{n-1} + 2c_{n-1}), \end{aligned} \tag{1}$$

where $r_i = a_i + ub_i + u^2c_i \in R$ and $a_i, b_i, c_i \in \mathbb{Z}_4$ for $i = 0, 1, \ldots, n-1$.

Keeping in view of the Section 3. of [12], we recall that the Lee weight $w_L(x)$ of any $x \in \mathbb{Z}_4$ is $\min\{|x|, |4-x|\}$. Thus, the Lee weights of 0, 1, 2, 3 are, respectively, 0, 1, 2, 1. The Lee weight of a vector $v \in \mathbb{Z}_4^n$ is defined as the rational sum of the Lee weight of its coordinates. The Lee weight for any $r \in R$ is defined as $w_L(r) = w_L(\psi_1(r))$ and for $r = (r_0, r_1, \ldots, r_{n-1}) \in R^n$ is given by $w_L(r) = \sum_{i=0}^{n-1} w_L(r_i)$. And, the Lee distance for the code C is defined by $d(C) = \min\{d_L(r, r') \mid r \neq r', r, r' \in C\}$, where $d_L(r, r') = w_L(r - r')$. Now, $d_L(r, r') = w_L(r - r') = w_L(\psi_1(r - r')) = w_L(\psi_1(r) - \psi_1(r')) = d_L(\psi_1(r), \psi_1(r')), \forall\, r, r' \in R^n$. Hence, ψ_1 is a distance preserving map from R^n(Lee distance) to $\mathbb{Z}_4^{2n}$(Lee distance).

Proposition 3.1. *For any $r \in R^n$, we have $\psi_1\tau_{(1+2u+2u^2)}(r) = \sigma\psi_1(r)$, where ψ_1, $\tau_{(1+2u+2u^2)}$ and σ are introduced in above.*

Proof. Let $r = (r_0, r_1, \ldots, r_{n-1}) \in R^n$, where $r_i = a_i + ub_i + u^2c_i \in R$ and $a_i, b_i, c_i \in \mathbb{Z}_4$ for $i = 0, 1, \ldots, n-1$. Clearly, $(1 + 2u + 2u^2)(a_{n-1} + ub_{n-1} + u^2c_{n-1}) = a_{n-1} + u(2a_{n-1} + b_{n-1}) +$

$u^2(2a_{n-1}+2b_{n-1}+c_{n-1})$. Therefore,

$$\begin{aligned}\psi_1\tau_{(1+2u+2u^2)}(r) &= \psi_1\left((1+2u+2u^2)r_{n-1}, r_0, \ldots, r_{n-2}\right)\\ &= (2a_{n-1}+b_{n-1}+2c_{n-1}, b_0+2c_0, \ldots, b_{n-2}+2c_{n-2}, b_{n-1}+2c_{n-1},\\ &\quad 2a_0+b_0+2c_0, \ldots, 2a_{n-2}+b_{n-2}+2c_{n-2}).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\sigma\psi_1(r) &= \sigma(b_0+2c_0, b_1+2c_1, \ldots, b_{n-1}+2c_{n-1}, 2a_0+b_0+2c_0, 2a_1+b_1+2c_1, \ldots,\\ &\quad 2a_{n-1}+b_{n-1}+2c_{n-1})\\ &= (2a_{n-1}+b_{n-1}+2c_{n-1}, b_0+2c_0, \ldots, b_{n-2}+2c_{n-2}, b_{n-1}+2c_{n-1}, 2a_0+b_0+2c_0, \ldots,\\ &\quad 2a_{n-2}+b_{n-2}+2c_{n-2}).\end{aligned}$$

Hence, $\psi_1\tau_{(1+2u+2u^2)}(r) = \sigma\psi_1(r)$. □

Theorem 3.2. *The Gray image, $\psi_1(C)$ of a $(1+2u+2u^2)$-constacyclic code C of length n over R is a cyclic code of length $2n$ over $\mathbb{Z}_4$.*

Proof. Since C is a $(1+2u+2u^2)$-constacyclic code of length n over R, $\tau_{(1+2u+2u^2)}(C) = C$. Applying ψ_1 on both sides and using Proposition 3.1, we have $\sigma\psi_1(C) = \psi_1(C)$. This shows that $\psi_1(C)$ is a cyclic code of length $2n$ over $\mathbb{Z}_4$. □

Again, we define another Gray map ψ_2 from R^n to $\mathbb{Z}_4^{2n}$ as

$$\psi_2 : R^n \to \mathbb{Z}_4^{2n},$$

given by

$$\begin{aligned}\psi_2(r_0, r_1, \ldots, r_{n-1}) &= (a_0, a_1, \ldots, a_{n-1}, a_0+2b_0+2c_0, a_1+2b_1+2c_1, \ldots,\\ &\quad a_{n-1}+2b_{n-1}+2c_{n-1}), \qquad (2)\end{aligned}$$

where $r_i = a_i + ub_i + u^2c_i \in R$ and $a_i, b_i, c_i \in \mathbb{Z}_4$ for $i = 0, 1, \ldots, n-1$.

Proposition 3.3. *For any $r \in R^n$, we have $\psi_2\tau_{(1+2u+2u^2)}(r) = \rho_2\psi_2(r)$, where ψ_2, $\tau_{(1+2u+2u^2)}$ and ρ_2 are introduced in above.*

Proof. Let $r = (r_0, r_1, \ldots, r_{n-1}) \in R^n$, where $r_i = a_i + ub_i + u^2c_i \in R$ and $a_i, b_i, c_i \in \mathbb{Z}_4$ for $i = 0, 1, \ldots, n-1$. Then

$$\begin{aligned}\psi_2\tau_{(1+2u+2u^2)}(r) &= \psi_2((1+2u+2u^2)r_{n-1}, r_0, \ldots, r_{n-2})\\ &= (a_{n-1}, a_0, \ldots, a_{n-2}, a_{n-1} + 2b_{n-1} + 2c_{n-1}, a_0 + 2b_0 + 2c_0, \ldots,\\ &\quad a_{n-2} + 2b_{n-2} + 2c_{n-2}).\end{aligned}$$

And, we have

$$\begin{aligned}\rho_2\psi_2(r) &= \rho_2(a_0, a_1, \ldots, a_{n-1}, a_0 + 2b_0 + 2c_0, a_1 + 2b_1 + 2c_1, \ldots, a_{n-1} + 2b_{n-1} + 2c_{n-1})\\ &= (a_{n-1}, a_0, \ldots, a_{n-2}, a_{n-1} + 2b_{n-1} + 2c_{n-1}, a_0 + 2b_0 + 2c_0, \ldots, a_{n-2} + 2b_{n-2} + 2c_{n-2}).\end{aligned}$$

Hence, $\psi_2\tau_{(1+2u+2u^2)}(r) = \rho_2\psi_2(r)$. □

Theorem 3.4. *The Gray image, $\psi_2(C)$ of a $(1+2u+2u^2)$-constacyclic code C of length n over R is a quasi-cyclic code of length $2n$ and index 2 over $\mathbb{Z}_4$.*

Proof. Since C is a $(1+2u+2u^2)$-constacyclic code of length n over R, $\tau_{(1+2u+2u^2)}(C) = C$. Applying ψ_2 on both sides and by Proposition 3.3, we have $\rho_2\psi_2(C) = \psi_2(C)$. This shows that $\psi_2(C)$ is a quasi-cyclic code of length $2n$ and index 2 over $\mathbb{Z}_4$. □

Further, we define another Gray map

$$\psi_3 : R^n \to \mathbb{Z}_4^{3n},$$

by

$$\begin{aligned}\psi_3(r_0, r_1, \ldots, r_{n-1}) &= (2a_0 + c_0, 2a_1 + c_1, \ldots, 2a_{n-1} + c_{n-1}, 2b_0 + c_0, 2b_1 + c_1, \ldots,\\ &\quad 2b_{n-1} + c_{n-1}, 2c_0, 2c_1, \ldots, 2c_{n-1}),\end{aligned} \tag{3}$$

where $r_i = a_i + ub_i + u^2c_i \in R$ and $a_i, b_i, c_i \in \mathbb{Z}_4$ for $i = 0, 1, \ldots, n-1$.

Proposition 3.5. *For any $r \in R^n$, we have $\psi_3\tau_{(1+2u+2u^2)}(r) = \delta\rho_3\psi_2(r)$, where ψ_3, $\tau_{(1+2u+2u^2)}$ and ρ_3 are introduced in above and δ is the permutation on $\mathbb{Z}_4^{3n}$ defined by $\delta(p_1, p_2, \ldots, p_{3n}) = (p_{\varepsilon(1)}, p_{\varepsilon(2)}, \ldots, p_{\varepsilon(3n)})$ with permutation $\varepsilon = (1, n+1)$ of $\{1, 2, \ldots, 3n\}$.*

Proof. Let $r = (r_0, r_1, \dots, r_{n-1}) \in R^n$, where $r_i = a_i + ub_i + u^2c_i \in R$ and $a_i, b_i, c_i \in \mathbb{Z}_4$ for $i = 0, 1, \dots, n-1$. Then

$$\begin{aligned}\psi_3\tau_{(1+2u+2u^2)}(r) &= \psi_3((1+2u+2u^2)r_{n-1}, r_0, \dots, r_{n-2})\\ &= (2b_{n-1}+c_{n-1}, 2a_0+c_0, \dots, 2a_{n-2}+c_{n-2}, 2a_{n-1}+c_{n-1}, 2b_0+c_0, \dots,\\ &\quad 2b_{n-2}+c_{n-2}, 2c_{n-1}, 2c_0, \dots, 2c_{n-2}),\end{aligned}$$

and, we have

$$\begin{aligned}\rho_3\psi_3(r) = \rho_3(&2a_0+c_0, 2a_1+c_1, \dots, 2a_{n-1}+c_{n-1}, 2b_0+c_0, 2b_1+c_1, \dots, 2b_{n-1}+c_{n-1},\\ &2c_0, 2c_1, \dots, 2c_{n-1})\\ = (&2a_{n-1}+c_{n-1}, 2a_0+c_0, \dots, 2a_{n-2}+c_{n-2}, 2b_{n-1}+c_{n-1}, 2b_0+c_0, \dots,\\ &2b_{n-2}+c_{n-2}, 2c_{n-1}, 2c_0, \dots, 2c_{n-2}).\end{aligned}$$

On applying the permutation δ, we get

$$\begin{aligned}\delta\rho_3\psi_3(r) = (&2b_{n-1}+c_{n-1}, 2a_0+c_0, \dots, 2a_{n-2}+c_{n-2}, 2a_{n-1}+c_{n-1}, 2b_0+c_0, \dots,\\ &2b_{n-2}+c_{n-2}, 2c_{n-1}, 2c_0, \dots, 2c_{n-2}).\end{aligned}$$

Hence, $\psi_3\tau_{(1+2u+2u^2)}(r) = \delta\rho_3\psi_3(r)$. □

Theorem 3.6. *The Gray image, $\psi_3(C)$ of a $(1+2u+2u^2)$-constacyclic code C of length n over R is a permutation equivalent to a quasi-cyclic code of length $3n$ and index 3 over $\mathbb{Z}_4$.*

Proof. Since C is a $(1+2u+2u^2)$-constacyclic code of length n over R, $\tau_{(1+2u+2u^2)}(C) = C$. Applying ψ_3 on both sides and using Proposition 3.5, we have $\delta\rho_3\psi_3(C) = \psi_3(C)$. This shows that $\psi_3(C)$ is a permutation equivalent to a quasi-cyclic code of length $3n$ and index 3 over $\mathbb{Z}_4$. □

The permutation version of the above Gray map ψ_1, denoting by, $\psi_{1,\pi}$ is defined as follows

$$\begin{aligned}\psi_{1,\pi}(r_0, r_1, \dots, r_{n-1}) = (&b_0+2c_0, 2a_0+b_0+2c_0, b_1+2c_1, 2a_1+b_1+2c_1, \dots,\\ &b_{n-1}+2c_{n-1}, 2a_{n-1}+b_{n-1}+2c_{n-1}),\end{aligned} \tag{4}$$

where $r_i = a_i + ub_i + u^2c_i \in R$ and $a_i, b_i, c_i \in \mathbb{Z}_4$ for $i = 0, 1, \dots, n-1$.

Proposition 3.7. *For any $r \in R^n$, we have $\psi_{1,\pi}\sigma(r) = \sigma^2\psi_{1,\pi}(r)$, where $\psi_{1,\pi}$ and σ are introduced in above.*

Proof. Let $r = (r_0, r_1, \dots, r_{n-1}) \in R^n$, where $r_i = a_i + ub_i + u^2c_i \in R$ and $a_i, b_i, c_i \in \mathbb{Z}_4$ for $i = 0, 1, \dots, n-1$. Then

$$\begin{aligned}\psi_{1,\pi}\sigma(r) &= \psi_{1,\pi}(r_{n-1}, r_0, \dots, r_{n-2}) \\ &= (b_{n-1} + 2c_{n-1}, 2a_{n-1} + b_{n-1} + 2c_{n-1}, b_0 + 2c_0, 2a_0 + b_0 + 2c_0, \dots, b_{n-2} + 2c_{n-2}, \\ &\quad 2a_{n-2} + b_{n-2} + 2c_{n-2}),\end{aligned}$$

and, we have

$$\begin{aligned}\sigma^2\psi_{1,\pi}(r) &= \sigma^2(b_0 + 2c_0, 2a_0 + b_0 + 2c_0, b_1 + 2c_1, 2a_1 + b_1 + 2c_1, \dots, b_{n-1} + 2c_{n-1}, \\ &\quad 2a_{n-1} + b_{n-1} + 2c_{n-1}) \\ &= (b_{n-1} + 2c_{n-1}, 2a_{n-1} + b_{n-1} + 2c_{n-1}, b_0 + 2c_0, 2a_0 + b_0 + 2c_0, \dots, b_{n-2} + 2c_{n-2}, \\ &\quad 2a_{n-2} + b_{n-2} + 2c_{n-2}).\end{aligned}$$

Hence, $\psi_{1,\pi}\sigma(r) = \sigma^2\psi_{1,\pi}(r)$. □

Theorem 3.8. *The Gray image, $\psi_{1,\pi}(C)$ of a cyclic code C of length n over R is equivalent to a 2-quasi-cyclic code of length $2n$ over $\mathbb{Z}_4$.*

Proof. Since C is a cyclic code of length n over R, $\sigma(C) = C$. Applying $\psi_{1,\pi}$ on both sides and using Proposition 3.7, we have $\sigma^2\psi_{1,\pi}(C) = \psi_{1,\pi}(C)$. This shows that $\psi_{1,\pi}(C)$ is equivalent to a 2-quasi-cyclic code of length $2n$ over $\mathbb{Z}_4$. □

Remark 3.9. Note that the other Gray maps ψ_2 and ψ_3 permutation versions can be defined analogously to obtain the similar results.

3.2. $(3+2u+2u^2)$**-constacyclic codes over** R **and their** $\mathbb{Z}_4$**-images.** In this part, we study the $(3 + 2u + 2u^2)$-constacyclic codes of length n over R by defining another three distinct Gray maps and show that Gray images of such constacyclic codes are cyclic, quasi-cyclic or permutation equivalent to quasi-cyclic codes.

Firstly, we define a Gray map φ_1 from R to $\mathbb{Z}_4^2$ as

$$\varphi_1 : R \to \mathbb{Z}_4^2,$$

by

$$\varphi_1(a + ub + u^2c) = (a + b + c, 3a + b + 3c) \quad \forall\ a, b, c \in \mathbb{Z}_4.$$

Clearly, φ_1 is a $\mathbb{Z}_4$-linear map but not bijective. This map can be extended to R^n component-wise as follows:

$$\varphi_1 : R^n \to \mathbb{Z}_4^{2n},$$

$$\varphi_1(r_0, r_1, \ldots, r_{n-1}) = (a_0 + b_0 + c_0, a_1 + b_1 + c_1, \ldots, a_{n-1} + b_{n-1} + c_{n-1}, 3a_0 + b_0 + 3c_0,$$
$$3a_1 + b_1 + 3c_1, \ldots, 3a_{n-1} + b_{n-1} + 3c_{n-1}), \quad (5)$$

where $r_i = a_i + ub_i + u^2c_i \in R$ and $a_i, b_i, c_i \in \mathbb{Z}_4$ for $i = 0, 1, \ldots, n-1$.

Similarly, we consider another two Gray maps as given below:

$$\varphi_2 : R^n \to \mathbb{Z}_4^{2n},$$

defined by

$$\varphi_2(r_0, r_1, \ldots, r_{n-1}) = (2a_0, 2a_1, \ldots, 2a_{n-1}, 2b_0 + 2c_0, 2b_1 + 2c_1, \ldots, 2b_{n-1} + 2c_{n-1}), \quad (6)$$

and

$$\varphi_3 : R^n \to \mathbb{Z}_4^{3n},$$

defined by

$$\varphi_3(r_0, r_1, \ldots, r_{n-1}) = (2a_0 + 2b_0 + 3c_0, 2a_1 + 2b_1 + 3c_1, \ldots, 2a_{n-1} + 2b_{n-1} + 3c_{n-1},$$
$$c_0, c_1, \ldots, c_{n-1}, 2a_0 + 2b_0 + 2c_0, 2a_1 + 2b_1 + 2c_1, \ldots,$$
$$2a_{n-1} + 2b_{n-1} + 2c_{n-1}), \quad (7)$$

where $r_i = a_i + ub_i + u^2c_i \in R$ and $a_i, b_i, c_i \in \mathbb{Z}_4$ for $i = 0, 1, \ldots, n-1$.

Proposition 3.10. *For any $r \in R^n$, we have $\varphi_1\tau_{(3+2u+2u^2)}(r) = \sigma\varphi_1(r)$, where φ_1, $\tau_{(3+2u+2u^2)}$ and σ are introduced in above.*

Proof. Let $r = (r_0, r_1, \ldots, r_{n-1}) \in R^n$, where $r_i = a_i + ub_i + u^2c_i \in R$ and $a_i, b_i, c_i \in \mathbb{Z}_4$ for $i = 0, 1, \ldots, n-1$. Clearly, $(3+2u+2u^2)r_{n-1} = 3a_{n-1}+u(2a_{n-1}+3b_{n-1})+u^2(2a_{n-1}+2b_{n-1}+3c_{n-1})$. Then

$$\varphi_1\tau_{(3+2u+2u^2)}(r) = \varphi_1((3 + 2u + 2u^2)r_{n-1}, r_0, \ldots, r_{n-2})$$
$$= (3a_{n-1} + b_{n-1} + 3c_{n-1}, a_0 + b_0 + c_0, \ldots, a_{n-2} + b_{n-2} + c_{n-2}, a_{n-1} + b_{n-1}+$$
$$c_{n-1}, 3a_0 + b_0 + 3c_0, \ldots, 3a_{n-2} + b_{n-2} + 3c_{n-2}),$$

and, we have

$$\begin{aligned}\sigma\varphi_1(r) &= \sigma(a_0+b_0+c_0, a_1+b_1+c_1, \ldots, a_{n-1}+b_{n-1}+c_{n-1}, 3a_0+b_0+3c_0, 3a_1+b_1+3c_1,\\ &\quad \ldots, 3a_{n-1}+b_{n-1}+3c_{n-1})\\ &= (3a_{n-1}+b_{n-1}+3c_{n-1}, a_0+b_0+c_0, \ldots, a_{n-2}+b_{n-2}+c_{n-2}, a_{n-1}+b_{n-1}+c_{n-1},\\ &\quad 3a_0+b_0+3c_0, \ldots, 3a_{n-2}+b_{n-2}+3c_{n-2}).\end{aligned}$$

Hence, $\varphi_1\tau_{(3+2u+2u^2)}(r) = \sigma\varphi_1(r)$. □

Theorem 3.11. *The Gray image, $\varphi_1(C)$ of a $(3+2u+2u^2)$-constacyclic code C of length n over R is a cyclic code of length $2n$ over $\mathbb{Z}_4$.*

Proof. Since C is a $(3+2u+2u^2)$-constacyclic code of length n over R, $\tau_{(3+2u+2u^2)}(C) = C$. Applying φ_1 on both sides and using Proposition 3.9, we have $\sigma\varphi_1(C) = \varphi_1(C)$. This shows that $\varphi_1(C)$ is a cyclic code of length $2n$ over $\mathbb{Z}_4$. □

Proposition 3.12. *For any $r \in R^n$, we have $\varphi_2\tau_{(3+2u+2u^2)}(r) = \rho_2\varphi_2(r)$, where φ_2, $\tau_{(3+2u+2u^2)}$ and ρ_2 are introduced in above.*

Proof. Let $r = (r_0, r_1, \ldots, r_{n-1}) \in R^n$, where $r_i = a_i + ub_i + u^2c_i \in R$ and $a_i, b_i, c_i \in \mathbb{Z}_4$ for $i = 0, 1, \ldots, n-1$. Then

$$\begin{aligned}\varphi_2\tau_{(3+2u+2u^2)}(r) &= \varphi_2((3+2u+2u^2)r_{n-1}, r_0, \ldots, r_{n-2})\\ &= (2a_{n-1}, 2a_0, \ldots, 2a_{n-2}, 2b_{n-1}+2c_{n-1}, 2b_0+2c_0, \ldots, 2b_{n-2}+2c_{n-2}),\end{aligned}$$

and, we have

$$\begin{aligned}\rho_2\varphi_2(r) &= \rho_2(2a_0, 2a_1, \ldots, 2a_{n-1}, 2b_0+2c_0, 2b_1+2c_1, \ldots, 2b_{n-1}+2c_{n-1})\\ &= (2a_{n-1}, 2a_0, \ldots, 2a_{n-2}, 2b_{n-1}+2c_{n-1}, 2b_0+2c_0, \ldots, 2b_{n-2}+2c_{n-2}).\end{aligned}$$

Hence, $\varphi_2\tau_{(3+2u+2u^2)}(r) = \rho_2\varphi_2(r)$. □

Theorem 3.13. *The Gray image, $\varphi_2(C)$ of a $(3+2u+2u^2)$-constacyclic code C of length n over R is a quasi-cyclic code of length $2n$ and index 2 over $\mathbb{Z}_4$.*

Proof. Since C is a $(3+2u+2u^2)$-constacyclic code of length n over R, $\tau_{(3+2u+2u^2)}(C) = C$. Applying φ_2 on both sides and using Proposition 3.12, we have $\rho_2\varphi_2(C) = \varphi_2(C)$. This shows that $\varphi_2(C)$ is a quasi-cyclic code of length $2n$ and index 2 over $\mathbb{Z}_4$. □

Proposition 3.14. *For any $r \in R^n$, we have $\varphi_3\tau_{(3+2u+2u^2)}(r) = \delta\rho_3\varphi_3(r)$, where φ_3, $\tau_{(3+2u+2u^2)}$, ρ_3 and δ are introduced in above.*

Proof. Let $r = (r_0, r_1, \ldots, r_{n-1}) \in R^n$, where $r_i = a_i + ub_i + u^2c_i \in R$ and $a_i, b_i, c_i \in \mathbb{Z}_4$ for $i = 0, 1, \ldots, n-1$. Then

$$\begin{aligned}\varphi_3\tau_{(3+2u+2u^2)}(r) &= \varphi_3((3+2u+2u^2)r_{n-1}, r_0, \ldots, r_{n-2})\\ &= (c_{n-1}, 2a_0 + 2b_0 + 3c_0, \ldots, 2a_{n-2} + 2b_{n-2} + 3c_{n-2},\\ &\quad 2a_{n-1} + 2b_{n-1} + 3c_{n-1}, c_0, \ldots, c_{n-2}, 2a_{n-1} + 2b_{n-1}\\ &\quad + 2c_{n-1}, 2a_0 + 2b_0 + 2c_0, \ldots, 2a_{n-2} + 2b_{n-2} + 2c_{n-2}),\end{aligned}$$

and, we have

$$\begin{aligned}\rho_3\varphi_3(r) &= \rho_3(2a_0 + 2b_0 + 3c_0, 2a_1 + 2b_1 + 3c_1, \ldots, 2a_{n-1} + 2b_{n-1} + 3c_{n-1}, c_0, c_1, \ldots, c_{n-1},\\ &\quad 2a_0 + 2b_0 + 2c_0, 2a_1 + 2b_1 + 2c_1, \ldots, 2a_{n-1} + 2b_{n-1} + 2c_{n-1})\\ &= (2a_{n-1} + 2b_{n-1} + 3c_{n-1}, 2a_0 + 2b_0 + 3c_0, \ldots, 2a_{n-2} + 2b_{n-2} + 3c_{n-2}, c_{n-1}, c_0,\\ &\quad \ldots, c_{n-2}, 2a_{n-1} + 2b_{n-1} + 2c_{n-1}, 2a_0 + 2b_0 + 2c_0, \ldots, 2a_{n-2} + 2b_{n-2} + 2c_{n-2}).\end{aligned}$$

On applying the permutation δ on both sides, we get

$$\begin{aligned}\delta\rho_3\varphi_3(r) &= (c_{n-1}, 2a_0 + 2b_0 + 3c_0, \ldots, 2a_{n-2} + 2b_{n-2} + 3c_{n-2}, 2a_{n-1} + 2b_{n-1} + 3c_{n-1}, c_0,\\ &\quad \ldots, c_{n-2}, 2a_{n-1} + 2b_{n-1} + 2c_{n-1}, 2a_0 + 2b_0 + 2c_0, \ldots, 2a_{n-2} + 2b_{n-2} + 2c_{n-2}).\end{aligned}$$

Hence, $\varphi_3\tau_{(3+2u+2u^2)}(r) = \delta\rho_3\varphi_3(r)$. □

Theorem 3.15. *The Gray image, $\varphi_3(C)$ of a $(3+2u+2u^2)$-constacyclic code C of length n over R is a permutation equivalent to a quasi-cyclic code of length $3n$ and index 3 over $\mathbb{Z}_4$.*

Proof. Since C is a $(3+2u+2u^2)$-constacyclic code of length n over R, $\tau_{(3+2u+2u^2)}(C) = C$. Applying φ_3 on both sides and using Proposition 3.14, we have $\delta\rho_3\varphi_3(C) = \varphi_3(C)$. This shows that $\varphi_3(C)$ is a permutation equivalent to a quasi-cyclic code of length $3n$ and index 3 over $\mathbb{Z}_4$. □

Let $\varphi_{1,\pi}$ be the permutation version of the above Gray map φ_1, which is defined as follows

$$\begin{aligned}\varphi_{1,\pi}(r_0, r_1, \ldots, r_{n-1}) &= (a_0 + b_0 + c_0, 3a_0 + b_0 + 3c_0, a_1 + b_1 + c_1, 3a_1 + b_1 + 3c_1,\\ &\quad \ldots, a_{n-1} + b_{n-1} + c_{n-1}, 3a_{n-1} + b_{n-1} + 3c_{n-1}),\end{aligned} \tag{8}$$

where $r_i = a_i + ub_i + u^2c_i \in R$ and $a_i, b_i, c_i \in \mathbb{Z}_4$ for $i = 0, 1, \ldots, n-1$.

Proposition 3.16. *For any $r \in R^n$, we have $\varphi_{1,\pi}\sigma(r) = \sigma^2\varphi_{1,\pi}(r)$, where $\varphi_{1,\pi}$ and σ are introduced in above.*

Proof. With a minor change in the permutation version of the Gray map, the proof is the same as given in Proposition 3.7. □

Theorem 3.17. *The Gray image, $\varphi_{1,\pi}(C)$ of a cyclic code C of length n over R is equivalent to a 2-quasi-cyclic code of length $2n$ over $\mathbb{Z}_4$.*

Proof. Similar to the proof of Theorem 3.8. □

4. Skew constacyclic codes and their $\mathbb{Z}_4$-images

We define an automorphism on the ring R by $\theta(a + ub + u^2c) = a + uc + u^2b \ \ \forall\, a, b, c \in \mathbb{Z}_4$, where $\theta(a) = a$, $\theta(u) = u^2$ and $\theta(u^2) = u$. Clearly, the order of the automorphism is 2 as $\theta^2(r) = r\ \forall\, r \in R$. The set $R[x; \theta] = \{a_0 + a_1x + \cdots + a_{n-1}x^{n-1} \mid a_i \in R,\ i = 0, 1, \ldots, n-1\}$ is a non-commutative skew polynomial ring under the usual addition of polynomials and multiplication of polynomials, which is defined as $(ax^s)(bx^t) = a\theta^s(b)x^{s+t}$. By taking $\lambda = (1 + 2u + 2u^2)$ and $(3 + 2u + 2u^2)$, we can identify each vector $r = (r_0, r_1, r_2, \ldots, r_{n-1}) \in R^n$ with a polynomial $r(x) \in R[x; \theta]/\langle x^n - \lambda\rangle$ by the following correspondence

$$r = (r_0, r_1, \ldots, r_{n-1}) \to r(x) = r_0 + r_1x + \cdots + r_{n-1}x^{n-1}(\mathrm{mod}\langle x^n - \lambda\rangle).$$

Definition 4.1. [10] A non-empty subset C of R^n is called a skew λ-constacyclic code of length n over R if it satisfies the following conditions:

(i) C is an R-submodule of R^n, and

(ii) if $(c_0, c_1, \ldots, c_{n-1}) \in C$, then

$$\tau_{\theta,\lambda}(c_0, c_1, \ldots, c_{n-1}) = (\theta(\lambda c_{n-1}), \theta(c_0), \ldots, \theta(c_{n-2})) \in C.$$

Theorem 4.2. [10] *Let C be a linear code of length n over R. Then C is a skew λ-constacyclic code over R if and only if C is a left $R[x; \theta]$-submodule of $R[x; \theta]/\langle x^n - \lambda\rangle$.*

Proposition 4.3. *For any $r \in R^n$, we have $\psi_2\tau_{\theta,\lambda} = \rho_2\psi_2$, where ψ_2, ρ_2 and $\tau_{\theta,\lambda}$ with $\lambda = (1 + 2u + 2u^2)$ are introduced in above.*

Proof. Let $r = (r_0, r_1, \ldots, r_{n-1}) \in R^n$, where $r_i = a_i + ub_i + u^2c_i \in R$ and $a_i, b_i, c_i \in \mathbb{Z}_4$ for $i = 0, 1, \ldots, n-1$. Now, $\theta(a_i + ub_i + u^2c_i) = a_i + uc_i + u^2b_i$ and

$\theta\left((1+2u+2u^2)(a_{n-1}+ub_{n-1}+u^2c_{n-1})\right) = a_{n-1}+u(2a_{n-1}+2b_{n-1}+c_{n-1})+u^2(2a_{n-1}+b_{n-1})$. Therefore,

$$\begin{aligned}\psi_2\tau_{\theta,\lambda}(r) &= \psi_2\left(\theta(\lambda r_{n-1}), \theta(r_0), \ldots, \theta(r_{n-2})\right)\\ &= (a_{n-1}, a_0, \ldots, a_{n-2}, a_{n-1}+2b_{n-1}+2c_{n-1}, a_0+2b_0+2c_0, \ldots,\\ &\quad a_{n-2}+2b_{n-2}+2c_{n-2}).\end{aligned}$$

From Proposition 3.3, we have

$$\begin{aligned}\rho_2\psi_2(r) = (&a_{n-1}, a_0, \ldots, a_{n-2}, a_{n-1}+2b_{n-1}+2c_{n-1}, a_0+2b_0+2c_0, \ldots,\\ &a_{n-2}+2b_{n-2}+2c_{n-2}).\end{aligned}$$

Hence, $\psi_2\tau_{\theta,\lambda}(r) = \rho_2\psi_2(r)$. □

Theorem 4.4. *The Gray image, $\psi_2(C)$ of a skew λ-constacyclic code C of length n over R with $\lambda = (1+2u+2u^2)$ is a quasi-cyclic code of length $2n$ and index 2 over $\mathbb{Z}_4$.*

Proof. Since C is a skew λ-constacyclic code of length n over R with $\lambda = (1+2u+2u^2)$, $\tau_{\theta,\lambda}(C) = C$. Applying ψ_2 on both sides and by Proposition 4.3, we have $\rho_2\psi_2(C) = \psi_2(C)$. This shows that $\psi_2(C)$ is a quasi-cyclic code of length $2n$ and index 2 over $\mathbb{Z}_4$. □

Proposition 4.5. *For any $r \in R^n$, we have $\varphi_2\tau_{\theta,\lambda}(r) = \rho_2\varphi_2(r)$, where φ_2, ρ_2 and $\tau_{\theta,\lambda}$ with $\lambda = (3+2u+2u^2)$ are introduced in above.*

Proof. Let $r = (r_0, r_1, \ldots, r_{n-1}) \in R^n$, where $r_i = a_i + ub_i + u^2c_i \in R$ and $a_i, b_i, c_i \in \mathbb{Z}_4$ for $i = 0, 1, \ldots, n-1$. Now, $\theta(a_i + ub_i + u^2c_i) = a_i + uc_i + u^2b_i$ and $\theta\left((3+2u+2u^2)(a_{n-1}+ub_{n-1}+u^2c_{n-1})\right) = 3a_{n-1}+u(2a_{n-1}+2b_{n-1}+3c_{n-1})+u^2(2a_{n-1}+3b_{n-1})$. Then

$$\begin{aligned}\varphi_2\tau_{\theta,\lambda}(r) &= \varphi_2(\theta(\lambda r_{n-1}), \theta(r_0), \ldots, \theta(r_{n-2}))\\ &= (2a_{n-1}, 2a_0, \ldots, 2a_{n-2}, 2b_{n-1}+2c_{n-1}, 2b_0+2c_0, \ldots, 2b_{n-2}+2c_{n-2}).\end{aligned}$$

From Proposition 3.12, we have

$$\rho_3\varphi_2(r) = (2a_{n-1}, 2a_0, \ldots, 2a_{n-2}, 2b_{n-1}+2c_{n-1}, 2b_0+2c_0, \ldots, 2b_{n-2}+2c_{n-2}).$$

Hence, $\varphi_2\tau_{\theta,\lambda}(r) = \rho_2\varphi_2(r)$. □

Theorem 4.6. *The Gray image, $\varphi_2(C)$ of a skew λ-constacyclic code C of length n over R with $\lambda = (3+2u+2u^2)$ is a quasi-cyclic code of length $2n$ and index 2 over $\mathbb{Z}_4$.*

Proof. Since C is a skew λ-constacyclic code of length n over R with $\lambda = (3 + 2u + 2u^2)$, $\tau_{\theta,\lambda}(C) = C$. Applying φ_2 on both sides and using Proposition 4.5, we have $\rho_2\varphi_2(C) = \varphi_2(C)$. This shows that $\varphi_2(C)$ is a quasi-cyclic code of length $2n$ and index 2 over $\mathbb{Z}_4$. □

5. Constacyclic codes of odd length n over R and their generators

In this section, we discuss λ-constacyclic codes of odd length n over R with $\lambda = (1+2u+2u^2)$ and $(3+2u+2u^2)$. Note that $\lambda^n = 1$ if n is an even integer and $\lambda^n = \lambda$ if n is an odd integer. Based on the results established in [3, 5, 10, 11, 12, 14, 18], analogous results are given below without proofs.

Theorem 5.1. *A mapping $\beta : R[x]/\langle x^n - 1\rangle \longrightarrow R[x]/\langle x^n - \lambda\rangle$ defined by $\beta(a(x)) = a(\lambda x)$ is a ring isomorphism, if n is an odd integer.*

Corollary 5.2. *For any odd integer n, I is an ideal of $R[x]/\langle x^n - 1\rangle$ if and only if $\beta(I)$ is an ideal of $R[x]/\langle x^n - \lambda\rangle$.*

Corollary 5.3. *Let $\overline{\beta}$ be a permutation of R^n, defined by $\overline{\beta}(c_0, c_1, \ldots, c_{n-1}) = (c_0, \lambda c_1, \ldots, \lambda^{n-1}c_{n-1})$. Then a subset C of R^n is a cyclic code of odd length n over R if and only if $\overline{\beta}(C)$ is a λ-constacyclic code over R.*

Theorem 5.4. [13] *Let C be a cyclic code of odd length n over R. Then $C = \langle g_1(x)+2a_1(x)+ ug(x) + u^2h(x), u(g_2(x) + 2a_2(x)) + u^2b(x), u^2(g_3(x) + 2a_3(x))\rangle$, where $a_i(x)|g_i(x)|(x^n - 1) \mod 2$, and $g_i(x) + 2a_i(x)$ is a generator of a cyclic code over $\mathbb{Z}_4$ for $i = 1, 2, 3$.*

Using Theorem 5.4, we can construct the generators for λ- constacyclic codes of odd length n over R as follows.

Theorem 5.5. *Let C be a cyclic code of odd length n over R. Then C is an ideal of $R_{n,\lambda}$ given by $C = \langle g_1(\widehat{x})+2a_1(\widehat{x})+ug(\widehat{x})+u^2h(\widehat{x}), u(g_2(\widehat{x})+2a_2(\widehat{x}))+u^2b(\widehat{x}), u^2(g_3(\widehat{x})+2a_3(\widehat{x}))\rangle$, where $a_i(x)|g_i(x)|(x^n - 1) \mod 2$, and $g_i(x) + 2a_i(x)$ is a generator of a cyclic code over $\mathbb{Z}_4$ for $i = 1, 2, 3$ and $\widehat{x} = \lambda x$.*

Proof. The result follows from Corollary 5.3 and Theorem 5.4. □

Theorem 5.6. *Let C be a λ-constacyclic code of length n over R and $C = \langle a(x) + ub(x) + u^2c(x)\rangle$, where $a(x), b(x), c(x) \in \mathbb{Z}_4[x]$ with degree less than n. Then $\psi_1(C)$ is a cyclic code of length $2n$ over $\mathbb{Z}_4$ generated by the polynomials $[b(x)+2c(x)]+x^n[2a(x)+b(x)+2c(x)]$, $[a(x)+2b(x)] + x^n[a(x) + 2b(x)]$ and $[2a(x)] + x^n[2a(x)]$.*

Proof. The polynomial that corresponds to the Gray map ψ_1 of (1) can be defined as

$$\psi_1 : \frac{R[x]}{\langle x^n - 1\rangle} \to \frac{\mathbb{Z}_4[x]}{\langle x^n - 1\rangle} \times \frac{\mathbb{Z}_4[x]}{\langle x^n - 1\rangle},$$

$$\psi_1(a(x) + ub(x) + u^2c(x)) = (b(x) + 2c(x), 2a(x) + b(x) + 2c(x)),$$

where $a(x), b(x), c(x) \in \mathbb{Z}_4[x]$.

For any $r_1(x), r_2(x), r_3(x) \in \mathbb{Z}_4[x]$, it can be shown that

$$\begin{aligned} &\psi_1[(r_1(x) + ur_2(x) + u^2r_3(x))(a(x) + ub(x) + u^2c(x))] \\ &= r_1(x)[b(x) + 2c(x), 2a(x) + b(x) + 2c(x)] + r_2(x)[a(x) + 2b(x), \\ &\quad a(x) + 2b(x)] + r_3(x)[2a(x), 2a(x)], \end{aligned}$$

and the vector $(a, b) \in \frac{\mathbb{Z}_4[x]}{\langle x^n-1\rangle} \times \frac{\mathbb{Z}_4[x]}{\langle x^n-1\rangle}$ corresponds to the same vector $(a + x^n b) \in \frac{\mathbb{Z}_4[x]}{\langle x^{2n}-1\rangle}$.

Hence, the polynomials $[b(x)+2c(x)]+x^n[2a(x)+b(x)+2c(x)]$, $[a(x)+2b(x)]+x^n[a(x)+2b(x)]$ and $[2a(x)] + x^n[2a(x)]$ generate $\psi_1(C)$. □

Theorem 5.7. *Let C be a λ-constacyclic code of length n over R and $C = \langle a(x) + ub(x) + u^2c(x)\rangle$, where $a(x), b(x), c(x) \in \mathbb{Z}_4[x]$ with degree less than n.Then $\varphi_1(C)$ is a cyclic code of length $2n$ over $\mathbb{Z}_4$ generated by the polynomials $[a(x) + b(x) + c(x)] + x^n[3a(x) + b(x) + 3c(x)]$, $[a(x) + b(x)] + x^n[a(x) + 3b(x)]$ and $[a(x)] + x^n[3a(x)]$.*

Proof. Similar to the proof of Theorem 5.6. □

Example 5.8. If $C = \langle x^4+(u+u^2)x^3+3ux+1+u+u^2\rangle$ is a $(1+2u+2u^2)$-constacyclic code of length 5 over R. In view of Theorem 5.6, $\psi_1(C)$ is a cyclic code of length 10 over $\mathbb{Z}_4$ generated by the polynomials $2x^9 + 3x^8 + 3x^6 + x^5 + 3x^3 + 3x + 3$, $x^9 + 2x^8 + 2x^6 + 3x^5 + x^4 + 2x^3 + 2x + 3$ and $2x^9 + 2x^5 + 2x^4 + 2$ with minimum Lee distance 8.

Example 5.9. If $C = \langle x^3 + (1 + u + u^2)x^2 + (2 + u)x + u + u^2\rangle$ is a $(3 + 2u + 2u^2)$-constacyclic code of length 4 over R. By Theorem 5.7, $\varphi_1(C)$ is a cyclic code of length 8 over $\mathbb{Z}_4$ generated by the polynomials $3x^7 + 3x^6 + 3x^5 + x^3 + 3x^2 + 3x + 2$, $x^7 + x^5 + 3x^4 + x^3 + 2x^2 + 3x + 1$ and $3x^7 + 3x^6 + 2x^5 + x^3 + x^2 + 2x$ with minimum Lee distance 8.

Definition 5.10. [16] Let n be an odd integer and $\Upsilon = (1, n + 1)(3, n + 3)...(2i + 1, n + 2i + 1)...(n - 2, 2n - 2)$ be a permutation of the set $\{0, 1, 2, \ldots, 2n - 1\}$. Then the Nechaev permutation Π is the permutation of $\mathbb{Z}_4^{2n}$ defined by

$$\Pi(r_0, r_1, \ldots, r_{2n-1}) = (r_{\Upsilon(0)}, r_{\Upsilon(1)}, \ldots, r_{\Upsilon(2n-1)}).$$

Theorem 5.11. *For any $r \in R^n$, we have $\psi_1 \overline{\beta}(r) = \Pi\, \psi_1(r)$, where ψ_1, $\overline{\beta}$ with $\lambda = (1 + 2u + 2u^2)$ and Π are introduced in above.*

Proof. Let $r = (r_0, r_1, \ldots, r_{n-1}) \in R^n$, where $r_i = a_i + ub_i + u^2c_i \in R$ and $a_i, b_i, c_i \in \mathbb{Z}_4$ for $i = 0, 1, \ldots, n-1$. Now, $(1 + 2u + 2u^2)(a_i + ub_i + u^2c_i) = a_i + u(2a_i + b_i) + u^2(2a_i + 2b_i + c_i)$ and $\psi_1(a_i + u(2a_i + b_i) + u^2(2a_i + 2b_i + c_i)) = (2a_i + b_i + 2c_i, b_i + 2c_i)$. Then

$$\begin{aligned}
\psi_1\overline{\beta}(r) &= \psi_1(r_0, \lambda r_1, \lambda^2 r_2, \ldots, \lambda^{n-2} r_{n-2}, \lambda^{n-1} r_{n-1}) \\
&= (b_0 + 2c_0, 2a_1 + b_1 + 2c_1, b_2 + 2c_2, \ldots, 2a_{n-2} + b_{n-2} + 2c_{n-2}, b_{n-1} + 2c_{n-1}, \\
&\quad 2a_0 + b_0 + 2c_0, b_1 + 2c_1, 2a_2 + b_2 + 2c_2, \ldots, b_{n-2} + 2c_{n-2}, 2a_{n-1} + b_{n-1} + 2c_{n-1}),
\end{aligned}$$

and, we have

$$\begin{aligned}
\Pi\psi_1(r) &= \Pi(b_0 + 2c_0, b_1 + 2c_1, b_2 + 2c_2, \ldots, b_{n-2} + 2c_{n-2}, b_{n-1} + 2c_{n-1}, 2a_0 + b_0 + 2c_0, \\
&\quad 2a_1 + b_1 + 2c_1, 2a_2 + b_2 + 2c_2, .., 2a_{n-2} + b_{n-2} + 2c_{n-2}, 2a_{n-1} + b_{n-1} + 2c_{n-1}) \\
&= (b_0 + 2c_0, 2a_1 + b_1 + 2c_1, b_2 + 2c_2, \ldots, 2a_{n-2} + b_{n-2} + 2c_{n-2}, b_{n-1} + 2c_{n-1}, \\
&\quad 2a_0 + b_0 + 2c_0, b_1 + 2c_1, 2a_2 + b_2 + 2c_2, \ldots, b_{n-2} + 2c_{n-2}, 2a_{n-1} + b_{n-1} + 2c_{n-1}).
\end{aligned}$$

Hence, $\psi_1\overline{\beta}(r) = \Pi\psi_1(r)$. □

Corollary 5.12. *If $\widetilde{C}$ is the Gray image of a cyclic code C of odd length n over R (i.e., $\psi_1(C) = \widetilde{C}$), then $\Pi(\widetilde{C})$ is a cyclic code of length $2n$ over $\mathbb{Z}_4$.*

Proof. Since C is a cyclic code over R, $\overline{\beta}(C)$ is a $(1 + 2u + 2u^2)$-constacyclic code over R by Corollary 5.3. From Theorem 3.2, we see that $\psi_1 \overline{\beta}(C)$ is a cyclic code of length $2n$ over $\mathbb{Z}_4$. Also, from Theorem 5.11, we have $\Pi\, \psi_1(C) = \Pi(\widetilde{C}) = \psi_1 \overline{\beta}(C)$. This implies that $\Pi(\widetilde{C})$ is a cyclic code of length $2n$ over $\mathbb{Z}_4$. □

Theorem 5.13. *For any $r \in R^n$, we have $\varphi_1\overline{\beta}(r) = \Pi\varphi_1(r)$, where φ_1, $\overline{\beta}$ with $\lambda = (3+2u+2u^2)$ and Π are introduced in above.*

Proof. Let $r = (r_0, r_1, \ldots, r_{n-1}) \in R^n$, where $r_i = a_i + ub_i + u^2c_i \in R$ and $a_i, b_i, c_i \in \mathbb{Z}_4$ for $i = 0, 1, \ldots, n-1$. Now, $(3+2u+2u^2)(a_i + ub_i + u^2c_i) = 3a_i + u(2a_i + 3b_i) + u^2(2a_i + 2b_i + 3c_i)$ and $\varphi_1(3a_i + u(2a_i + 3b_i) + u^2(2a_i + 2b_i + 3c_i)) = (3a_i + b_i + 3c_i, a_i + b_i + c_i)$. Then

$$\begin{aligned}
\varphi_1\overline{\beta}(r) &= \varphi_1(r_0, \lambda r_1, \lambda^2 r_2, \ldots, \lambda^{n-2} r_{n-2}, \lambda^{n-1} r_{n-1}) \\
&= (a_0 + b_0 + c_0, 3a_1 + b_1 + 3c_1, \ldots, 3a_{n-2} + b_{n-2} + 3c_{n-2}, a_{n-1} + b_{n-1} + c_{n-1}, \\
&\quad 3a_0 + b_0 + 3c_0, a_1 + b_1 + c_1, \ldots, a_{n-2} + b_{n-2} + c_{n-2}, 3a_{n-1} + b_{n-1} + 3c_{n-1}),
\end{aligned}$$

and, we have

$$\begin{aligned}\Pi\varphi_1(r) &= \Pi(a_0+b_0+c_0, a_1+b_1+c_1, \ldots, a_{n-2}+b_{n-2}+c_{n-2}, a_{n-1}+b_{n-1}+c_{n-1},\\ &\quad 3a_0+b_0+3c_0, 3a_1+b_1+3c_1, \ldots, 3a_{n-2}+b_{n-2}+3c_{n-2}, 3a_{n-1}+b_{n-1}+3c_{n-1})\\ &= (a_0+b_0+c_0, 3a_1+b_1+3c_1, \ldots, 3a_{n-2}+b_{n-2}+3c_{n-2}, a_{n-1}+b_{n-1}+c_{n-1},\\ &\quad 3a_0+b_0+3c_0, a_1+b_1+c_1, \ldots, a_{n-2}+b_{n-2}+c_{n-2}, 3a_{n-1}+b_{n-1}+3c_{n-1}).\end{aligned}$$

Hence, $\varphi_1\overline{\beta}(r) = \Pi\varphi_1(r)$. □

Corollary 5.14. *If $\widetilde{C}$ is the Gray image of a cyclic code C of odd length n over R (i.e., $\varphi_1(C) = \widetilde{C}$), then $\Pi(\widetilde{C})$ is a cyclic code of length $2n$ over $\mathbb{Z}_4$.*

Proof. Since C is a cyclic code over R, $\overline{\beta}(C)$ is a $(3+2u+2u^2)$-constacyclic code over R by Corollary 5.3. From Theorem 3.11, we see that $\varphi_1\overline{\beta}(C)$ is a cyclic code of length $2n$ over $\mathbb{Z}_4$. Also, from Theorem 5.13, we have $\Pi\,\varphi_1(C) = \Pi(\widetilde{C}) = \varphi_1\,\overline{\beta}(C)$. This implies that $\Pi(\widetilde{C})$ is a cyclic code of length $2n$ over $\mathbb{Z}_4$. □

Theorem 5.15. *For any $r \in R^n$, we have $\psi_3\overline{\beta}(r) = \eta\psi_3(r)$, where ψ_3 and $\overline{\beta}$ with $\lambda = (1+2u+2u^2)$ are introduced in above and η is a permutation of $\mathbb{Z}_4^{3n}$ defined by $\eta(c_1, c_2, \ldots, c_{3n}) = (c_{\zeta(1)}, c_{\zeta(2)}, \ldots, c_{\zeta(3n)})$ with the permutation $\zeta = (2, n+2)(4, n+4)...(n-1, 2n-1)$ of $\{1, 2, 3, \ldots, 3n\}$.*

Proof. Let $r = (r_0, r_1, \ldots, r_{n-1}) \in R^n$, where $r_i = a_i + ub_i + u^2c_i \in R$ and $a_i, b_i, c_i \in \mathbb{Z}_4$ for $i = 0, 1, \ldots, n-1$. Now, $(1+2u+2u^2)(a_i+ub_i+u^2c_i) = a_i + u(2a_i+b_i) + u^2(2a_i+2b_i+c_i)$ and $\psi_3(a_i + u(2a_i+b_i) + u^2(2a_i+2b_i+c_i)) = (2b_i+c_i, 2a_i+c_i, 2c_i)$. Then

$$\begin{aligned}\psi_3\overline{\beta}(r) &= \psi_3(r_0, \lambda r_1, r_2, \ldots, \lambda r_{n-2}, r_{n-1})\\ &= (2a_0+c_0, 2b_1+c_1, 2a_2+c_2, \ldots, 2b_{n-2}+c_{n-2}, 2a_{n-1}+c_{n-1}, 2b_0+c_0, 2a_1+c_1,\\ &\quad 2b_2+c_2, \ldots, 2a_{n-2}+c_{n-2}, 2b_{n-1}+c_{n-1}, 2c_0, 2c_1, 2c_2, \ldots, 2c_{n-2}, 2c_{n-1}),\end{aligned}$$

and, we have

$$\begin{aligned}\eta\psi_3(r) &= \eta(2a_0+c_0, 2a_1+c_1, 2a_2+c_2, \ldots, 2a_{n-2}+c_{n-2}, 2a_{n-1}+c_{n-1}, 2b_0+c_0, 2b_1+c_1,\\ &\quad 2b_2+c_2, \ldots, 2b_{n-2}+c_{n-2}, 2b_{n-1}+c_{n-1}, 2c_0, 2c_1, \ldots, 2c_{n-2}, 2c_{n-1})\\ &= (2a_0+c_0, 2b_1+c_1, 2a_2+c_2, \ldots, 2b_{n-2}+c_{n-2}, 2a_{n-1}+c_{n-1}, 2b_0+c_0, 2a_1+c_1,\\ &\quad 2b_2+c_2, \ldots, 2a_{n-2}+c_{n-2}, 2b_{n-1}+c_{n-1}, 2c_0, 2c_1, 2c_2, \ldots, 2c_{n-2}, 2c_{n-1}).\end{aligned}$$

Hence, $\psi_3\overline{\beta}(r) = \eta\psi_3(r)$. □

Corollary 5.16. *If $\widetilde{C}$ is the Gray image of a cyclic code C of odd length n over R (i.e., $\psi_3(C) = \widetilde{C}$), then $\eta(\widetilde{C})$ is the permutation equivalent to a quasi-cyclic code of index 3 and length $3n$ over $\mathbb{Z}_4$.*

Proof. Since C is a cyclic code over R, $\overline{\beta}(C)$ is a $(1+2u+2u^2)$-constacyclic code over R by Corollary 5.3. From Theorem 3.6, we see that $\psi_3\,\overline{\beta}(C)$ is permutation equivalent to a quasi-cyclic code of index 3 and length $3n$ over $\mathbb{Z}_4$. By Theorem 5.15, we have $\eta\,\psi_3(C) = \eta(\widetilde{C}) = \psi_3\,\overline{\beta}(C)$. This implies that $\eta(\widetilde{C})$ is permutation equivalent to a quasi-cyclic code of index 3 and length $3n$ over $\mathbb{Z}_4$. □

Theorem 5.17. *For any $r \in R^n$, we have $\varphi_3\,\overline{\beta}(r) = \eta\,\varphi_3(r)$, where φ_3, $\overline{\beta}$ with $\lambda = (3+2u+2u^2)$ and η are introduced in above.*

Proof. Let $r = (r_0, r_1, \ldots, r_{n-1}) \in R^n$, where $r_i = a_i + ub_i + u^2c_i \in R$ and $a_i, b_i, c_i \in \mathbb{Z}_4$ for $i = 0, 1, \ldots, n-1$. Now, $(3+2u+2u^2)(a_i+ub_i+u^2c_i) = 3a_i + u(2a_i+3b_i) + u^2(2a_i+2b_i+3c_i)$ and $\varphi_3(3a_i + u(2a_i+3b_i) + u^2(2a_i+2b_i+3c_i)) = (c_i, 2a_i+2b_i+3c_i, 2a_i+2b_i+2c_i)$. Then

$$\begin{aligned}\varphi_3\,\overline{\beta}(r) &= \varphi_3(r_0, \lambda r_1, r_2, \ldots, \lambda r_{n-2}, r_{n-1})\\ &= (2a_0+2b_0+3c_0, c_1, 2a_2+2b_2+3c_2, \ldots, c_{n-2}, 2a_{n-1}+2b_{n-1}+3c_{n-1}, c_0,\\ &\quad 2a_1+2b_1+3c_1, c_2, \ldots, 2a_{n-2}+2b_{n-2}+3c_{n-2}, c_{n-1}, 2a_0+2b_0+2c_0, 2a_1+2b_1\\ &\quad +2c_1, 2a_2+2b_2+2c_2, \ldots, 2a_{n-2}+2b_{n-2}+2c_{n-2}, 2a_{n-1}+2b_{n-1}+2c_{n-1}),\end{aligned}$$

and, we have

$$\begin{aligned}\eta\varphi_3(r) &= \eta(2a_0+2b_0+3c_0, 2a_1+2b_1+3c_1, \ldots, 2a_{n-1}+2b_{n-1}+3c_{n-1}, c_0, c_1, \ldots, c_{n-1},\\ &\quad 2a_0+2b_0+2c_0, 2a_1+2b_1+2c_1, \ldots, 2a_{n-1}+2b_{n-1}+2c_{n-1})\\ &= (2a_0+2b_0+3c_0, c_1, 2a_2+2b_2+3c_2, \ldots, c_{n-2}, 2a_{n-1}+2b_{n-1}+3c_{n-1}, c_0, 2a_1+2b_1\\ &\quad +3c_1, c_2, \ldots, 2a_{n-2}+2b_{n-2}+3c_{n-2}, c_{n-1}, 2a_0+2b_0+2c_0, 2a_1+2b_1+2c_1, 2a_2+\\ &\quad 2b_2+2c_2, \ldots, 2a_{n-2}+2b_{n-2}+2c_{n-2}, 2a_{n-1}+2b_{n-1}+2c_{n-1}).\end{aligned}$$

Hence, $\varphi_3\,\overline{\beta}(r) = \eta\,\varphi_3(r)$. □

Corollary 5.18. *If $\widetilde{C}$ is the Gray image of a cyclic code C of odd length n over R (i.e., $\varphi_3(C) = \widetilde{C}$), then $\eta(\widetilde{C})$ is permutation equivalent to a quasi-cyclic code of index 3 and length $3n$ over $\mathbb{Z}_4$.*

Proof. Since C is a cyclic code over R, $\overline{\beta}(C)$ is a $(3+2u+2u^2)$-constacyclic code over R by Corollary 5.3. From Theorem 3.15, we see that $\varphi_3\,\overline{\beta}(C)$ is permutation equivalent to a quasi-cyclic code of index 3 and length $3n$ over $\mathbb{Z}_4$. By Theorem 5.17, we have $\eta(\widetilde{C}) = \varphi_3\,\overline{\beta}(C)$. This implies that $\eta(\widetilde{C})$ is permutation equivalent to a quasi-cyclic code of index 3 and length $3n$ over $\mathbb{Z}_4$. □

6. Conclusion

In this article, we discussed the λ-constacyclic codes over the ring $R = \mathbb{Z}_4 + u\mathbb{Z}_4 + u^2\mathbb{Z}_4$, $u^3 = 0$ with $\lambda = (1+2u+2u^2)$ and $(3+2u+2u^2)$. We have shown that the Gray images of λ-constacyclic codes over R are cyclic, quasi-cyclic and permutation equivalent to quasi-cyclic codes over $\mathbb{Z}_4$ similar to the results obtained in [10, 11, 13]. It is also proved that Gray images of skew λ-constacyclic codes are quasi-cyclic codes over $\mathbb{Z}_4$. Furthermore, the structure of λ-constacyclic codes of odd length n over R are determined with some suitable examples.

7. Acknowledgements

The first author is grateful to the Council of Scientific and Industrial Research, Government of India for financial support through fellowship with Award no. 09/476(0089)/2019-EMR-I, and the Department of Mathematics, Manipur University, for providing research facilities. The authors also sincerely thank the referees for the useful comments which help to improve this paper.

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