

On chromatic number and clique number in k -step Hamiltonian graphs

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Abstract: A graph G of order n is called k -step Hamiltonian for $k \geq 1$ if we can label the vertices of G as v_1, v_2, \dots, v_n such that $d(v_n, v_1) = d(v_i, v_{i+1}) = k$ for $i = 1, 2, \dots, n-1$. The (vertex) chromatic number of a graph G is the minimum number of colors needed to color the vertices of G so that no pair of adjacent vertices receive the same color. The clique number of G is the maximum cardinality of a set of pairwise adjacent vertices in G . In this paper, we study the chromatic number and the clique number in k -step Hamiltonian graphs for $k \geq 2$. We present upper bounds for the chromatic number in k -step Hamiltonian graphs and give characterizations of graphs achieving the equality of the bounds. We also present an upper bound for the clique number in k -step Hamiltonian graphs and characterize graphs achieving equality of the bound.

Keywords: Hamiltonian graph, k -step Hamiltonian graph, chromatic number, clique number

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1. Introduction

Throughout this paper, $G = (V(G), E(G))$ is a simple graph with $V(G)$ as its vertex set and $E(G)$ as its edge set. The open *neighborhood* of a vertex $v \in V(G)$, denoted by $N_G(v)$ (or just $N(v)$) is the set $\{u : uv \in E(G)\}$. The *degree* of a vertex v , $\deg_G(v)$ (or just $\deg(v)$) is the number of neighbors of v in G , that is, $\deg(v) = |N_G(v)|$. We refer to $\delta(G)$ and $\Delta(G)$ as the minimum and maximum degree among all vertices of G , respectively. Also, a vertex $v \in V(G)$ is called a *pendant vertex* if $\deg(v) = 1$. Let K_n , C_n and P_n be a complete graph, a path and a cycle with n vertices, respectively. The *distance* between two vertices u and v in G , $d(u, v)$, is the minimum length among all paths between u and v and the maximum distance $d(u, v)$ among two vertices u, v of G is the *diameter* of G and is denoted by $\text{diam}(G)$. For a set $A \subseteq V(G)$, $G[A]$ is the subgraph of G induced by A . A *circulant graph* $C_m(a_1, a_2, \dots, a_k)$ with $0 < a_1 < a_2 < \dots < a_k < \frac{m+1}{2}$ is a graph of order m with vertices $\{v_1, v_2, \dots, v_m\}$ such that v_i is adjacent to v_{i+a_j} for all $a_j \in \{a_1, a_2, \dots, a_k\}$, where the summation $i + a_j$ is taken modulo m . For a graph G , the *corona graph* of G , $\text{cor}(G)$, is the graph obtained by adding a pendant vertex to every vertex of G . For other notation and terminology not defined here, we refer to [15].

A proper *vertex coloring* of a graph G is an assignment of colors to the vertices of G such that every pair of adjacent vertices receives different colors. The *chromatic number* of a graph G , denoted by $\chi(G)$, is the minimum number of colors required in a proper vertex coloring of G . If G has a proper vertex coloring of k colors, then $\chi(G) \leq k$. The study of chromatic number of graph is an active area of research, see for example [6, 11–14]. A *clique* in a graph G is a set S of pairwise adjacent vertices and the number of vertices in the maximum clique is referred to as the *clique number* of G , denoted by $\omega(G)$.

A graph G is said to be *Hamiltonian* if G has a spanning cycle referred as a *Hamiltonian cycle*. Although the Hamiltonicity problem is a widely studied subject in graph theory, no exact characterization for the existence of the Hamiltonian cycle has been found. A good survey on the developments of Hamiltonicity problem can be found in [4]. The concept of Hamiltonicity has been extended by Lau et al. [9] to k -step Hamiltonicity as follows: For a graph G of order n , if we can arrange the vertices as v_1, v_2, \dots, v_n such that $d(v_n, v_1) = d(v_i, v_{i+1}) = k$ for $i = 1, 2, \dots, n-1$ and $k \geq 1$, then we call G a k -step Hamiltonian (or just k -SH) graph with $v_1, v_2, \dots, v_n, v_1$ as the k -step Hamiltonian (or just k -SH) walk of G . The k -step Hamiltonicity of some family of graphs including trees, tripartite graphs, cycles, grid graphs, torus graphs, cubic graphs and subdivision of cycles, have been studied, see [1, 2, 5, 7–10].

In this paper, we continue the study of k -SH graphs by proving bounds for the chromatic number and the clique number in k -SH graphs, where $k \geq 2$. In Section 2 we give a proof for the fact that a k -SH graph has at least $2k + 1$ vertices. In

Section 3, we present upper bounds for the chromatic number in k -SH graphs and give characterizations of graphs achieving the equality of the bounds. In Section 4, we present an upper bound for the clique number in k -SH graphs and characterize graphs achieving equality of the bound. We make use of the following known results.

Theorem 1 (Brooks' Theorem). *For every connected graph G other than an odd cycle or a complete graph, $\chi(G) \leq \Delta(G)$.*

Theorem 2 (Chartrand et al. [3]). *If G is a connected graph of order n and diameter d , then $\chi(G) \leq n - d + 1$.*

2. Preliminary

The following theorem has played an important role in several works on the subject of k -step Hamiltonian graphs, while the proof given in [8] does not have any argument for the bound $n \geq 2k + 1$.

Theorem 3 (Lau et al. [8]). *The cycle C_n for $n \geq 3$ is k -SH for $k \geq 2$ if and only if $n \geq 2k + 1$ and $\gcd(n, k) = 1$.*

We provide in the following a proof for the above bound.

Theorem 4. *If G is a k -SH graph of order n for $k \geq 1$, then $n \geq 2k + 1$.*

Proof. The result is obvious if $k = 1$. Thus assume that $k \geq 2$. Let G be a k -SH graph of order n , and let $W : v_1, v_2, \dots, v_n, v_1$ be a k -SH walk in G . Thus $d(v_n, v_1) = d(v_i, v_{i+1}) = k$ for each $i = 1, 2, \dots, n - 1$. Since $d(v_1, v_2) = k$, let x_0, x_1, \dots, x_k be a shortest path between v_1 and v_2 , where $x_0 = v_1$ and $x_k = v_2$. Clearly each of x_i , ($i = 1, 2, \dots, k - 1$) lies on W . We follow the walk W starting from v_1 . We relabel the vertices x_1, \dots, x_{k-1} according to their place in W . Let the relabeled vertices be $x_{j_1}, x_{j_2}, \dots, x_{j_{k-1}}$, where x_{j_r} is before x_{j_s} if $r < s$. For each $r \in \{1, 2, \dots, k - 1\}$, x_{j_r} has two consecutive vertices on W namely x'_{j_r} and x''_{j_r} , and without loss of generality, assume that x'_{j_r} is on the left side of x_{j_r} and x''_{j_r} is on the right side of x_{j_r} in W . Clearly $\{x'_{j_1}, x''_{j_1}\} \cap \{v_1, v_2\} = \emptyset$. For each $r = 2, \dots, k - 1$, $\{x'_{j_r}, x''_{j_r}\} \not\subseteq \{v_1, v_2, x_{j_s}, x'_{j_s}, x''_{j_s}\}$ for $s < r$. So for each $r = 2, \dots, k - 1$, $\{x'_{j_r}, x''_{j_r}\} - \{v_1, v_2, x_{j_s}, x'_{j_s}, x''_{j_s} : s < r\} \neq \emptyset$. So $n \geq k + 1 + 2 + k - 2 = 2k + 1$. \square

For the sharpness of the bound in Theorem 4, consider the graph $G = C_{2k+1}$ for $k \geq 1$. By Theorem 3, G is k -SH.

3. Chromatic number

We begin with the following bound.

Theorem 5. *If G is a k -SH graph of order n for $k \geq 2$, then $\chi(G) \leq \lceil \frac{n}{2} \rceil$. If equality holds, then $k = 2$.*

Proof. Let G be a k -SH graph of order n for $k \geq 2$. Without loss of generality, assume that $v_1, v_2, \dots, v_n, v_1$ is a k -SH walk of G . Clearly, $d(v_i, v_{i+1}) = d(v_n, v_1) = k$ for $i = 1, 2, \dots, n-1$. We define a vertex coloring c of G as follows: If n is even, then for $i = 0, 1, \dots, \frac{n}{2} - 1$, we let $c(v_{2i+1}) = c(v_{2i+2}) = i + 1$. If n is odd, then for $i = 0, 1, \dots, \frac{n-1}{2} - 1$, we let $c(v_{2i+1}) = c(v_{2i+2}) = i + 1$ and $c(v_n) = \frac{n-1}{2} + 1$. Clearly no pair of adjacent vertices receive the same color. The number of colors used in this proper vertex coloring c is $\lceil \frac{n}{2} \rceil$. Therefore, $\chi(G) \leq \lceil \frac{n}{2} \rceil$, as required.

Assume now that $\chi(G) = \lceil \frac{n}{2} \rceil$. Since G is k -SH for $k \geq 2$, clearly G is connected and G is not a complete graph. If G is an odd cycle, then $\chi(G) = 3$, that is, $\frac{n+1}{2} = 3$. This means $G = C_5$. By Theorem 3, C_5 is 2-SH. Thus assume that G is not an odd cycle. Since G is not a complete graph or an odd cycle, by Brooks' Theorem, $\chi(G) \leq \Delta(G)$. Therefore we have $\lceil \frac{n}{2} \rceil \leq \Delta(G)$. Let v be a vertex of maximum degree, that is $\deg(v) = \Delta(G)$ and let $A = V(G) - N[v]$.

Assume that n is even. Then, $n \leq 2\Delta(G)$ and $|A| = n - \Delta(G) - 1 \leq \Delta(G) - 1$. Since G is k -SH, there exist two vertices $y, z \in A$ such that y, v, z are consecutive vertices in a k -SH walk of G . Let W be such k -SH walk and $W' = W - \{y, v, z\}$. Then it remains $\Delta(G)$ vertices from $N(v)$ in W' and some vertices of A . Thus, clearly there exist two consecutive vertices α, β in W with $\alpha, \beta \in N(v)$. Therefore, $k = 2$. The case n odd is similarly verified. \square

We next show that for each $n \geq 5$, there exists a graph achieving equality of the bound in Theorem 5.

Proposition 1. *For each $n \geq 5$, there exists a 2-SH graph G of order n with $\chi(G) = \lceil \frac{n}{2} \rceil$.*

Proof. If G is a graph of order $n \leq 4$, then by Theorem 4, G is not 2-SH. Thus, we consider $n \geq 5$. Let $G = G_n$ be a graph obtained from the complete graph K_n , ($n \geq 5$) with $V(K_n) = \{v_1, v_2, \dots, v_n\}$ by removing the edges of the Hamiltonian cycle $v_1, v_2, \dots, v_n, v_1$. The graph G is connected since K_n , $n \geq 5$ has $\lfloor \frac{n-1}{2} \rfloor$ edge-disjoint Hamiltonian cycles. Note that for each $i = 1, 2, \dots, n$, v_i is adjacent to every v_j for $j \notin \{i+1, i-1\}$ with the summations $i+1$ and $i-1$ are taken in modulo n and so $d(v_i, v_{i+1}) = 2$. Therefore, $v_1, v_2, \dots, v_n, v_1$ is a 2-SH walk of G and thus G is 2-SH. By Theorem 5, $\chi(G) \leq \lceil \frac{n}{2} \rceil$.

We next prove that $\chi(G) \geq \lceil \frac{n}{2} \rceil$. For even n , clearly $\{v_{2i} : 1 \leq i \leq \frac{n}{2}\}$ is a clique. Therefore, $\chi(G) \geq \omega(G) \geq \frac{n}{2}$. Now, we prove by induction on n that for each odd $n \geq 5$, $\chi(G) \geq \lceil \frac{n}{2} \rceil = \frac{n+1}{2}$. For the base step assume that $n = 5$. Then $G = G_5 = C_5$. Clearly, $\chi(G) = 3 \geq \frac{n+1}{2}$. Assume the result holds for all odd n' with $5 \leq n' < n$.

Now, consider the graph $G = G_n$ for odd n . Let c be a proper vertex coloring of G . Since G is not a complete graph, there exist i, j such that $c(v_i) = c(v_j)$. Clearly, $j = i - 1$ or $j = i + 1$. Without loss of generality, we assume that $j = i + 1$. Now, remove v_i and v_{i+1} and also the edge $v_{i-1}v_{i+2}$ to obtain G_{n-2} . Clearly, the restriction of c on G_{n-2} is a proper vertex coloring of G_{n-2} . By the induction hypothesis, we have $|\{c(v) : v \in V(G_{n-2})\}| \geq \lceil \frac{n-2}{2} \rceil = \frac{n-1}{2}$. Therefore, for the graph $G = G_n$, we have $|\{c(v) : v \in V(G)\}| \geq 1 + \lceil \frac{n-2}{2} \rceil = \frac{n+1}{2}$, as desired. \square

As another example of families of graphs achieving the equality in the bound in Theorem 5, consider the complete graph $K_{\frac{n}{2}}$ for even $n \geq 6$ with vertices $v_1, v_2, \dots, v_{\frac{n}{2}}$. Let $G = \text{cor}(K_{\frac{n}{2}})$ with $V(G) = V(K_{\frac{n}{2}}) \cup \{u_1, u_2, \dots, u_{\frac{n}{2}}\}$ such that u_i is adjacent to v_i in G for $i = 1, 2, \dots, \frac{n}{2}$. Clearly, $d(v_i, u_j) = 2$ for $i \neq j$. Also, it is clear that $\chi(G) = \chi(K_{\frac{n}{2}}) = \frac{n}{2}$ because we can color each vertex u_i for $i = 1, 2, \dots, \frac{n}{2}$ with one of the color in the set $\{1, 2, \dots, \frac{n}{2}\}$ that is different from the color of v_i . The 2-SH walk is then given by the sequence of vertices $v_1, u_2, v_3, u_4, \dots, u_{\frac{n}{2}-1}, v_{\frac{n}{2}}, u_1, v_2, \dots, v_{\frac{n}{2}-1}, u_{\frac{n}{2}}, v_1$ when $\frac{n}{2}$ is odd and $u_1, v_2, u_3, v_4, \dots, u_{\frac{n}{2}-1}, v_{\frac{n}{2}}, u_2, v_1, u_{\frac{n}{2}}, v_{\frac{n}{2}-1}, \dots, u_4, v_3, u_1$ when $\frac{n}{2}$ is even.

Now, we propose the following problem.

Problem 1. Characterize all 2-SH graphs G of order n with $\chi(G) = \lceil \frac{n}{2} \rceil$.

Theorem 6. *There is no forbidden induced subgraph characterization for 2-SH graphs of order n with chromatic number $\lceil \frac{n}{2} \rceil$.*

Proof. Let G be a graph of order a . The result is obvious if $a \leq 2$. Thus, assume that $a \geq 3$. Let $b = 2 \lceil \frac{a}{2} \rceil$. Then we form the graph $\text{cor}(K_b)$. Identify each vertex of G with a pendant vertex of $\text{cor}(K_b)$ to obtain a graph H of order $2b$. Since the adding edges are between pendant vertices of $\text{cor}(K_b)$, clearly $\chi(H) = \chi(\text{cor}(K_b)) = b$. As before, one can easily see that the graph $\text{cor}(K_b)$ is 2-SH and no two pendant vertices of $\text{cor}(K_b)$ are consecutive in the 2-SH walk. Therefore, a 2-SH walk in $\text{cor}(K_b)$ is also a 2-SH walk in H and thus H is 2-SH. Thus G is an induced subgraph of H , where H is a 2-SH graph with $\chi(H) = \lceil \frac{|V(H)|}{2} \rceil$. \square

Proposition 2. *The difference $\lceil \frac{|V(G)|}{2} \rceil - \chi(G)$ can be arbitrarily large in a 2-SH graph G .*

Proof. Let $n \geq 7$ be an odd integer, and $r = \frac{n+1}{2} - 3$. Consider the graph $G = C_{2(r+3)-1}$. By Theorem 3, G is 2-SH. Then, $\lceil \frac{|V(G)|}{2} \rceil - \chi(G) = \lceil \frac{2(r+3)-1}{2} \rceil - 3 = r = \frac{n-5}{2}$. \square

Theorem 7. *For each $k \geq 3$, there exists a k -SH graph G of order n with $\chi(G) = \lceil \frac{n}{k} \rceil$.*

Proof. Let $k \geq 3$ and consider the graph $G = C_{2k+1}$. By Theorem 3, G is k -SH. Since G is an odd cycle of order $2k + 1$, then $\chi(G) = 3 = \lceil \frac{2k+1}{k} \rceil$. \square

We propose the following conjecture.

Conjecture 1. If G is a k -SH graph of order n for $k \geq 1$, then $\chi(G) \leq \lceil \frac{n}{k} \rceil$.

We next present another upper bound for the chromatic number in a k -SH graph.

Theorem 8. If G is a k -SH graph of order n for $k \geq 2$, then $\chi(G) \leq n - k$, with equality if and only if $k = 2$ and $G = C_5$.

Proof. Let G be a k -SH graph of order n for $k \geq 2$. Clearly, G is connected. Since G is k -SH, we have $\text{diam}(G) \geq k$ and thus, by Theorem 2, we have $\chi(G) \leq n - k + 1$. If $\chi(G) = n - k + 1$, then we obtain the contradiction $2k + 1 \leq n \leq 2k - 1$ from Theorems 4 and 5. Thus $\chi(G) \leq n - k$.

We next prove the equality part. Assume that $\chi(G) = n - k$. Since G is k -SH, it follows by Theorem 4 that $n \geq 2k + 1$. If $k \geq 3$, then by Theorem 5, $\chi(G) < \lceil \frac{n}{2} \rceil$ and so $n < 2k + 1$, a contradiction. Therefore, $k = 2$ and thus $n \geq 5$. Now, we have $\chi(G) = n - 2 \leq \lceil \frac{n}{2} \rceil$. If n is even, then $n \leq 4$, a contradiction. If n is odd, then $n \leq 5$ and thus $n = 5$. Therefore, G is a 2-SH graph of order 5. We can easily check that $G = C_5$.

The converse is clear. \square

Let $C_m(1, 2)$ be the circulant graph of order m . Abd Aziz et al. [1] obtained the following sufficient condition for the graph $C_m(1, 2)$ to be k -SH.

Theorem 9 (Abd Aziz et al. [1]). If $\gcd(m, 2j - 1) = 1$ for $m \geq 6$ and $2 \leq j \leq \lceil \frac{m-1}{4} \rceil$, then $C_m(1, 2)$ is j -SH.

They then gave a construction namely B -construction that produces a $(k+1)$ -SH graph from any given k -SH graph G . The construction is as follows:

B -Construction. Let G be a k -SH graph of order n for $k \geq 1$ with a given k -SH walk $v_1, v_2, \dots, v_n, v_1$. Consider the graph $\text{cor}(G)$ with the new n vertices u_1, u_2, \dots, u_n such that u_i is adjacent to v_i for $i = 1, 2, \dots, n$. Then, the B -construction produces a graph $B(G)$ from G as follows:

- (i) For odd n , $B(G) = \text{cor}(G)$.
- (ii) For even n , $B(G)$ is obtained from $\text{cor}(G)$ by the following scheme:

Step 1. For an integer m , $m \geq 6$ and $k \leq \lceil \frac{m-1}{4} \rceil - 1$ with $\gcd(m, 2k + 1) = 1$, the circulant graph $C_m(1, 2)$ is $(k + 1)$ -SH by Theorem 9. Let $C_m^1(1, 2)$ and $C_m^2(1, 2)$ be two copies of $C_m(1, 2)$. Without loss of generality, assume that

$u_{1,1}, u_{1,2}, \dots, u_{1,m}, u_{1,1}$ (respectively $u_{2,1}, u_{2,2}, \dots, u_{2,m}, u_{2,1}$) is a $(k+1)$ -SH walk of $C_m^1(1, 2)$ (respectively $C_m^2(1, 2)$). Note that $d(u_{i,j}, u_{i,j+1}) = k+1$ for $i = 1, 2$ and for $j = 1, 2, \dots, m$, where the summation $j+1$ is taken in modulo m .

Step 2. Identify the vertices $u_{1,1}, u_{1,m}, u_{2,1}$ and $u_{2,m}$ to the vertices u_n, u_1, v_n and v_1 , respectively.

The i -th iterated construction B of G , $B^i(G)$ for any $i \geq 1$ is defined recursively by $B^1(G) = B(G)$, $B^2(G) = B(B(G))$ and $B^i(G) = B(B^{i-1}(G))$ for $i \geq 2$.

Theorem 10 (Abd Aziz et al. [1]). *If G is k -SH for $k \geq 1$, then $B(G)$ is $(k+1)$ -SH.*

From the B -construction above, we can obtain the following two results.

Lemma 1. *If G is a k -SH graph of order n with $\chi(G) \geq 5$ and H is a graph obtained from G by B -construction, then $\chi(H) = \chi(G)$.*

Proof. Let G be a k -SH graph of order n with $\chi(G) \geq 5$ and H be a graph obtained from G by applying the B -construction. If n is odd, then by the above construction, $H = B(G) = \text{cor}(G)$. Clearly, $\chi(H) = \chi(G)$ because in H , we can color every vertex u_i for $i = 1, 2, \dots, n$ with one of the color in the set $\{1, 2, \dots, \chi(G)\}$ that is not the color of v_i .

If n is even, then $H = B(G)$ is obtained from $\text{cor}(G)$ as described above. Since the circulant graph $C_m(1, 2)$ contains a triangle, $\chi(C_m(1, 2)) \geq 3$ and it is not difficult to see that for $m \geq 6$, $\chi(C_m(1, 2)) = 3$ when $m \equiv 0 \pmod{3}$ and $\chi(C_m(1, 2)) = 4$ when $m \not\equiv 0 \pmod{3}$. Note that in $B(G)$, we have $v_n = u_{2,1}$, $u_n = u_{1,1}$, $v_1 = u_{2,m}$ and $u_1 = u_{1,m}$. As before, we can color $\text{cor}(G)$ with $\chi(G)$ colors. Let c be this coloring. Now, we can color $C_m^1(1, 2)$ and $C_m^2(1, 2)$ in such a way that the vertices $u_{1,1}$ and $u_{1,m}$ receive $c(u_n)$ and $c(u_1)$, respectively, and the vertices $u_{2,1}$ and $u_{2,m}$ receive $c(v_n)$ and $c(v_1)$, respectively. Therefore, we have $\chi(H) = \chi(G)$. \square

Theorem 11. *For each $l \geq 5$, there exists a chain of graphs $H_2 \subseteq H_3 \subseteq H_4 \subseteq \dots$ such that $\chi(H_i) = l$ for each $i \geq 2$, H_i is i -SH, and $\frac{n(H_{i+1})}{n(H_i)} > 2$.*

Proof. Let $l \geq 5$ and $H_2 = G_{2l}$ be the graph defined in the proof of Proposition 1. Then, we know that H_2 is 2-SH with $\chi(H_2) = l$. Now, for each $i > 2$, let $H_i = B^{i-2}(H_2)$. By Theorem 10 and Lemma 1, for each $i > 2$, H_i is an i -SH graph with $\chi(H_i) = l$. Clearly, from the construction of H_i for $i \geq 2$, we have $\frac{n(H_{i+1})}{n(H_i)} > 2$. \square

Corollary 1. For each $k \geq 3$, there exists a k -SH graph G such that $\chi(G) < \frac{n(G)}{2^{k-1}}$.

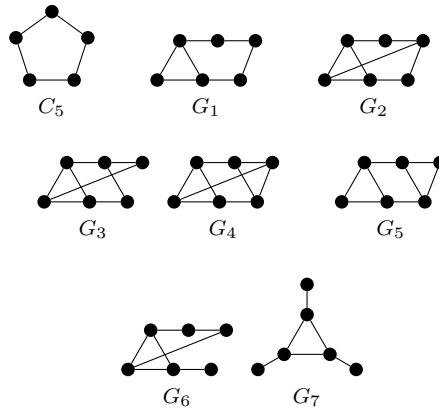


Figure 1. The graphs in \mathcal{F} .

4. Clique number

In this section, we present an upper bound for the clique number in a k -SH graph and characterize the graphs achieving the equality of the bound. Let \mathcal{F} be the family of graphs shown in Figure 1.

Theorem 12. If G is a k -SH graph of order n for $k \geq 2$, then $\omega(G) \leq n - k - 1$, with equality if and only if $k = 2$ and $G \in \mathcal{F}$.

Proof. Let G be a k -SH graph of order n for $k \geq 2$ and S be a maximum clique in G . Let v_1 and v_2 be two consecutive vertices on a k -SH walk of G and assume that $P : x_0, x_1, \dots, x_{k-1}, x_k$ is a shortest path in G from v_1 to v_2 , where $v_1 = x_0$ and $v_2 = x_k$. Clearly, $|S \cap \{x_0, \dots, x_k\}| \leq 2$, otherwise we will have a shorter v_1, v_2 -path. Therefore, $\omega(G) \leq n - k + 1$. Suppose that $\omega(G) = n - k + 1$. Then, $|S \cap \{x_0, \dots, x_k\}| = 2$. Let $x_i, x_{i+1} \in S \cap \{x_0, \dots, x_k\}$. Without loss of generality, we can assume that $x_i \neq v_1$. Then, there is no vertex at distance k from x_i , a contradiction. Thus, $\omega(G) \leq n - k$.

Suppose that $\omega(G) = n - k$. Clearly $1 \leq |S \cap \{x_0, \dots, x_k\}| \leq 2$. Suppose that $|S \cap \{x_0, \dots, x_k\}| = 1$. Let $x_i \in S \cap \{x_0, \dots, x_k\}$. If $x_i = v_1$, then v_2 is the only vertex at distance k from x_i in G , a contradiction. Thus $x_i \neq v_1$. Similarly, $x_i \neq v_2$. But then there is no vertex at distance k from x_i in G , a contradiction. Next suppose that $|S \cap \{x_0, \dots, x_k\}| = 2$. Let $x_i, x_{i+1} \in S \cap \{x_0, \dots, x_k\}$. Since $|S| = n - k$, clearly there exists a vertex $y \notin S \cup V(P)$. Without loss of generality, assume that $x_i \neq v_1$.

Then there exists at most one vertex at distance k from x_i (possibly $d(x_i, y) = k$), a contradiction.

We conclude that, $\omega(G) \leq n - k - 1$, as desired. We next prove the equality part. Assume that $\omega(G) = n - k - 1$. Let $v_1, v_2, x_0, x_1, \dots, x_k$ and S be as described above.

Claim 1. $k = 2$.

Proof of Claim 1. Suppose $k \geq 3$. Clearly $|S| \geq 2$. According to $|S \cap \{x_0, \dots, x_k\}| \leq 2$, we have three possibilities.

Suppose that $|S \cap \{x_0, \dots, x_k\}| = 0$. If there exists a vertex x_i for $i = 1, 2, \dots, k - 1$ such that x_i is adjacent to some vertex $y \in S$, then there is no vertex at distance k from x_i in G , a contradiction. Therefore, every vertex x_i for $i = 1, 2, \dots, k - 1$ has no neighbor in S . But then, v_2 is the only vertex at distance k from v_1 in G , a contradiction.

Next suppose that $|S \cap \{x_0, \dots, x_k\}| = 1$. Let $x_i \in S \cap \{x_0, \dots, x_k\}$. Since $|S| = n - k - 1$, there exists a vertex $y \notin S \cup V(P)$. Suppose $x_i = v_1$. If $d(v_1, y) \neq k$, then v_2 is the only vertex at distance two from v_1 in G , a contradiction. Therefore, $d(v_1, y) = k$ and thus y is adjacent to x_{k-1} . Clearly, y is not adjacent to x_0, x_1, \dots, x_{k-2} . But now, x_1 is at distance at most $k - 1$ to other vertices of G , a contradiction. Therefore, $x_i \neq v_1$ and similarly $x_i \neq v_2$. But then, x_i is at distance k to at most one vertex of G (possibly $d(x_i, y) = k$), a contradiction.

Next, suppose that $|S \cap \{x_0, \dots, x_k\}| = 2$. Let $x_i, x_{i+1} \in S \cap \{x_0, \dots, x_k\}$. Since $|S| = n - k - 1$, there exist two vertices $y, z \notin S \cup V(P)$. Assume that $x_i = v_1$. Since G is k -SH, there is another vertex at distance k from v_1 different from v_2 . Clearly, that vertex is either y or z . Without loss of generality, assume that $d(v_1, y) = k$. Since $k \geq 3$, y has no neighbor in S . Also, it is clear that y is not adjacent to x_1, x_2, \dots, x_{k-2} . Thus, y is adjacent to x_{k-1} . But now, x_{i+1} is at distance k to at most one vertex of G (possibly $d(x_{i+1}, z) = k$), a contradiction. Therefore, $x_i \neq v_1$. Similarly, $x_i \neq x_{k-1}$. Now, assume that $x_i = x_1$. Clearly $d(x_i, y) = d(x_i, z) = k$ since G is k -SH. Again, since $k \geq 3$, y and z have no neighbor in S . Also, y and z are not adjacent to v_1, x_1, \dots, x_{k-1} . Thus, both y and z are adjacent to v_2 . But now, v_2 is the only vertex at distance k from v_1 , a contradiction. Therefore, $x_i \neq x_1$. Similarly, $x_i \neq x_{k-2}$ ($k \geq 4$). Next, consider $x_i \neq x_1$ or $x_i \neq x_{k-2}$ ($k \geq 5$). Again $d(x_i, y) = d(x_i, z) = k$. Clearly, y and z are adjacent to some vertex in $\{v_1, x_1, \dots, x_{k-1}, v_2\} - \{x_i\}$. But, every x_i, y -path and every x_i, z -path created by joining y and z to any of those vertices has length at most $k - 1$, a contradiction. So the proof of Claim 1 is complete. \diamond

We now prove that $G \in \mathcal{F} = \{C_5, G_1, G_2, G_3, G_4, G_5, G_6, G_7\}$. Note that $|S| = \omega(G) = n - 3$. Let v_1 and v_2 be two consecutive vertices on a 2-SH walk of G and assume that $P : x_0, x_1, x_2$ is a shortest path in G from v_1 to v_2 , where $v_1 = x_0$ and $v_2 = x_2$. Since $k = 2$, clearly $|S| \geq 2$. According to $|S \cap \{x_0, x_1, x_2\}| \leq 2$, we have three cases.

Case 1. $|S \cap \{x_0, x_1, x_2\}| = 0$.

Since G is 2-SH, there exist two vertices y_1, y_2 at distance two from x_1 and clearly, $y_1, y_2 \in S$. Therefore, x_1 is not adjacent to y_1 and also not adjacent to y_2 . Assume that $|S| \geq 3$ and consider $y_3 \in S$. Since v_1, v_2 are consecutive vertices in the 2-SH walk, we have at least four other vertices x_1, y_1, y_2, y_3 in the 2-SH walk of G . Clearly, two vertices in S will be consecutive in the 2-SH walk, a contradiction. Thus, we assume that $|S| = 2$. Since G is 2-SH, there exists a vertex at distance two from v_1 different from v_2 and clearly, that vertex is from S . Without loss of generality, let $d(v_1, y_2) = 2$. Thus v_1 is adjacent to y_1 . Since $d(x_1, y_2) = 2$ and y_2 is not adjacent to v_1 , clearly y_2 is adjacent to v_2 . If y_1 is adjacent to v_2 , then x_1 is the only vertex at distance two from y_1 , a contradiction. Therefore, y_1 is not adjacent to v_2 . Thus, we have $G = C_5$.

Case 2. $|S \cap \{x_0, x_1, x_2\}| = 1$.

Let $x_i \in S \cap \{x_0, x_1, x_2\}$. Since $|S| = n - 3$, there exists a vertex $y \notin S \cup V(P)$. If $x_1 \in S$, then there exists at most one vertex at distance two from x_1 in G (possibly $d(x_1, y) = 2$), a contradiction. Therefore, $x_1 \notin S$. Without loss of generality, assume that $x_0 \in S$. Clearly, $d(x_0, y) = 2$ since G is 2-SH. Now, follow the 2-SH walk starting from x_2, x_0, y . Assume that $|S| \geq 4$ and consider $y_1, y_2, y_3 \in S \setminus \{x_0\}$. The next vertex after y in the 2-SH walk is either x_1 or some vertex in $S \setminus \{x_0\}$. Suppose the next vertex after y is x_1 . Then the next vertex after x_1 should be in $S \setminus \{x_0\}$, say y_1 . But then the next vertex after y_1 in the 2-SH walk does not exist, a contradiction. Therefore, the next vertex after y in the 2-SH walk is from $S \setminus \{x_0\}$, say y_1 . Then the next vertex after y_1 should be x_1 and the next vertex after x_1 is from $S \setminus \{x_0, y_1\}$, say y_2 . But again there is no next vertex after y_2 in the 2-SH walk, a contradiction. Therefore $2 \leq |S| \leq 3$.

Assume that $|S| = 2$. Let $S = \{x_0, y_1\}$. Since G is 2-SH, the two vertices at distance two from x_1 are y_1 and y . Therefore, x_1 is not adjacent to y_1 and also not adjacent to y . Since $d(x_0, y) = 2$, y is adjacent to both y_1 and x_2 . And since $\omega(G) = 2$, x_2 is not adjacent to y_1 . Thus we have $G = C_5$.

Next assume that $|S| = 3$. Let $S = \{x_0, y_1, y_2\}$.

Assume that y is adjacent to x_1 . Then, the next vertex after y in the 2-SH walk is either y_1 or y_2 . Without loss of generality, assume that the next vertex after y is y_1 . The next vertex after y_1 should be x_1 . Thus y_1 is not adjacent to x_1 . Then, the next vertex after x_1 should be y_2 and thus x_1 is not adjacent to y_2 . Now, the vertices $x_2, x_0, y, y_1, x_1, y_2$ are consecutive in the 2-SH walk. Assume that y is adjacent to x_2 . Suppose y is not adjacent to y_2 . Since $d(y, y_1) = 2$, y_1 is adjacent to x_2 . If y_2 is adjacent to x_2 , then x_0 is the only vertex at distance two from x_2 in G , a contradiction. Therefore y_2 is not adjacent to x_2 and thus we have $G = G_5$. Suppose next y is adjacent to y_2 . If y_2 is adjacent to x_2 , then x_1 is the only vertex at distance two from y_2 in G , a contradiction. Therefore y_2 is not adjacent to x_2 . If y_1 is adjacent to x_2 , then we have $G = G_4$, otherwise we have $G = G_5$. Assume next y is not adjacent to x_2 . Since $d(y, y_1) = 2$, y is adjacent to y_2 . If y_2 is adjacent to x_2 , then x_1 is the only vertex at distance two from y_2 in G , a contradiction. Therefore y_2 is not adjacent to x_2 . If y_1 is not adjacent to x_2 , again x_1 is the only vertex at

distance two from y_2 in G , a contradiction. Therefore y_1 is adjacent to x_2 . Thus we have $G = G_3$.

Next, assume that y is not adjacent to x_1 . Then y is adjacent to at least one of y_1 or y_2 since $d(x_0, y) = 2$. Without loss of generality, assume that y is adjacent to y_2 . If x_1 is adjacent to y_2 , then y_2 is at distance two to at most one vertex of G , a contradiction. Therefore, x_1 is not adjacent to y_2 . If y_2 is adjacent to x_2 , then x_1 is the only vertex at distance two from y_2 in G , a contradiction. Thus, y_2 is not adjacent to x_2 . Suppose y is adjacent to y_1 . Then clearly y is adjacent to x_2 , for otherwise there is no next vertex after y in the 2-SH walk. Then, x_1 is the next vertex after y in the 2-SH walk. Then, the next vertex after x_1 is either y_1 or y_2 . But then the next vertex in the 2-SH walk does not exist, a contradiction. Therefore, y is not adjacent to y_1 . Assume that y is not adjacent to x_2 . Then the next vertex after y in the 2-SH walk is y_1 . Then the next vertex after y_1 is x_1 . Thus y_1 is not adjacent to x_1 . Then the next vertex after x_1 is y_2 . If y_1 is not adjacent to x_2 , then, x_1 is the only vertex at distance two from y_2 in G , a contradiction. Therefore, y_1 is adjacent to x_2 . Thus we have $G = G_6$. Now, assume that y is adjacent to x_2 . Then, the next vertex after y in the 2-SH walk is either x_1 or y_1 . Suppose the next vertex after y is x_1 . Then, the next vertex after x_1 is either y_1 or y_2 . But in either case, there exists no next vertex in the 2-SH walk, a contradiction. Therefore, the next vertex after y is y_1 . Then, the next vertex after y_1 must be x_1 and so y_1 is not adjacent to x_1 . Then clearly the next vertex after x_1 is y_2 . If y_1 is adjacent to x_2 , then $G = G_2$, otherwise $G = G_1$.

Case 3. $|S \cap \{x_0, x_1, x_2\}| = 2$.

Without loss of generality, assume that $x_0, x_1 \in S$. Since $|S| = n - 3$, there exist two vertices $y_1, y_2 \notin S \cup V(P)$. Clearly, the vertices y_1, x_1, y_2 are consecutive in the 2-SH walk. Also, x_0 is consecutive with one of y_1 or y_2 in the 2-SH walk. Without loss of generality, assume that x_0 is consecutive with y_1 . Now, follow the 2-SH walk starting from x_2, x_0, y_1, x_1, y_2 . Assume that $|S| \geq 4$ and consider $z_1, z_2 \in S \setminus \{x_0, x_1\}$. Then the next vertex after y_2 must be in $S \setminus \{x_0, x_1\}$. Without loss of generality, assume that the next vertex after y_2 in the 2-SH walk is z_1 . But then the next vertex after z_1 in the 2-SH walk does not exist, a contradiction. Therefore $2 \leq |S| \leq 3$.

Assume that $|S| = 2$. Since $d(x_0, y_1) = d(x_1, y_1) = d(x_1, y_2) = 2$, it follows that x_0 is adjacent to y_2 , x_2 is adjacent to y_1 and y_1 is adjacent to y_2 . If x_2 is adjacent to y_2 , then x_1 is the only vertex at distance two from y_2 in G , a contradiction. Therefore, x_2 is not adjacent to y_2 . Thus we have $G = C_5$.

Next assume that $|S| = 3$. Let $S = \{x_0, x_1, z\}$. Then, clearly the next vertex after y_2 in the 2-SH walk is z and so y_2 is not adjacent to z . If z is adjacent to x_2 , then y_2 is the only vertex at distance two from z in G , a contradiction. Therefore, z is not adjacent to x_2 .

Assume that y_1 is not adjacent to y_2 . Since $d(x_0, y_1) = d(z, y_2) = 2$, clearly y_1 is adjacent to z and y_2 is adjacent to x_0 . Suppose now y_2 is adjacent to x_2 . If y_1 is adjacent to x_2 , then $G = G_3$, otherwise $G = G_6$. Suppose next y_2 is not adjacent to x_2 . If y_1 is adjacent to x_2 , then $G = G_6$, otherwise $G = G_7$.

Assume next y_1 is adjacent to y_2 . Now, look at the adjacency between x_0 and y_2 . First, assume that x_0 is adjacent to y_2 . Consider the adjacency between y_1 and z . Suppose y_1 is not adjacent to z . Since $d(x_1, y_1) = 2$, y_1 is adjacent to x_2 . If x_2 is adjacent to y_2 , then $G = G_5$, otherwise $G = G_1$. Suppose next y_1 is adjacent to z . Now, assume that y_1 is not adjacent to x_2 . If x_2 is adjacent to y_2 , then $G = G_2$, otherwise $G = G_6$. Next, assume that y_1 is adjacent to x_2 . If x_2 is adjacent to y_2 , then $G = G_4$, otherwise $G = G_3$. Now, assume that x_0 is not adjacent to y_2 . Since $d(x_0, y_1) = d(x_1, y_2) = 2$, it follows that y_1 is adjacent to z and y_2 is adjacent to x_2 . If y_1 is adjacent to x_2 , then $G = G_5$, otherwise $G = G_1$.

For the converse, it is not difficult to show that any graph $G \in \mathcal{F}$ is 2-SH with $\omega(G) = n - 3$. \square

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Data Availability. Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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