

A NOTE ON SUM FORMULAS $\sum_{k=0}^n kx^k W_k$ AND $\sum_{k=1}^n kx^k W_{-k}$ OF GENERALIZED HEXANACCI NUMBERS

Y. SOYKAN ^{ORCID}, E. TAŞDEMİR ^{ORCID} ✉, AND T.E. TAŞDEMİR ^{ORCID}

Article type: Research Article

(Received: 01 June 2022, Received in revised form: 18 August 2022)

(Accepted: 30 August 2022, Published Online: 31 August 2022)

ABSTRACT. In this paper, closed forms of the sum formulas $\sum_{k=0}^n kx^k W_k$ and $\sum_{k=1}^n kx^k W_{-k}$ for generalized Hexanacci numbers are presented. As special cases, we give summation formulas of Hexanacci, Hexanacci-Lucas, and other sixth-order recurrence sequences.

Keywords: Hexanacci numbers, Hexanacci-Lucas numbers, sum formulas, summing formulas.

2020 MSC: Primary 11B37, 11B39, 11B83.

1. Introduction

The generalized Hexanacci sequence

$$\{W_n(W_0, W_1, W_2, W_3, W_4, W_5; r, s, t, u, v, y)\}_{n \geq 0}$$

(or shortly $\{W_n\}_{n \geq 0}$) is defined as follows:

$$(1) \quad \begin{aligned} W_n &= rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4} + vW_{n-5} + yW_{n-6}, \\ W_0 &= c_0, W_1 = c_1, W_2 = c_2, W_3 = c_3, W_4 = c_4, W_5 = c_5, n \geq 6, \end{aligned}$$

where $W_0, W_1, W_2, W_3, W_4, W_5$ are arbitrary real or complex numbers and r, s, t, u, v, y are real numbers. The sequence $\{W_n\}_{n \geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{v}{y}W_{-n+1} - \frac{u}{y}W_{-n+2} - \frac{t}{y}W_{-n+3} - \frac{s}{y}W_{-n+4} - \frac{r}{y}W_{-n+5} + \frac{1}{y}W_{-n+6}$$

for $n = 1, 2, 3, \dots$ when $y \neq 0$. Therefore, recurrence (1) holds for all integer n . Hexanacci sequence has been studied by many authors, see for example [28, 40, 95] and references therein.

As $\{W_n\}$ is a sixth-order recurrence sequence (difference equation), its characteristic equation is

$$(2) \quad x^6 - rx^5 - sx^4 - tx^3 - ux^2 - vx - y = 0$$

whose roots are $\alpha, \beta, \gamma, \delta, \lambda, \mu$.

✉ erkantademir@hotmail.com, ORCID: 0000-0002-5002-3193

DOI: 10.22103/jmmr.2022.19594.1273

Publisher: Shahid Bahonar University of Kerman

How to cite: Y. Soykan, E. Taşdemir, T.E. Taşdemir, *A Note on Sum Formulas*

$\sum_{k=0}^n kx^k W_k$ and $\sum_{k=1}^n kx^k W_{-k}$ of Generalized Hexanacci Numbers, *J. Mahani Math.*

Res. 2023; 12(2): 137-164.



© the Authors

Note that we have the following identities:

$$\begin{aligned} \alpha + \beta + \gamma + \delta + \lambda + \mu &= r, \\ \alpha\beta + \alpha\lambda + \alpha\gamma + \alpha\mu + \beta\lambda + \alpha\delta + \beta\gamma + \beta\mu + \lambda\gamma + \lambda\mu + \beta\delta + \lambda\delta + \gamma\mu + \gamma\delta + \mu\delta &= -s, \\ \alpha\beta\lambda + \alpha\beta\gamma + \alpha\beta\mu + \alpha\lambda\gamma + \alpha\lambda\mu + \alpha\beta\delta + \alpha\lambda\delta + \alpha\gamma\mu + \beta\lambda\gamma + \beta\lambda\mu + \alpha\gamma\delta + \\ \alpha\mu\delta + \beta\lambda\delta + \beta\gamma\mu + \lambda\gamma\mu + \beta\gamma\delta + \beta\mu\delta + \lambda\gamma\delta + \lambda\mu\delta + \gamma\mu\delta &= t, \\ \alpha\beta\lambda\gamma + \alpha\beta\lambda\mu + \alpha\beta\lambda\delta + \alpha\beta\gamma\mu + \alpha\lambda\gamma\mu + \alpha\beta\gamma\delta + \alpha\beta\mu\delta + \alpha\lambda\gamma\delta + \alpha\lambda\mu\delta + \\ \beta\lambda\gamma\mu + \alpha\gamma\mu\delta + \beta\lambda\gamma\delta + \beta\lambda\mu\delta + \beta\gamma\mu\delta + \lambda\gamma\mu\delta &= -u \\ \alpha\beta\lambda\gamma\mu + \alpha\beta\lambda\gamma\delta + \alpha\beta\lambda\mu\delta + \alpha\beta\gamma\mu\delta + \alpha\lambda\gamma\mu\delta + \beta\lambda\gamma\mu\delta &= v, \\ \alpha\beta\lambda\gamma\mu\delta &= -y. \end{aligned}$$

Generalized Hexanacci numbers can be expressed, for all integers n , using Binet's formula.

Theorem 1.1. [95] (Binet's formula of generalized (r, s, t, u, v, y) numbers (generalized Hexanacci numbers))

$$\begin{aligned} (3) \quad W_n &= \frac{p_1\alpha^n}{(\alpha - \beta)(\alpha - \gamma)(\alpha - \delta)(\alpha - \lambda)(\alpha - \mu)} \\ &+ \frac{p_2\beta^n}{(\beta - \alpha)(\beta - \gamma)(\beta - \delta)(\beta - \lambda)(\beta - \mu)} \\ &+ \frac{p_3\gamma^n}{(\gamma - \alpha)(\gamma - \beta)(\gamma - \delta)(\gamma - \lambda)(\gamma - \mu)} \\ &+ \frac{p_4\delta^n}{(\delta - \alpha)(\delta - \beta)(\delta - \gamma)(\delta - \lambda)(\delta - \mu)} \\ &+ \frac{p_5\lambda^n}{(\lambda - \alpha)(\lambda - \beta)(\lambda - \gamma)(\lambda - \delta)(\lambda - \mu)} \\ &+ \frac{p_6\mu^n}{(\mu - \alpha)(\mu - \beta)(\mu - \gamma)(\mu - \delta)(\mu - \lambda)}, \end{aligned}$$

where

$$\begin{aligned} p_1 &= W_5 - (\beta + \gamma + \delta + \lambda + \mu)W_4 \\ &+ (\beta\lambda + \beta\gamma + \beta\mu + \lambda\gamma + \lambda\mu + \beta\delta + \lambda\delta + \gamma\mu + \gamma\delta + \mu\delta)W_3 \\ &- (\beta\lambda\gamma + \beta\lambda\mu + \beta\lambda\delta + \beta\gamma\mu + \lambda\gamma\mu + \beta\gamma\delta + \beta\mu\delta + \lambda\gamma\delta + \lambda\mu\delta + \gamma\mu\delta)W_2 \\ &+ (\beta\lambda\gamma\mu + \beta\lambda\gamma\delta + \beta\lambda\mu\delta + \beta\gamma\mu\delta + \lambda\gamma\mu\delta)W_1 - \beta\lambda\gamma\mu\delta W_0, \end{aligned}$$

$$\begin{aligned} p_2 &= W_5 - (\alpha + \gamma + \delta + \lambda + \mu)W_4 \\ &+ (\alpha\lambda + \alpha\gamma + \alpha\mu + \alpha\delta + \lambda\gamma + \lambda\mu + \lambda\delta + \gamma\mu + \gamma\delta + \mu\delta)W_3 \\ &- (\alpha\lambda\gamma + \alpha\lambda\mu + \alpha\lambda\delta + \alpha\gamma\mu + \alpha\gamma\delta + \alpha\mu\delta + \lambda\gamma\mu + \lambda\gamma\delta + \lambda\mu\delta + \gamma\mu\delta)W_2 \\ &+ (\alpha\lambda\gamma\mu + \alpha\lambda\gamma\delta + \alpha\lambda\mu\delta + \alpha\gamma\mu\delta + \lambda\gamma\mu\delta)W_1 - \alpha\lambda\gamma\mu\delta W_0, \end{aligned}$$

$$\begin{aligned}
 p_3 = & W_5 - (\alpha + \beta + \delta + \lambda + \mu)W_4 \\
 & + (\alpha\beta + \alpha\lambda + \alpha\mu + \beta\lambda + \alpha\delta + \beta\mu + \lambda\mu + \beta\delta + \lambda\delta + \mu\delta)W_3 \\
 & - (\alpha\beta\lambda + \alpha\beta\mu + \alpha\lambda\mu + \alpha\beta\delta + \alpha\lambda\delta + \beta\lambda\mu + \alpha\mu\delta + \beta\lambda\delta + \beta\mu\delta + \lambda\mu\delta)W_2 \\
 & + (\alpha\beta\lambda\mu + \alpha\beta\lambda\delta + \alpha\beta\mu\delta + \alpha\lambda\mu\delta + \beta\lambda\mu\delta)W_1 - \alpha\beta\lambda\mu\delta W_0,
 \end{aligned}$$

$$\begin{aligned}
 p_4 = & W_5 - (\alpha + \beta + \gamma + \lambda + \mu)W_4 \\
 & + (\alpha\beta + \alpha\lambda + \alpha\gamma + \alpha\mu + \beta\lambda + \beta\gamma + \beta\mu + \lambda\gamma + \lambda\mu + \gamma\mu)W_3 \\
 & - (\alpha\beta\lambda + \alpha\beta\gamma + \alpha\beta\mu + \alpha\lambda\gamma + \alpha\lambda\mu + \alpha\gamma\mu + \beta\lambda\gamma + \beta\lambda\mu + \beta\gamma\mu + \lambda\gamma\mu)W_2 \\
 & + (\alpha\beta\lambda\gamma + \alpha\beta\lambda\mu + \alpha\beta\gamma\mu + \alpha\lambda\gamma\mu + \beta\lambda\gamma\mu)W_1 - \alpha\beta\lambda\gamma\mu W_0,
 \end{aligned}$$

$$\begin{aligned}
 p_5 = & W_5 - (\alpha + \beta + \gamma + \delta + \mu)W_4 \\
 & + (\alpha\beta + \alpha\gamma + \alpha\mu + \alpha\delta + \beta\gamma + \beta\mu + \beta\delta + \gamma\mu + \gamma\delta + \mu\delta)W_3 \\
 & - (\alpha\beta\gamma + \alpha\beta\mu + \alpha\beta\delta + \alpha\gamma\mu + \alpha\gamma\delta + \alpha\mu\delta + \beta\gamma\mu + \beta\gamma\delta + \beta\mu\delta + \gamma\mu\delta)W_2 \\
 & + (\alpha\beta\gamma\mu + \alpha\beta\gamma\delta + \alpha\beta\mu\delta + \alpha\gamma\mu\delta + \beta\gamma\mu\delta)W_1 - \alpha\beta\gamma\mu\delta W_0,
 \end{aligned}$$

$$\begin{aligned}
 p_6 = & W_5 - (\alpha + \beta + \gamma + \delta + \lambda)W_4 \\
 & + (\alpha\beta + \alpha\lambda + \alpha\gamma + \beta\lambda + \alpha\delta + \beta\gamma + \lambda\gamma + \beta\delta + \lambda\delta + \gamma\delta)W_3 \\
 & - (\alpha\beta\lambda + \alpha\beta\gamma + \alpha\lambda\gamma + \alpha\beta\delta + \alpha\lambda\delta + \beta\lambda\gamma + \alpha\gamma\delta + \beta\lambda\delta + \beta\gamma\delta + \lambda\gamma\delta)W_2 \\
 & + (\alpha\beta\lambda\gamma + \alpha\beta\lambda\delta + \alpha\beta\gamma\delta + \alpha\lambda\gamma\delta + \beta\lambda\gamma\delta)W_1 - \alpha\beta\lambda\gamma\delta W_0.
 \end{aligned}$$

Usually, it is customary to choose r, s, t, u, v, y so that the Equ. (2) has at least one real (say α) solution.

(3) can be written in the following form:

$$W_n = A_1\alpha^n + A_2\beta^n + A_3\gamma^n + A_4\delta^n + A_5\lambda^n + A_6\mu^n,$$

where

$$\begin{aligned}
 A_1 &= \frac{p_1}{(\alpha - \beta)(\alpha - \gamma)(\alpha - \delta)(\alpha - \lambda)(\alpha - \mu)}, \\
 A_2 &= \frac{p_2}{(\beta - \alpha)(\beta - \gamma)(\beta - \delta)(\beta - \lambda)(\beta - \mu)}, \\
 A_3 &= \frac{p_3}{(\gamma - \alpha)(\gamma - \beta)(\gamma - \delta)(\gamma - \lambda)(\gamma - \mu)}, \\
 A_4 &= \frac{p_4}{(\delta - \alpha)(\delta - \beta)(\delta - \gamma)(\delta - \lambda)(\delta - \mu)}, \\
 A_5 &= \frac{p_5}{(\lambda - \alpha)(\lambda - \beta)(\lambda - \gamma)(\lambda - \delta)(\lambda - \mu)}, \\
 A_6 &= \frac{p_6}{(\mu - \alpha)(\mu - \beta)(\mu - \gamma)(\mu - \delta)(\mu - \lambda)}.
 \end{aligned}$$

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} W_n x^n$ of the sequence $\{W_n\}$.

Lemma 1.2. [95] Suppose that $f_{W_n}(x) = \sum_{n=0}^{\infty} W_n x^n$ is the ordinary generating function of the generalized (r, s, t, u, v, y) sequence $\{W_n\}_{n \geq 0}$. Then, $\sum_{n=0}^{\infty} W_n x^n$ is given by

$$(4) \quad \sum_{n=0}^{\infty} W_n x^n = \frac{\Lambda}{1 - rx - sx^2 - tx^3 - ux^4 - vx^5 - yx^6},$$

where

$$\begin{aligned} \Lambda = & W_0 + (W_1 - rW_0)x + (W_2 - rW_1 - sW_0)x^2 \\ & + (W_3 - rW_2 - sW_1 - tW_0)x^3 \\ & + (W_4 - rW_3 - sW_2 - tW_1 - uW_0)x^4 \\ & + (W_5 - rW_4 - sW_3 - tW_2 - uW_1 - vW_0)x^5. \end{aligned}$$

No	Sequences (Numbers)	Notation	Ref.
1	Gen. Hexanacci	$\{V_n\} = \{W_n(W_0, W_1, W_2, W_3, W_4, W_5; 1, 1, 1, 1, 1, 1)\}$	[96]
2	Gen. Sixth order Pell	$\{V_n\} = \{W_n(W_0, W_1, W_2, W_3, W_4, W_5; 2, 1, 1, 1, 1, 1)\}$	[97]
3	Gen. Sixth order Jacobsthal	$\{V_n\} = \{W_n(W_0, W_1, W_2, W_3, W_4, W_5; 1, 1, 1, 1, 1, 2)\}$	[98]
4	Gen. 6-primes	$\{V_n\} = \{W_n(W_0, W_1, W_2, W_3, W_4, W_5; 2, 3, 5, 7, 11, 13)\}$	[99]

TABLE 1. A few special case of generalized Hexanacci sequences.

For some specific values of $W_0, W_1, W_2, W_3, W_4, W_5$ and r, s, t, u, v, y it is worth presenting these special Hexanacci numbers in a table as a specific name. In literature, for example, the following names and notations (see Table 2) are used for the special cases of r, s, t, u, v, y and initial values.

For easy writing, from now on, we drop the superscripts from the sequences, for example we write P_n for $P_n^{(6)}$.

The following theorem presents some linear summing formulas of generalized Hexanacci numbers with positive subscripts.

Theorem 1.3. For $n \geq 0$ we have the following formulas: If $sx^2 + tx^3 + ux^4 + vx^5 + x^6y + rx - 1 \neq 0$, then

$$\sum_{k=0}^n x^k W_k = \frac{\Theta_1(x)}{sx^2 + tx^3 + ux^4 + vx^5 + x^6y + rx - 1},$$

Sequences (Numbers)	Notation	Ref
Hexanacci	$\{H_n\} = \{W_n(0, 1, 1, 2, 4, 8; 1, 1, 1, 1, 1, 1)\}$	[96]
Hexanacci-Lucas	$\{E_n\} = \{W_n(6, 1, 3, 7, 15, 31; 1, 1, 1, 1, 1, 1)\}$	[96]
sixth order Pell	$\{P_n^{(6)}\} = \{W_n(0, 1, 2, 5, 13, 34; 2, 1, 1, 1, 1, 1)\}$	[97]
sixth order Pell-Lucas	$\{Q_n^{(6)}\} = \{W_n(6, 2, 6, 17, 46, 122; 2, 1, 1, 1, 1, 1)\}$	[97]
modified sixth order Pell	$\{E_n^{(6)}\} = \{W_n(0, 1, 1, 3, 8, 21; 2, 1, 1, 1, 1, 1)\}$	[97]
sixth order Jacobsthal	$\{J_n^{(6)}\} = \{W_n(0, 1, 1, 1, 1, 1; 1, 1, 1, 1, 1, 2)\}$	[6,98]
sixth order Jacobsthal-Lucas	$\{j_n^{(6)}\} = \{W_n(2, 1, 5, 10, 20, 40; 1, 1, 1, 1, 1, 2)\}$	[6,98]
modified sixth order Jacobsthal	$\{K_n^{(6)}\} = \{W_n(3, 1, 3, 10, 20, 40; 1, 1, 1, 1, 1, 2)\}$	[98]
sixth-order Jacobsthal Perrin	$\{Q_n^{(6)}\} = \{W_n(3, 0, 2, 8, 16, 32; 1, 1, 1, 1, 1, 2)\}$	[98]
adjusted sixth-order Jacobsthal	$\{S_n^{(6)}\} = \{W_n(0, 1, 1, 2, 4, 8; 1, 1, 1, 1, 1, 2)\}$	[98]
modified sixth-order Jacobsthal-Lucas	$\{R_n^{(6)}\} = \{W_n(6, 1, 3, 7, 15, 31; 1, 1, 1, 1, 1, 2)\}$	[98]
6-primes	$\{G_n\} = \{W_n(0, 0, 0, 0, 1, 2; 2, 3, 5, 7, 11, 13)\}$	[99]
Lucas 6-primes	$\{H_n\} = \{W_n(6, 2, 10, 41, 150, 542; 2, 3, 5, 7, 11, 13)\}$	[99]
modified 6-primes	$\{E_n\} = \{W_n(0, 0, 0, 0, 1, 1; 2, 3, 5, 7, 11, 13)\}$	[99]

TABLE 2. A few members of generalized Hexanacci sequences.

where

$$\Theta_1(x) = x^{n+5}W_{n+5} - (rx - 1)x^{n+4}W_{n+4} - (sx^2 + rx - 1)x^{n+3}W_{n+3} - (sx^2 + tx^3 + rx - 1)x^{n+2}W_{n+2} - (sx^2 + tx^3 + ux^4 + rx - 1)x^{n+1}W_{n+1} + yx^{n+6}W_n - x^5W_5 + x^4(rx - 1)W_4 + x^3(sx^2 + rx - 1)W_3 + x^2(sx^2 + tx^3 + rx - 1)W_2 + x(sx^2 + tx^3 + ux^4 + rx - 1)W_1 + (sx^2 + tx^3 + ux^4 + vx^5 + rx - 1)W_0.$$

Proof. It is given in Soykan [85, Theorem 2.1]. □

The following theorem presents some linear summing formulas of generalized Hexanacci numbers with negative subscripts.

Theorem 1.4. *Let x be a complex number. For $n \geq 1$ we have the following formulas: If $y + rx^5 + sx^4 + tx^3 + ux^2 + vx - x^6 \neq 0$, then*

$$\sum_{k=1}^n x^k W_{-k} = \frac{\Theta_4(x)}{y + rx^5 + sx^4 + tx^3 + ux^2 + vx - x^6},$$

where

$$\Theta_4(x) = -x^{n+1}W_{-n+5} + (r - x)x^{n+1}W_{-n+4} + (s + rx - x^2)x^{n+1}W_{-n+3} + (t + rx^2 + sx - x^3)x^{n+1}W_{-n+2} + (u + rx^3 + sx^2 + tx - x^4)x^{n+1}W_{-n+1} + (v +$$

$$rx^4 + sx^3 + tx^2 + ux - x^5)x^{n+1}W_{-n} + xW_5 - x(r-x)W_4 + x(-s-rx+x^2)W_3 + x(-t-rx^2-sx+x^3)W_2 + x(-u-rx^3-sx^2-tx+x^4)W_1 + x(-v-rx^4-sx^3-tx^2-ux+x^5)W_0.$$

Proof. It is given in Soykan [85, Theorem 4.1]. □

In this work, we investigate summation formulas of generalized Hexanacci numbers.

2. An Application of the Sum of the Numbers

An application of the sum of the numbers is circulant matrix. Computations of the Frobenius norm, spectral norm, maximum column length norm and maximum row length norm of circulant (r-circulant, geometric circulant, semicirculant) matrices with the generalized m -step Fibonacci sequences require the sum of the squares of the numbers of the sequences. For generalized m -step Fibonacci sequences see for example Soykan [50]. If $m = 2, m = 3$ and $m = 4$, we get the generalized Fibonacci sequence, generalized Tribonacci sequence and generalized Tetranacci sequence, respectively. Next, we recall some information on circulant (r-circulant, geometric circulant) matrices and Frobenius norm, spectral norm, maximum column length norm and maximum row length norm.

Circulant matrices have been around for a long time and have been extensively used in many scientific areas. In some scientific areas such as image processing, coding theory and signal processing we often encounter circulant matrices. These matrices also have many applications in numerical analysis, optimization, digital image processing, mathematical statistics and modern technology.

Let $n \geq 2$ be an integer and r be any real or complex number. An $n \times n$ matrix C_r is called a r -circulant matrix if it of the form

$$C_r = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ rc_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ rc_{n-2} & rc_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ rc_1 & rc_2 & rc_3 & \cdots & rc_{n-1} & c_0 \end{pmatrix}_{n \times n},$$

and the r -circulant matrix C_r is denoted by

$$C_r = \text{Circ}_r(c_0, c_1, \dots, c_{n-1}).$$

If $r = 1$ then 1-circulant matrix is called as circulant matrix and denoted by $C = \text{Circ}(c_0, c_1, \dots, c_{n-1})$. Circulant matrixs were first proposed by Davis in [8]. These matrixs have many interesting properties, and it is one of the most important research subject in the field of the computational and pure mathematics (see for example references given in Table 3). For instance, Shen and Cen [46] studied on the norms of r -circulant matrices with Fibonacci and

Lucas numbers. Then, later Kızılateş and Tuglu [24] defined a new geometric circulant matrix as follows:

$$C_{r^*} = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ r c_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ r^2 c_{n-2} & r c_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ r^{n-1} c_1 & r^{n-2} c_2 & r^{n-3} c_3 & \cdots & r c_{n-1} & c_0 \end{pmatrix}_{n \times n},$$

and then they obtained the bounds for the spectral norms of geometric circulant matrices with the generalized Fibonacci number and Lucas numbers. When the parameter satisfies $r = 1$, we get the classical circulant matrix. See also Polath [33] for the spectral norms of r-circulant matrices with a type of Catalan triangle numbers.

The Frobenius (or Euclidean) norm and spectral norm of a matrix $A = (a_{ij})_{m \times n} \in M_{m \times n}(\mathbb{C})$ are defined respectively as follows:

$$\|A\|_F = \left(\sum_{i=1}^m \sum_{j=1}^n |a_{ij}|^2 \right)^{1/2} \quad \text{and} \quad \|A\|_2 = \left(\max_{1 \leq i \leq n} |\lambda_i| \right)^{1/2},$$

where λ_i 's are the eigenvalues of the matrix A^*A and A^* is the conjugate of transpose of the matrix A . The maximum column length norm $c_1(\cdot)$ and the maximum row length norm $r_1(\cdot)$ of an matrix of order $n \times n$ are defined as follows:

$$c_1(A) = \max_{1 \leq j \leq n} \left(\sum_{i=1}^n |a_{ij}|^2 \right)^{1/2} \quad \text{and} \quad r_1(A) = \max_{1 \leq i \leq n} \left(\sum_{j=1}^n |a_{ij}|^2 \right)^{1/2}.$$

The following inequality holds for any matrix $A = M_{n \times n}(\mathbb{C})$:

$$\frac{1}{\sqrt{n}} \|A\|_F \leq \|A\|_2 \leq \|A\|_F.$$

In literature there are other types of norms of matrices. The maximum column sum matrix norm of $n \times n$ matrix $A = (a_{ij})$ is $\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^n |a_{ij}|$ and the maximum row sum matrix norm is $\|A\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^n |a_{ij}|$.

Calculations of the above norms $\|A\|_F$, $\|A\|_2$, $c_1(A)$ and $r_1(A)$ require the sum of the squares of the numbers a_{ij} and calculations of the above norms $\|A\|_1$ and $\|A\|_\infty$ require the linear sum the numbers a_{ij} . We also note that the sum of entries of (a_{ij}) require the linear sum the numbers a_{ij} . As in our case, the numbers a_{ij} can be chosen as elements of second, third or higher order linear recurrence sequences.

In Table 3, we present a few special study on the Frobenius norm, spectral norm, maximum column length norm and maximum row length norm of circulant (r-circulant, geometric circulant, semicirculant) matrices with the

generalized m -step Fibonacci sequences which require sum formulas of second powers of numbers in m -step Fibonacci sequences ($m = 2, 3, 4$).

Name of sequence	Papers
second order↓ Fibonacci, Lucas	second order↓ [9, 10, 20, 24, 27, 38, 43–49, 101]
Pell, Pell-Lucas Jacobsthal, Jacobsthal- Lucas	[2, 102] [36, 103–105]
third order↓ Tribonacci, Tribonacci- Lucas	third order↓ [23, 37, 39, 94]
Padovan, Perrin Third-Order Pell Num- bers	[7, 32, 42] [93]
fourth order↓ Tetranacci, Tetranacci- Lucas	fourth order↓ [29]

TABLE 3. Papers on the norms.

Linear summing formulas of the generalized m -step Fibonacci sequences are required for the computation of various norms of circulant matrices with the generalized m -step Fibonacci sequences. We present some works on summing formulas of the numbers in Table 4.

Name of sequence	Papers which deal with summing formulas
Pell and Pell-Lucas	[1, 18, 25, 26, 30]
Generalized Fibonacci	[19, 63–67, 69, 82]
Generalized Tribonacci	[13, 16, 31, 68, 81, 83]
Generalized Tetranacci	[70, 75, 106]
Generalized Pentanacci	[71, 72, 84]
Generalized Hexanacci	[73, 74, 85]

TABLE 4. A few special study of sum formulas.

Also, the sum of the squares of the generalized m -step Fibonacci sequences are required for the computation of various norms of circulant matrices with the generalized m -step Fibonacci sequences. We present some works on sum formulas of powers of the numbers in Table 5

Name of sequence	sums of powers	second sums of powers	third powers	sums of powers
Generalized Fibonacci	[3, 4, 17, 21, 22, 51, 57, 60, 61, 78, 89–92]		[15, 52, 54, 55, 58, 59, 79, 80, 86–88, 107]	[5, 14, 34]
Generalized Tribonacci	[39, 53, 56, 76]			
Generalized Tetranacci	[35, 41, 62, 77]			

TABLE 5. A few special study on sum formulas of second, third and arbitrary powers.

3. Sum Formulas of Generalized Hexanacci Numbers with Positive Subscripts

The following theorem presents some summing formulas of generalized Hexanacci numbers with positive subscripts.

Theorem 3.1. *Let x be a real (or complex) number. For $n \geq 0$ we have the following formula; If $sx^2 + tx^3 + ux^4 + vx^5 + x^6y + rx - 1 \neq 0$, then*

$$\sum_{k=0}^n kx^k W_k = \frac{\Omega_1}{(sx^2 + tx^3 + ux^4 + vx^5 + x^6y + rx - 1)^2},$$

where

$$\begin{aligned} \Omega_1 = & x^{n+5}(n(sx^2 + tx^3 + ux^4 + vx^5 + x^6y + rx - 1) - 5 + 3sx^2 + 2tx^3 + \\ & ux^4 - x^6y + 4rx)W_{n+5} + x^{n+4}(n(1 - rx)(sx^2 + tx^3 + ux^4 + vx^5 + x^6y + rx - 1) \\ & - 4 + 2sx^2 + tx^3 - vx^5 - 2x^6y - 4r^2x^2 + 8rx - 3rsx^3 - 2rtx^4 - rux^5 + rx^7y)W_{n+4} + \\ & x^{n+3}(-n(sx^2 + rx - 1)(sx^2 + tx^3 + ux^4 + vx^5 + x^6y + rx - 1) - 3 + 6sx^2 - ux^4 - \\ & 2vx^5 - 3x^6y - 3r^2x^2 - 3s^2x^4 + 6rx - 6rsx^3 - rtx^4 - 2stx^5 + rvx^6 - sux^6 + 2r \\ & x^7y + sx^8y)W_{n+3} + x^{n+2}(-n(sx^2 + tx^3 + rx - 1)(sx^2 + tx^3 + ux^4 + vx^5 + x^6y + rx - 1) \\ & - 2 + 4sx^2 + 4tx^3 - 2ux^4 - 3vx^5 - 4x^6y - 2r^2x^2 - 2s^2x^4 - 2t^2x^6 + 4rx - 4rsx^3 - \\ & 4rtx^4 + rux^5 - 4stx^5 + 2rvx^6 + svx^7 - tux^7 + 3rx^7y + 2sx^8y + tx^9y)W_{n+2} + \\ & x^{n+1}(-n(sx^2 + tx^3 + ux^4 + rx - 1)(sx^2 + tx^3 + ux^4 + vx^5 + x^6y + rx - 1) \\ & - 1 + 2sx^2 + 2tx^3 + 2ux^4 - 4vx^5 - 5x^6y - r^2x^2 - s^2x^4 - t^2x^6 - u^2x^8 + 2rx - \\ & 2rsx^3 - 2rtx^4 - 2rux^5 - 2stx^5 + 3rvx^6 - 2sux^6 + 2svx^7 - 2tux^7 + tvx^8 + 4rx^7y + \\ & 3sx^8y + 2tx^9y + ux^{10}y)W_{n+1} + yx^{n+6}(n(sx^2 + tx^3 + ux^4 + vx^5 + x^6y + rx - 1) - 6 + \\ & 4sx^2 + 3tx^3 + 2ux^4 + vx^5 + 5rx)W_n + x^5(yx^6 - ux^4 - 2tx^3 - 3sx^2 - 4rx + 5)W_5 + \\ & x^4(-2sx^2 - tx^3 + vx^5 + 2x^6y + 4r^2x^2 - 8rx + 3rsx^3 + 2rtx^4 + rux^5 - rx^7y + 4) \\ & W_4 + x^3(-6sx^2 + ux^4 + 2vx^5 + 3x^6y + 3r^2x^2 + 3s^2x^4 - 6rx + 6rsx^3 + rtx^4 + \\ & 2stx^5 - rvx^6 + sux^6 - 2rx^7y - sx^8y + 3)W_3 + x^2(-4sx^2 - 4tx^3 + 2ux^4 + 3vx^5 + \\ & 4x^6y + 2r^2x^2 + 2s^2x^4 + 2t^2x^6 - 4rx + 4rsx^3 + 4rtx^4 - rux^5 + 4stx^5 - 2rvx^6 - \\ & svx^7 + tux^7 - 3rx^7y - 2sx^8y - tx^9y + 2)W_2 + x(-2sx^2 - 2tx^3 - 2ux^4 + 4vx^5 + \\ & 5x^6y + r^2x^2 + s^2x^4 + t^2x^6 + u^2x^8 - 2rx + 2rsx^3 + 2rtx^4 + 2rux^5 + 2stx^5 - \end{aligned}$$

$$3rvx^6 + 2sux^6 - 2svx^7 + 2tux^7 - tvx^8 - 4rx^7y - 3sx^8y - 2tx^9y - ux^{10}y + 1)W_1 + yx^6(-vx^5 - 2ux^4 - 3tx^3 - 4sx^2 - 5rx + 6)W_0.$$

Proof. Using the recurrence relation

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4} + vW_{n-5} + yW_{n-6}$$

i.e.,

$$yW_{n-6} = W_n - rW_{n-1} - sW_{n-2} - tW_{n-3} - uW_{n-4} - vW_{n-5},$$

we obtain

$$\begin{aligned} y \times 0 \times x^0 W_0 &= 0 \times x^0 W_6 - r \times 0 \times x^0 W_5 - s \times 0 \times x^0 W_4 \\ &\quad - t \times 0 \times x^0 W_3 - u \times 0 \times x^0 W_2 - v \times 0 \times x^0 W_1 \\ y \times 1 \times x^1 W_1 &= 1 \times x^1 W_7 - r \times 1 \times x^1 W_6 - s \times 1 \times x^1 W_5 \\ &\quad - t \times 1 \times x^1 W_4 - u \times 1 \times x^1 W_3 - v \times 1 \times x^1 W_2 \\ y \times 2 \times x^2 W_2 &= 2 \times x^2 W_8 - r \times 2 \times x^2 W_7 - s \times 2 \times x^2 W_6 \\ &\quad - t \times 2 \times x^2 W_5 - u \times 2 \times x^2 W_4 - v \times 2 \times x^2 W_3 \\ &\quad \vdots \end{aligned}$$

$$\begin{aligned} y(n-2)x^{n-2}W_{n-2} &= (n-2)x^{n-2}W_{n+4} - r(n-2)x^{n-2}W_{n+3} \\ &\quad - s(n-2)x^{n-2}W_{n+2} - t(n-2)x^{n-2}W_{n+1} \\ &\quad - u(n-2)x^{n-2}W_n - v(n-2)x^{n-2}W_{n-1} \\ y(n-1)x^{n-1}W_{n-1} &= (n-1)x^{n-1}W_{n+5} - r(n-1)x^{n-1}W_{n+4} \\ &\quad - s(n-1)x^{n-1}W_{n+3} - t(n-1)x^{n-1}W_{n+2} \\ &\quad - u(n-1)x^{n-1}W_{n+1} - v(n-1)x^{n-1}W_n \\ y \times n \times x^n W_n &= n \times x^n W_{n+6} - r \times n \times x^n W_{n+5} - s \times n \times x^n W_{n+4} \\ &\quad - t \times n \times x^n W_{n+3} - u \times n \times x^n W_{n+2} \\ &\quad - v \times n \times x^n W_{n+1}. \end{aligned}$$

If we add the equations side by side we obtain

$$\begin{aligned}
 y \sum_{k=0}^n kx^k W_k &= (nx^n W_{n+6} + (n-1)x^{n-1} W_{n+5} + (n-2)x^{n-2} W_{n+4} \\
 &+ (n-3)x^{n-3} W_{n+3} + (n-4)x^{n-4} W_{n+2} + (n-5)x^{n-5} W_{n+1} \\
 &- (-1)x^{-1} W_5 - (-2)x^{-2} W_4 - (-3)x^{-3} W_3 - (-4)x^{-4} W_2 - (-5)x^{-5} W_1 \\
 &- (-6)x^{-6} W_0 + \sum_{k=0}^n (k-6)x^{k-6} W_k) - r(nx^n W_{n+5} + (n-1)x^{n-1} W_{n+4} \\
 &+ (n-2)x^{n-2} W_{n+3} + (n-3)x^{n-3} W_{n+2} + (n-4)x^{n-4} W_{n+1} \\
 &- (-1)x^{-1} W_4 - (-2)x^{-2} W_3 - (-3)x^{-3} W_2 - (-4)x^{-4} W_1 - (-5)x^{-5} W_0 \\
 &+ \sum_{k=0}^n (k-5)x^{k-5} W_k) - s(nx^n W_{n+4} + (n-1)x^{n-1} W_{n+3} \\
 &+ (n-2)x^{n-2} W_{n+2} + (n-3)x^{n-3} W_{n+1} - (-1)x^{-1} W_3 - (-2)x^{-2} W_2 \\
 &- (-3)x^{-3} W_1 - (-4)x^{-4} W_0 + \sum_{k=0}^n (k-4)x^{k-4} W_k) - t(nx^n W_{n+3} \\
 &+ (n-1)x^{n-1} W_{n+2} + (n-2)x^{n-2} W_{n+1} - (-1)x^{-1} W_2 - (-2)x^{-2} W_1 \\
 &- (-3)x^{-3} W_0 + \sum_{k=0}^n (k-3)x^{k-3} W_k) - u(nx^n W_{n+2} + (n-1)x^{n-1} W_{n+1} \\
 &- (-1)x^{-1} W_1 - (-2)x^{-2} W_0 + \sum_{k=0}^n (k-2)x^{k-2} W_k) \\
 &- v(nx^n W_{n+1} - (-1)x^{-1} W_0 + \sum_{k=0}^n (k-1)x^{k-1} W_k).
 \end{aligned}$$

Then, using Theorem 1.3, we get the required result. □

4. Special Cases

In this section, for the special cases of x , we present the closed form solutions (identities) of the sums $\sum_{k=0}^n kx^k W_k$, $\sum_{k=0}^n kx^k W_{2k}$ and $\sum_{k=0}^n kx^k W_{2k+1}$ for the specific case of sequence $\{W_n\}$.

4.1. The case $x = 1$. In this subsection we consider the special case $x = 1$. The case $x = 1$ of Theorem 3.1 is given in Soykan [100].

4.2. The case $x = -1$. In this subsection we consider the special case $x = -1$ and we present the closed form solutions (identities) of the sums $\sum_{k=0}^n k(-1)^k W_k$, $\sum_{k=0}^n k(-1)^k W_{2k}$ and $\sum_{k=0}^n k(-1)^k W_{2k+1}$ for the specific case of the sequence $\{W_n\}$.

Taking $r = s = t = u = v = y = 1$ in Theorem 3.1, we obtain the following proposition.

Proposition 4.1. *If $r = s = t = u = v = y = 1$, then for $n \geq 0$ we have the following formula:*

$$\sum_{k=0}^n k(-1)^k W_k = (-1)^n ((n+8)W_{n+5} - (2n+15)W_{n+4} + (n+5)W_{n+3} - (2n+12)W_{n+2} + (n+2)W_{n+1} - (n+9)W_n) - 8W_5 + 15W_4 - 5W_3 + 12W_2 - 2W_1 + 9W_0.$$

From the above proposition, we have the following corollary which gives sum formulas of Hexanacci and Hexanacci-Lucas numbers (take $W_n = H_n$ with $H_0 = 0, H_1 = 1, H_2 = 1, H_3 = 2, H_4 = 4, H_5 = 8$ and take $W_n = E_n$ with $E_0 = 6, E_1 = 1, E_2 = 3, E_3 = 7, E_4 = 15, E_5 = 31$, respectively).

Corollary 4.2. *For $n \geq 1$, we have the following properties:*

- (a): $\sum_{k=0}^n k(-1)^k H_k = (-1)^n ((n+8)H_{n+5} - (2n+15)H_{n+4} + (n+5)H_{n+3} - (2n+12)H_{n+2} + (n+2)H_{n+1} - (n+9)H_n) - 4.$
- (b): $\sum_{k=0}^n k(-1)^k E_k = (-1)^n ((n+8)E_{n+5} - (2n+15)E_{n+4} + (n+5)E_{n+3} - (2n+12)E_{n+2} + (n+2)E_{n+1} - (n+9)E_n) + 30.$

We present the next result as an example of the above corollary for $n = 8$.

Example 4.3. *For $n = 8$, we have the followings:*

- (a): $\sum_{k=0}^8 k(-1)^k H_k = 347.$
- (b): $\sum_{k=0}^8 k(-1)^k E_k = 1339.$

Taking $r = 2, s = t = u = v = y = 1$ in Theorem 3.1, we obtain the following proposition.

Proposition 4.4. *If $r = 2, s = t = u = v = y = 1$, then for $n \geq 1$ we have the following formula:*

$$\sum_{k=0}^n k(-1)^k W_k = \frac{1}{2}((-1)^n ((n+6)W_{n+5} - (3n+17)W_{n+4} + (2n+8)W_{n+3} - (3n+12)W_{n+2} + (2n+3)W_{n+1} - (n+7)W_n) - 6W_5 + 17W_4 - 8W_3 + 12W_2 - 3W_1 + 7W_0).$$

From the above proposition, we have the following corollary which gives sum formulas of sixth-order Pell and sixth-order Pell-Lucas numbers (take $W_n = P_n$ with $P_0 = 0, P_1 = 1, P_2 = 2, P_3 = 5, P_4 = 13, P_5 = 34$ and take $W_n = Q_n$ with $Q_0 = 6, Q_1 = 2, Q_2 = 6, Q_3 = 17, Q_4 = 46, Q_5 = 122$, respectively).

Corollary 4.5. *For $n \geq 1$, we have the following properties:*

- (a): $\sum_{k=0}^n k(-1)^k P_k = \frac{1}{2}((-1)^n ((n+6)P_{n+5} - (3n+17)P_{n+4} + (2n+8)P_{n+3} - (3n+12)P_{n+2} + (2n+3)P_{n+1} - (n+7)P_n) - 2).$
- (b): $\sum_{k=0}^n k(-1)^k Q_k = \frac{1}{2}((-1)^n ((n+6)Q_{n+5} - (3n+17)Q_{n+4} + (2n+8)Q_{n+3} - (3n+12)Q_{n+2} + (2n+3)Q_{n+1} - (n+7)Q_n) + 22).$

We present the next result as an example of the above corollary for $n = 8$.

Example 4.6. *For $n = 8$, we have the followings:*

- (a): $\sum_{k=0}^8 k(-1)^k P_k = 3645.$
- (b): $\sum_{k=0}^8 k(-1)^k Q_k = 13070.$

Observe that setting $x = -1, r = 1, s = 1, t = 1, u = 1, v = 1, y = 2$ (i.e., for the generalized sixth order Jacobsthal case) in Theorem 3.1, makes the right hand side of the sum formulas to be an indeterminate form. Application of L'Hospital rule however provides the evaluation of the sum formulas.

Theorem 4.7. *If $r = 1, s = 1, t = 1, u = 1, v = 2, y = 2$, then for $n \geq 1$ we have the following formulas:*

$$\sum_{k=0}^n k(-1)^k W_k = \frac{1}{54}((-1)^n (-(3n^2 + 11n - 118)W_{n+5} + 2(3n^2 + 8n - 122)W_{n+4} - (3n + 17)(n - 8)W_{n+3} + 2(3n^2 - n - 122)W_{n+2} - (3n^2 - 25n - 118)W_{n+1} + 2(3n^2 + 17n - 104)W_n) - 118W_5 + 244W_4 - 136W_3 + 244W_2 - 118W_1 + 208W_0).$$

Proof. We use Theorem 3.1. If we set $r = 1, s = 1, t = 1, u = 2$ in Theorem 3.1, then we have

$$\sum_{k=0}^n kx^k W_k = \frac{g_1(x)}{(2x - 1)^2(x + 1)^2(-x + x^2 + 1)^2(x + x^2 + 1)^2},$$

where

$$g_1(x) = x^{n+5}(4x + n(2x^6 + x^5 + x^4 + x^3 + x^2 + x - 1) + 3x^2 + 2x^3 + x^4 - 2x^6 - 5)W_{n+5} - x^{n+4}(2x^2 - 8x + 2x^3 + 2x^4 + 2x^5 + 4x^6 - 2x^7 + n(x - 1)(2x^6 + x^5 + x^4 + x^3 + x^2 + x - 1) + 4)W_{n+4} - x^{n+3}(6x^3 - 3x^2 - 6x + 5x^4 + 4x^5 + 6x^6 - 4x^7 - 2x^8 + n(x^2 + x - 1)(2x^6 + x^5 + x^4 + x^3 + x^2 + x - 1) + 3)W_{n+3} + x^{n+2}(4x + 2x^2 - 8x^4 - 6x^5 - 8x^6 + 6x^7 + 4x^8 + 2x^9 - n(x^3 + x^2 + x - 1)(2x^6 + x^5 + x^4 + x^3 + x^2 + x - 1) - 2)W_{n+2} + x^{n+1}(2x + x^2 - x^4 - 8x^5 - 10x^6 + 8x^7 + 6x^8 + 4x^9 + 2x^{10} - n(x^4 + x^3 + x^2 + x - 1)(2x^6 + x^5 + x^4 + x^3 + x^2 + x - 1) - 1)W_{n+1} + 2x^{n+6}(5x + n(2x^6 + x^5 + x^4 + x^3 + x^2 + x - 1) + 4x^2 + 3x^3 + 2x^4 + x^5 - 6)W_n - x^5(-2x^6 + x^4 + 2x^3 + 3x^2 + 4x - 5)W_5 + x^4(-2x^7 + 4x^6 + 2x^5 + 2x^4 + 2x^3 + 2x^2 - 8x + 4)W_4 + x^3(-2x^8 - 4x^7 + 6x^6 + 4x^5 + 5x^4 + 6x^3 - 3x^2 - 6x + 3)W_3 - x^2(2x^9 + 4x^8 + 6x^7 - 8x^6 - 6x^5 - 8x^4 + 2x^2 + 4x - 2)W_2 - x(2x^{10} + 4x^9 + 6x^8 + 8x^7 - 10x^6 - 8x^5 - x^4 + x^2 + 2x - 1)W_1 - 2x^6(x^5 + 2x^4 + 3x^3 + 4x^2 + 5x - 6)W_0.$$

For $x = 1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule. Then we get the required result using

$$\begin{aligned} \sum_{k=0}^n k(-1)^k W_k &= \frac{\frac{d^2}{dx^2}(g_1(x))}{\frac{d^2}{dx^2}((2x - 1)^2(x + 1)^2(-x + x^2 + 1)^2(x + x^2 + 1)^2)} \Big|_{x=-1} \\ &= \frac{1}{54}((-1)^n (-(3n^2 + 11n - 118)W_{n+5} \\ &\quad + 2(3n^2 + 8n - 122)W_{n+4} - (3n + 17)(n - 8)W_{n+3} \\ &\quad + 2(3n^2 - n - 122)W_{n+2} - (3n^2 - 25n - 118)W_{n+1} \\ &\quad + 2(3n^2 + 17n - 104)W_n) - 118W_5 \\ &\quad + 244W_4 - 136W_3 + 244W_2 - 118W_1 + 208W_0). \end{aligned}$$

□

Taking, respectively,

$W_n = J_n$ with $J_0 = 0, J_1 = 1, J_2 = 1, J_3 = 1, J_4 = 1, J_5 = 1$ (sixth-order Jacobsthal numbers),

$W_n = j_n$ with $j_0 = 2, j_1 = 1, j_2 = 5, j_3 = 10, j_4 = 20, j_5 = 40$ (sixth order Jacobsthal-Lucas numbers),

$W_n = K_n$ with $K_0 = 3, K_1 = 1, K_2 = 3, K_3 = 10, K_4 = 20, K_5 = 40$ (modified sixth order Jacobsthal numbers),

$W_n = Q_n$ with $Q_0 = 3, Q_1 = 0, Q_2 = 2, Q_3 = 8, Q_4 = 16, Q_5 = 32$ (sixth-order Jacobsthal Perrin numbers),

$W_n = S_n$ with $S_0 = 0, S_1 = 1, S_2 = 1, S_3 = 2, S_4 = 4, S_5 = 8$ (adjusted sixth-order Jacobsthal numbers),

$W_n = R_n$ with $R_0 = 6, R_1 = 1, R_2 = 3, R_3 = 7, R_4 = 15, R_5 = 31$ (modified sixth-order Jacobsthal-Lucas numbers),

in the last Theorem, we have the following corollary.

Corollary 4.8. *For $n \geq 0$, we have the following properties:*

- (a): $\sum_{k=0}^n k(-1)^k J_k = \frac{1}{54}((-1)^n (-3n^2 + 11n - 118)J_{n+5} + 2(3n^2 + 8n - 122)J_{n+4} - (3n + 17)(n - 8)J_{n+3} + 2(3n^2 - n - 122)J_{n+2} - (3n^2 - 25n - 118)J_{n+1} + 2(3n^2 + 17n - 104)J_n) + 116$.
- (b): $\sum_{k=0}^n k(-1)^k j_k = \frac{1}{54}((-1)^n (-3n^2 + 11n - 118)j_{n+5} + 2(3n^2 + 8n - 122)j_{n+4} - (3n + 17)(n - 8)j_{n+3} + 2(3n^2 - n - 122)j_{n+2} - (3n^2 - 25n - 118)j_{n+1} + 2(3n^2 + 17n - 104)j_n) + 318$.
- (c): $\sum_{k=0}^n k(-1)^k K_k = \frac{1}{54}((-1)^n (-3n^2 + 11n - 118)K_{n+5} + 2(3n^2 + 8n - 122)K_{n+4} - (3n + 17)(n - 8)K_{n+3} + 2(3n^2 - n - 122)K_{n+2} - (3n^2 - 25n - 118)K_{n+1} + 2(3n^2 + 17n - 104)K_n) + 38$.
- (d): $\sum_{k=0}^n k(-1)^k Q_k = \frac{1}{54}((-1)^n (-3n^2 + 11n - 118)Q_{n+5} + 2(3n^2 + 8n - 122)Q_{n+4} - (3n + 17)(n - 8)Q_{n+3} + 2(3n^2 - n - 122)Q_{n+2} - (3n^2 - 25n - 118)Q_{n+1} + 2(3n^2 + 17n - 104)Q_n) + 152$.
- (e): $\sum_{k=0}^n k(-1)^k S_k = \frac{1}{54}((-1)^n (-3n^2 + 11n - 118)S_{n+5} + 2(3n^2 + 8n - 122)S_{n+4} - (3n + 17)(n - 8)S_{n+3} + 2(3n^2 - n - 122)S_{n+2} - (3n^2 - 25n - 118)S_{n+1} + 2(3n^2 + 17n - 104)S_n) - 114$.
- (f): $\sum_{k=0}^n k(-1)^k R_k = \frac{1}{54}((-1)^n (-3n^2 + 11n - 118)R_{n+5} + 2(3n^2 + 8n - 122)R_{n+4} - (3n + 17)(n - 8)R_{n+3} + 2(3n^2 - n - 122)R_{n+2} - (3n^2 - 25n - 118)R_{n+1} + 2(3n^2 + 17n - 104)R_n) + 912$.

We present the next result as an example of the above corollary for $n = 8$.

Example 4.9. *For $n = 8$, we have the followings:*

- (a): $\sum_{k=0}^8 k(-1)^k J_k = 118$.
- (b): $\sum_{k=0}^8 k(-1)^k j_k = 1776$.
- (c): $\sum_{k=0}^8 k(-1)^k K_k = 1738$.
- (d): $\sum_{k=0}^8 k(-1)^k Q_k = 1382$.
- (e): $\sum_{k=0}^8 k(-1)^k S_k = 356$.
- (f): $\sum_{k=0}^8 k(-1)^k R_k = 1454$.

Taking $r = 2, s = 3, t = 5, u = 7, v = 11, y = 13$ in Theorem 3.1, we obtain the following proposition.

Proposition 4.10. *If $r = 2, s = 3, t = 5, u = 7, v = 11, y = 13$, then for $n \geq 1$ we have the following formula:*

$$\sum_{k=0}^n k(-1)^k W_k = \frac{1}{4}((-1)^n(-(n-5)W_{n+5} + (3n-16)W_{n+4} + 4W_{n+3} + (5n-29)W_{n+2} + (2n-1)W_{n+1} + 13(n-4)W_n) - 5W_5 + 16W_4 - 4W_3 + 29W_2 + W_1 + 52W_0).$$

From the above proposition, we have the following corollary which gives sum formulas of 6-primes, Lucas 6-primes and modified 6-primes numbers (take $W_n = G_n$ with $G_0 = 0, G_1 = 0, G_2 = 0, G_3 = 0, G_4 = 1, G_5 = 2$ and take $W_n = H_n$ with $H_0 = 6, H_1 = 2, H_2 = 10, H_3 = 41, H_4 = 150, H_5 = 542$ and take $W_n = E_n$ with $E_0 = 0, E_1 = 0, E_2 = 0, E_3 = 0, E_4 = 1, E_5 = 1$, respectively).

Corollary 4.11. *For $n \geq 1$, we have the following properties:*

- (a): $\sum_{k=0}^n k(-1)^k G_k = \frac{1}{4}((-1)^n(-(n-5)G_{n+5} + (3n-16)G_{n+4} + 4G_{n+3} + (5n-29)G_{n+2} + (2n-1)G_{n+1} + 13(n-4)G_n) + 6).$
- (b): $\sum_{k=0}^n k(-1)^k H_k = \frac{1}{4}((-1)^n(-(n-5)H_{n+5} + (3n-16)H_{n+4} + 4H_{n+3} + (5n-29)H_{n+2} + (2n-1)H_{n+1} + 13(n-4)H_n) + 130).$
- (c): $\sum_{k=0}^n k(-1)^k E_k = \frac{1}{4}((-1)^n(-(n-5)E_{n+5} + (3n-16)E_{n+4} + 4E_{n+3} + (5n-29)E_{n+2} + (2n-1)E_{n+1} + 13(n-4)E_n) + 11).$

We present the next result as an example of the above corollary for $n = 8$.

Example 4.12. *For $n = 8$, we have the followings:*

- (a): $\sum_{k=0}^8 k(-1)^k G_k = 565.$
- (b): $\sum_{k=0}^8 k(-1)^k H_k = 149336.$
- (c): $\sum_{k=0}^8 k(-1)^k E_k = 407.$

4.3. The case $x = i$. In this subsection we consider the special case $x = i$.

Taking $x = i, r = s = t = u = v = y = 1$ in Theorem 3.1, we obtain the following proposition.

Proposition 4.13. *If $r = s = t = u = v = y = 1$, then for $n \geq 0$ we have the following formula:*

$$\sum_{k=0}^n k i^k W_k = \frac{1}{3-4i}(i^n(-((1+2i)n + (2+6i))W_{n+5} - ((1-3i)n + (2-7i))W_{n+4} + ((4+3i)n + (6+7i))W_{n+3} + ((4-2i)n + (6+4i))W_{n+2} + ((8+4i) - (1+2i)n)W_{n+1} + ((2-i)n + (8-3i))W_n) + (2+6i)W_5 + (2-7i)W_4 - (6+7i)W_3 - (6+4i)W_2 - (8+4i)W_1 - (8-3i)W_0).$$

From the above proposition, we have the following corollary which gives sum formulas of Hexanacci and Hexanacci-Lucas numbers (take $W_n = H_n$ with $H_0 = 0, H_1 = 1, H_2 = 1, H_3 = 2, H_4 = 4, H_5 = 8$ and take $H_n = E_n$ with $E_0 = 6, E_1 = 1, E_2 = 3, E_3 = 7, E_4 = 15, E_5 = 31$, respectively).

Corollary 4.14. *For $n \geq 0$, we have the following properties:*

$$\begin{aligned}
 \text{(a): } \sum_{k=0}^n ki^k H_k &= \frac{1}{3-4i} (i^n - ((1+2i)n + (2+6i))H_{n+5} - ((1-3i)n + (2-7i))H_{n+4} + ((4+3i)n + (6+7i))H_{n+3} + ((4-2i)n + (6+4i))H_{n+2} + ((8+4i) - (1+2i)n)H_{n+1} + ((2-i)n + (8-3i))H_n) - (2+2i). \\
 \text{(b): } \sum_{k=0}^n ki^k E_k &= \frac{1}{3-4i} (i^n - ((1+2i)n + (2+6i))E_{n+5} - ((1-3i)n + (2-7i))E_{n+4} + ((4+3i)n + (6+7i))E_{n+3} + ((4-2i)n + (6+4i))E_{n+2} + ((8+4i) - (1+2i)n)E_{n+1} + ((2-i)n + (8-3i))E_n) + (-24+34i).
 \end{aligned}$$

Corresponding sums of the other sixth order generalized Hexanacci numbers can be calculated similarly.

5. Sum Formulas of Generalized Hexanacci Numbers with Negative Subscripts

The following theorem presents some summing formulas of generalized Hexanacci numbers with negative subscripts.

Theorem 5.1. *Let x be a real (or complex) number. For $n \geq 1$ we have the following formulas: If $y + rx^5 + sx^4 + tx^3 + ux^2 + vx - x^6 \neq 0$, then*

$$\sum_{k=1}^n kx^k W_{-k} = \frac{\Omega_2}{(y + rx^5 + sx^4 + tx^3 + ux^2 + vx - x^6)^2},$$

where

$$\begin{aligned}
 \Omega_2 &= x^{n+1}(n(-y - rx^5 - sx^4 - tx^3 - ux^2 - vx + x^6) + 4rx^5 + 3sx^4 + 2tx^3 + ux^2 - y - 5x^6)W_{-n+5} + x^{n+1}(n(r - x)(y + rx^5 + sx^4 + tx^3 + ux^2 + vx - x^6) + 8rx^6 + 2sx^5 + tx^4 - vx^2 - 4r^2x^5 + ry - 2xy - 4x^7 - 3rsx^4 - 2rtx^3 - rux^2)W_{-n+4} + x^{n+1}(n(s + rx - x^2)(y + rx^5 + sx^4 + tx^3 + ux^2 + vx - x^6) + 6rx^7 + 6sx^6 - ux^4 - 2vx^3 - 3x^2y - 3r^2x^6 - 3s^2x^4 + sy - 3x^8 - 6rsx^5 - rtx^4 - 2stx^3 + rvx^2 - sux^2 + 2rxy)W_{-n+3} + x^{n+1}(n(t + rx^2 + sx - x^3)(y + rx^5 + sx^4 + tx^3 + ux^2 + vx - x^6) + 4rx^8 + 4sx^7 + 4tx^6 - 2ux^5 - 3vx^4 - 4x^3y - 2r^2x^7 - 2s^2x^5 - 2t^2x^3 + ty - 2x^9 - 4rsx^6 - 4rtx^5 + rux^4 - 4stx^4 + 2rvx^3 + svx^2 - tux^2 + 3rx^2y + 2sxy)W_{-n+2} + x^{n+1}(n(u + rx^3 + sx^2 + tx - x^4)(y + rx^5 + sx^4 + tx^3 + ux^2 + vx - x^6) + 2rx^9 + 2sx^8 + 2tx^7 + 2ux^6 - 4vx^5 - 5x^4y - r^2x^8 - s^2x^6 - t^2x^4 - u^2x^2 + uy - x^{10} - 2rsx^7 - 2rtx^6 - 2rux^5 - 2stx^5 + 3rvx^4 - 2sux^4 + 2svx^3 - 2tux^3 + tvx^2 + 4rx^3y + 3sx^2y + 2txy)W_{-n+1} + x^{n+1}(n(v + rx^4 + sx^3 + tx^2 + ux - x^5)(y + rx^5 + sx^4 + tx^3 + ux^2 + vx - x^6) - 6x^5y + vy + 5rx^4y + 4sx^3y + 3tx^2y + 2uxy)W_{-n} + x(y - 4rx^5 - 3sx^4 - 2tx^3 - ux^2 + 5x^6)W_5 + x(-8rx^6 - 2sx^5 - tx^4 + vx^2 + 4r^2x^5 - ry + 2xy + 4x^7 + 3rsx^4 + 2rtx^3 + rux^2)W_4 + x(-6rx^7 - 6sx^6 + ux^4 + 2vx^3 + 3x^2y + 3r^2x^6 + 3s^2x^4 - sy + 3x^8 + 6rsx^5 + rtx^4 + 2stx^3 - rvx^2 + sux^2 - 2rxy)W_3 + x(-4rx^8 - 4sx^7 - 4tx^6 + 2ux^5 + 3vx^4 + 4x^3y + 2r^2x^7 + 2s^2x^5 + 2t^2x^3 - ty + 2x^9 + 4rsx^6 + 4rtx^5 - rux^4 + 4stx^4 - 2rvx^3 - svx^2 + tux^2 - 3rx^2y - 2sxy)W_2 + x(-2rx^9 - 2sx^8 - 2tx^7 - 2ux^6 + 4vx^5 + 5x^4y + r^2x^8 + s^2x^6 + t^2x^4 + u^2x^2 - uy + x^{10} + 2rsx^7 + 2rtx^6 + 2rux^5 + 2stx^5 - 3rvx^4 + 2sux^4 - 2svx^3 + 2tux^3 - tvx^2 - 4rx^3y - 3sx^2y - 2txy)W_1 + xy(-v - 5rx^4 - 4sx^3 - 3tx^2 - 2ux + 6x^5)W_0.
 \end{aligned}$$

Proof. Using the recurrence relation

$$W_{-n} = \frac{1}{y}W_{-n+6} - \frac{v}{y}W_{-n+1} - \frac{u}{y}W_{-n+2} - \frac{t}{y}W_{-n+3} - \frac{s}{y}W_{-n+4} - \frac{r}{y}W_{-n+5}$$

i.e.,

$$yW_{-n} = W_{-n+6} - rW_{-n+5} - sW_{-n+4} - tW_{-n+3} - uW_{-n+2} - vW_{-n+1},$$

we obtain

$$\begin{aligned} y \times n \times x^n W_{-n} &= n \times x^n W_{-n+6} - r \times n \times x^n W_{-n+5} \\ &\quad - s \times n \times x^n W_{-n+4} - t \times n \times x^n W_{-n+3} \\ &\quad - u \times n \times x^n W_{-n+2} - v \times n \times x^n W_{-n+1} \\ y(n-1)x^{n-1}W_{-n+1} &= (n-1)x^{n-1}W_{-n+7} - r(n-1)x^{n-1}W_{-n+6} \\ &\quad - s(n-1)x^{n-1}W_{-n+5} - t(n-1)x^{n-1}W_{-n+4} \\ &\quad - u(n-1)x^{n-1}W_{-n+3} - v(n-1)x^{n-1}W_{-n+2} \\ y(n-2)x^{n-2}W_{-n+2} &= (n-2)x^{n-2}W_{-n+8} - r(n-2)x^{n-2}W_{-n+7} \\ &\quad - s(n-2)x^{n-2}W_{-n+6} - t(n-2)x^{n-2}W_{-n+5} \\ &\quad - u(n-2)x^{n-2}W_{-n+4} - v(n-2)x^{n-2}W_{-n+3} \\ &\quad \vdots \end{aligned}$$

$$\begin{aligned} y \times 3 \times x^3 W_{-3} &= 3 \times x^3 W_3 - r \times 3 \times x^3 W_2 - s \times 3 \times x^3 W_1 \\ &\quad - t \times 3 \times x^3 W_0 - u \times 3 \times x^3 W_{-1} - v \times 3 \times x^3 W_{-2} \\ y \times 2 \times x^2 W_{-2} &= 2 \times x^2 W_4 - r \times 2 \times x^2 W_3 - s \times 2 \times x^2 W_2 \\ &\quad - t \times 2 \times x^2 W_1 - u \times 2 \times x^2 W_0 - v \times 2 \times x^2 W_{-1} \\ y \times 1 \times x^1 W_{-1} &= 1 \times x^1 W_5 - r \times 1 \times x^1 W_4 - s \times 1 \times x^1 W_3 \\ &\quad - t \times 1 \times x^1 W_2 - u \times 1 \times x^1 W_1 - v \times 1 \times x^1 W_0. \end{aligned}$$

If we add the equations side by side we obtain

$$\begin{aligned}
 y \sum_{k=1}^n kx^k W_{-k} = & \quad (-(n+1)x^{n+1}W_{-n+5} - (n+2)x^{n+2}W_{-n+4} \\
 & \quad -(n+3)x^{n+3}W_{-n+3} - (n+4)x^{n+4}W_{-n+2} \\
 & \quad -(n+5)x^{n+5}W_{-n+1} - (n+6)x^{n+6}W_{-n} + 1 \times x^1 W_5 \\
 & \quad + 2x^2 W_4 + 3x^3 W_3 + 4x^4 W_2 + 5x^5 W_1 + 6x^6 W_0 \\
 & \quad + \sum_{k=1}^n (k+6)x^{k+6}W_{-k} - r(-(n+1)x^{n+1}W_{-n+4} \\
 & \quad -(n+2)x^{n+2}W_{-n+3} - (n+3)x^{n+3}W_{-n+2} \\
 & \quad -(n+4)x^{n+4}W_{-n+1} - (n+5)x^{n+5}W_{-n} + 1 \times x^1 W_4 \\
 & \quad + 2x^2 W_3 + 3x^3 W_2 + 4x^4 W_1 + 5x^5 W_0 + \sum_{k=1}^n (k+5)x^{k+5}W_{-k}) \\
 & \quad -s(-(n+1)x^{n+1}W_{-n+3} - (n+2)x^{n+2}W_{-n+2} \\
 & \quad -(n+3)x^{n+3}W_{-n+1} - (n+4)x^{n+4}W_{-n} + 1 \times x^1 W_3 \\
 & \quad + 2x^2 W_2 + 3x^3 W_1 + 4x^4 W_0 + \sum_{k=1}^n (k+4)x^{k+4}W_{-k}) \\
 & \quad -t(-(n+1)x^{n+1}W_{-n+2} - (n+2)x^{n+2}W_{-n+1} \\
 & \quad -(n+3)x^{n+3}W_{-n} + 1 \times x^1 W_2 + 2x^2 W_1 + 3x^3 W_0 \\
 & \quad + \sum_{k=1}^n (k+3)x^{k+3}W_{-k} - u(-(n+1)x^{n+1}W_{-n+1} \\
 & \quad -(n+2)x^{n+2}W_{-n} + 1 \times x^1 W_1 + 2x^2 W_0 \\
 & \quad + \sum_{k=1}^n (k+2)x^{k+2}W_{-k} - v(-(n+1)x^{n+1}W_{-n} \\
 & \quad + 1 \times x^1 W_0 + \sum_{k=1}^n (k+1)x^{k+1}W_{-k})
 \end{aligned}$$

Then, using Theorem 1.4, we get the required result. □

6. Specific Cases

In this section, for the specific cases of x , we present the closed form solutions (identities) of the sums $\sum_{k=1}^n kx^k W_{-k}$, $\sum_{k=1}^n kx^k W_{-2k}$ and $\sum_{k=1}^n kx^k W_{-2k+1}$ for the specific case of sequence $\{W_n\}$.

6.1. The case $x = 1$. In this subsection we consider the special case $x = 1$. The case $x = 1$ of Theorem 5.1 is given in Soykan [100].

6.2. **The case $x = -1$.** In this subsection we consider the special case $x = -1$.

Taking $r = s = t = u = v = y = 1$ in Theorem 5.1, we obtain the following proposition.

Proposition 6.1. *If $r = s = t = u = v = y = 1$, then for $n \geq 0$ we have the following formula:*

$$\sum_{k=1}^n k(-1)^k W_{-k} = (-1)^n (-(n-8)W_{-n+5} + (2n-15)W_{-n+4} - (n-5)W_{-n+3} + (2n-12)W_{-n+2} - (n-2)W_{-n+1} + (2n-9)W_{-n}) - 8W_5 + 15W_4 - 5W_3 + 12W_2 - 2W_1 + 9W_0.$$

From the above proposition, we have the following corollary which gives sum formulas of Hexanacci and Hexanacci-Lucas numbers (take $W_n = H_n$ with $H_0 = 0, H_1 = 1, H_2 = 1, H_3 = 2, H_4 = 4, H_5 = 8$ and take $W_n = E_n$ with $E_0 = 6, E_1 = 1, E_2 = 3, E_3 = 7, E_4 = 15, E_5 = 31$, respectively).

Corollary 6.2. *For $n \geq 1$, we have the following properties:*

- (a): $\sum_{k=1}^n k(-1)^k H_{-k} = (-1)^n (-(n-8)H_{-n+5} + (2n-15)H_{-n+4} - (n-5)H_{-n+3} + (2n-12)H_{-n+2} - (n-2)H_{-n+1} + (2n-9)H_{-n}) - 4.$
- (b): $\sum_{k=1}^n k(-1)^k E_{-k} = (-1)^n (-(n-8)E_{-n+5} + (2n-15)E_{-n+4} - (n-5)E_{-n+3} + (2n-12)E_{-n+2} - (n-2)E_{-n+1} + (2n-9)E_{-n}) + 30.$

Taking $r = 2, s = t = u = v = y = 1$ in Theorem 5.1, we obtain the following Proposition.

Proposition 6.3. *If $r = 2, s = t = u = v = y = 1$, then for $n \geq 1$ we have the following formula:*

$$\sum_{k=1}^n k(-1)^k W_{-k} = \frac{1}{2}((-1)^n (-(n-6)W_{-n+5} + (3n-17)W_{-n+4} - (2n-8)W_{-n+3} + (3n-12)W_{-n+2} - (2n-3)W_{-n+1} + (3n-7)W_{-n}) - 6W_5 + 17W_4 - 8W_3 + 12W_2 - 3W_1 + 7W_0).$$

From the above proposition, we have the following corollary which gives sum formulas of sixth-order Pell and sixth-order Pell-Lucas numbers (take $W_n = P_n$ with $P_0 = 0, P_1 = 1, P_2 = 2, P_3 = 5, P_4 = 13, P_5 = 34$ and take $W_n = Q_n$ with $Q_0 = 6, Q_1 = 2, Q_2 = 6, Q_3 = 17, Q_4 = 46, Q_5 = 122$, respectively).

Corollary 6.4. *For $n \geq 1$, we have the following properties:*

- (a): $\sum_{k=1}^n k(-1)^k P_{-k} = \frac{1}{2}((-1)^n (-(n-6)P_{-n+5} + (3n-17)P_{-n+4} - (2n-8)P_{-n+3} + (3n-12)P_{-n+2} - (2n-3)P_{-n+1} + (3n-7)P_{-n}) - 2).$
- (b): $\sum_{k=1}^n k(-1)^k Q_{-k} = \frac{1}{2}((-1)^n (-(n-6)Q_{-n+5} + (3n-17)Q_{-n+4} - (2n-8)Q_{-n+3} + (3n-12)Q_{-n+2} - (2n-3)Q_{-n+1} + (3n-7)Q_{-n}) + 22).$

Observe that setting $x = -1, r = 1, s = 1, t = 1, u = 1, v = 1, y = 2$ (i.e., for the generalized sixth order Jacobsthal case) in Theorem 5.1, makes the right hand side of the sum formulas to be an indeterminate form. Application of L'Hospital rule however provides the evaluation of the sum formulas.

Theorem 6.5. *If $r = 1, s = 1, t = 1, u = 1, v = 2, y = 2$, then for $n \geq 1$ we have the following formulas:*

$$\sum_{k=1}^n k(-1)^k W_{-k} = \frac{1}{54}((-1)^n (-(3n^2 - 11n - 96)W_{-n+5} + 2(3n^2 - 8n - 100)W_{-n+4} - (3n^2 + 7n - 114)W_{-n+3} + 2(3n^2 + n - 100)W_{-n+2} - (3n^2 + 25n - 96)W_{-n+1} + 2(3n^2 + 10n - 82)W_{-n}) - 96W_5 + 200W_4 - 114W_3 + 200W_2 - 96W_1 + 164W_0).$$

Proof. We use Theorem 5.1. If we set $r = 1, s = 1, t = 1, u = 2$ in Theorem 5.1, then we have

$$\sum_{k=1}^n kx^k W_{-k} = \frac{g_2(x)}{(x-2)^2(x+1)^2(x+x^2+1)^2(-x+x^2+1)^2},$$

where

$$g_2(x) = x^{n+1}(x^2 - n(-x^6 + x^5 + x^4 + x^3 + x^2 + x + 2) + 2x^3 + 3x^4 + 4x^5 - 5x^6 - 2)W_{5-n} - x^{n+1}(4x + 2x^2 + 2x^3 + 2x^4 + 2x^5 - 8x^6 + 4x^7 + n(x-1)(-x^6 + x^5 + x^4 + x^3 + x^2 + x + 2) - 2)W_{4-n} - x^{n+1}(6x^2 - 4x + 4x^3 + 5x^4 + 6x^5 - 3x^6 - 6x^7 + 3x^8 - n(-x^2 + x + 1)(-x^6 + x^5 + x^4 + x^3 + x^2 + x + 2) - 2)W_{3-n} + x^{n+1}(4x + 6x^2 - 8x^3 - 6x^4 - 8x^5 + 2x^7 + 4x^8 - 2x^9 + n(-x^3 + x^2 + x + 1)(-x^6 + x^5 + x^4 + x^3 + x^2 + x + 2) + 2)W_{2-n} + x^{n+1}(4x + 6x^2 + 8x^3 - 10x^4 - 8x^5 - x^6 + x^8 + 2x^9 - x^{10} + n(-x^4 + x^3 + x^2 + x + 1)(-x^6 + x^5 + x^4 + x^3 + x^2 + x + 2) + 2)W_{1-n} + x^{n+1}(4x + 6x^2 + 8x^3 + 10x^4 - 12x^5 + n(-x^5 + x^4 + x^3 + x^2 + x + 1)(-x^6 + x^5 + x^4 + x^3 + x^2 + x + 2) + 2)W_{-n} - x(-5x^6 + 4x^5 + 3x^4 + 2x^3 + x^2 - 2)W_5 + x(4x^7 - 8x^6 + 2x^5 + 2x^4 + 2x^3 + 2x^2 + 4x - 2)W_4 + x(3x^8 - 6x^7 - 3x^6 + 6x^5 + 5x^4 + 4x^3 + 6x^2 - 4x - 2)W_3 - x(-2x^9 + 4x^8 + 2x^7 - 8x^5 - 6x^4 - 8x^3 + 6x^2 + 4x + 2)W_2 - x(-x^{10} + 2x^9 + x^8 - x^6 - 8x^5 - 10x^4 + 8x^3 + 6x^2 + 4x + 2)W_1 - 2x(-6x^5 + 5x^4 + 4x^3 + 3x^2 + 2x + 1)W_0$$

For $x = -1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule. Then we get the required result using

$$\begin{aligned} \sum_{k=1}^n k(-1)^k W_{-k} &= \frac{\frac{d^2}{dx^2}(g_2(x))}{\frac{d^2}{dx^2}((x-2)^2(x+1)^2(x+x^2+1)^2(-x+x^2+1)^2)} \Big|_{x=-1} \\ &= \frac{1}{54}((-1)^n (-(3n^2 - 11n - 96)W_{-n+5} \\ &\quad + 2(3n^2 - 8n - 100)W_{-n+4} - (3n^2 + 7n - 114)W_{-n+3} \\ &\quad + 2(3n^2 + n - 100)W_{-n+2} - (3n^2 + 25n - 96)W_{-n+1} \\ &\quad + 2(3n^2 + 10n - 82)W_{-n}) - 96W_5 + 200W_4 - 114W_3 \\ &\quad + 200W_2 - 96W_1 + 164W_0). \end{aligned}$$

□

Taking, respectively,

$W_n = J_n$ with $J_0 = 0, J_1 = 1, J_2 = 1, J_3 = 1, J_4 = 1, J_5 = 1$ (sixth-order Jacobsthal numbers),

$W_n = j_n$ with $j_0 = 2, j_1 = 1, j_2 = 5, j_3 = 10, j_4 = 20, j_5 = 40$ (sixth order Jacobsthal-Lucas numbers),

$W_n = K_n$ with $K_0 = 3, K_1 = 1, K_2 = 3, K_3 = 10, K_4 = 20, K_5 = 40$ (modified sixth order Jacobsthal numbers),

$W_n = Q_n$ with $Q_0 = 3, Q_1 = 0, Q_2 = 2, Q_3 = 8, Q_4 = 16, Q_5 = 32$ (sixth-order Jacobsthal Perrin numbers),

$W_n = S_n$ with $S_0 = 0, S_1 = 1, S_2 = 1, S_3 = 2, S_4 = 4, S_5 = 8$ (adjusted sixth-order Jacobsthal numbers),

$W_n = R_n$ with $R_0 = 6, R_1 = 1, R_2 = 3, R_3 = 7, R_4 = 15, R_5 = 31$ (modified sixth-order Jacobsthal-Lucas numbers),

in the last theorem, we have the following corollary.

Corollary 6.6. *For $n \geq 1$, we have the following properties:*

- (a): $\sum_{k=1}^n k(-1)^k J_{-k} = \frac{1}{54}((-1)^n (-3n^2 - 11n - 96)J_{-n+5} + 2(3n^2 - 8n - 100)J_{-n+4} - (3n^2 + 7n - 114)J_{-n+3} + 2(3n^2 + n - 100)J_{-n+2} - (3n^2 + 25n - 96)J_{-n+1} + 2(3n^2 + 10n - 82)J_{-n}) + 94$.
- (b): $\sum_{k=1}^n k(-1)^k j_{-k} = \frac{1}{54}((-1)^n (-3n^2 - 11n - 96)j_{-n+5} + 2(3n^2 - 8n - 100)j_{-n+4} - (3n^2 + 7n - 114)j_{-n+3} + 2(3n^2 + n - 100)j_{-n+2} - (3n^2 + 25n - 96)j_{-n+1} + 2(3n^2 + 10n - 82)j_{-n}) + 252$.
- (c): $\sum_{k=1}^n k(-1)^k K_{-k} = \frac{1}{54}((-1)^n (-3n^2 - 11n - 96)K_{-n+5} + 2(3n^2 - 8n - 100)K_{-n+4} - (3n^2 + 7n - 114)K_{-n+3} + 2(3n^2 + n - 100)K_{-n+2} - (3n^2 + 25n - 96)K_{-n+1} + 2(3n^2 + 10n - 82)K_{-n}) + 16$.
- (d): $\sum_{k=1}^n k(-1)^k Q_{-k} = \frac{1}{54}((-1)^n (-3n^2 - 11n - 96)Q_{-n+5} + 2(3n^2 - 8n - 100)Q_{-n+4} - (3n^2 + 7n - 114)Q_{-n+3} + 2(3n^2 + n - 100)Q_{-n+2} - (3n^2 + 25n - 96)Q_{-n+1} + 2(3n^2 + 10n - 82)Q_{-n}) + 108$.
- (e): $\sum_{k=1}^n k(-1)^k S_{-k} = \frac{1}{54}((-1)^n (-3n^2 - 11n - 96)S_{-n+5} + 2(3n^2 - 8n - 100)S_{-n+4} - (3n^2 + 7n - 114)S_{-n+3} + 2(3n^2 + n - 100)S_{-n+2} - (3n^2 + 25n - 96)S_{-n+1} + 2(3n^2 + 10n - 82)S_{-n}) - 92$.
- (f): $\sum_{k=1}^n k(-1)^k R_{-k} = \frac{1}{54}((-1)^n (-3n^2 - 11n - 96)R_{-n+5} + 2(3n^2 - 8n - 100)R_{-n+4} - (3n^2 + 7n - 114)R_{-n+3} + 2(3n^2 + n - 100)R_{-n+2} - (3n^2 + 25n - 96)R_{-n+1} + 2(3n^2 + 10n - 82)R_{-n}) + 714$.

Taking $r = 2, s = 3, t = 5, u = 7, v = 11, y = 13$ in Theorem 5.1, we obtain the following proposition.

Proposition 6.7. *If $r = 2, s = 3, t = 5, u = 7, v = 11, y = 13$, then for $n \geq 1$ we have the following formula:*

$$\sum_{k=1}^n k(-1)^k W_{-k} = \frac{1}{4}((-1)^n ((n+5)W_{-n+5} - (3n+16)W_{-n+4} + 4W_{-n+3} - (5n+29)W_{-n+2} - (2n+1)W_{-n+1} - (9n+52)W_{-n}) - 5W_5 + 16W_4 - 4W_3 + 29W_2 + W_1 + 52W_0).$$

From the above proposition, we have the following corollary which gives sum formulas of 6-primes, Lucas 6-primes and modified 6-primes numbers (take $W_n = G_n$ with $G_0 = 0, G_1 = 0, G_2 = 0, G_3 = 0, G_4 = 1, G_5 = 2$ and take $W_n = H_n$ with $H_0 = 6, H_1 = 2, H_2 = 10, H_3 = 41, H_4 = 150, H_5 = 542$ and take $W_n = E_n$ with $E_0 = 0, E_1 = 0, E_2 = 0, E_3 = 0, E_4 = 1, E_5 = 1$, respectively).

Corollary 6.8. For $n \geq 1$, we have the following properties:

- (a): $\sum_{k=1}^n k(-1)^k G_{-k} = \frac{1}{4}((-1)^n ((n+5)G_{-n+5} - (3n+16)G_{-n+4} + 4G_{-n+3} - (5n+29)G_{-n+2} - (2n+1)G_{-n+1} - (9n+52)G_{-n}) + 6)$.
- (b): $\sum_{k=1}^n k(-1)^k H_{-k} = \frac{1}{4}((-1)^n ((n+5)H_{-n+5} - (3n+16)H_{-n+4} + 4H_{-n+3} - (5n+29)H_{-n+2} - (2n+1)H_{-n+1} - (9n+52)H_{-n}) + 130)$.
- (c): $\sum_{k=1}^n k(-1)^k E_{-k} = \frac{1}{4}((-1)^n ((n+5)E_{-n+5} - (3n+16)E_{-n+4} + 4E_{-n+3} - (5n+29)E_{-n+2} - (2n+1)E_{-n+1} - (9n+52)E_{-n}) + 11)$.

6.3. **The case $x = i$.** In this subsection, we consider the special case $x = i$.

Taking $r = s = t = u = v = y = 1$ in Theorem 5.1, we obtain the following proposition.

Proposition 6.9. If $r = s = t = u = v = y = 1$, then for $n \geq 1$ we have the following formula:

$$\sum_{k=1}^n ki^k W_{-k} = \frac{1}{3+4i}(i^n(((1-2i)n - (2-6i))W_{5-n} + ((1+3i)n - (2+7i))W_{4-n} + ((6-7i) - (4-3i)n)W_{3-n} + ((6-4i) - (4+2i)n)W_{2-n} + ((8-4i) + (1-2i)n)W_{1-n} + ((8+3i) + (1+3i)n)W_{-n}) + (2-6i)W_5 + (2+7i)W_4 - (6-7i)W_3 - (6-4i)W_2 - (8-4i)W_1 - (8+3i)W_0).$$

From the above proposition, we have the following corollary which gives sum formulas of Hexanacci and Hexanacci-Lucas numbers (take $W_n = H_n$ with $H_0 = 0, H_1 = 1, H_2 = 1, H_3 = 2, H_4 = 4, H_5 = 8$ and take $H_n = E_n$ with $E_0 = 6, E_1 = 1, E_2 = 3, E_3 = 7, E_4 = 15, E_5 = 31$, respectively).

Corollary 6.10. For $n \geq 1$, we have the following properties:

- (a): $\sum_{k=1}^n ki^k H_{-k} = \frac{1}{3+4i}(i^n(((1-2i)n - (2-6i))H_{5-n} + ((1+3i)n - (2+7i))H_{4-n} + ((6-7i) - (4-3i)n)H_{3-n} + ((6-4i) - (4+2i)n)H_{2-n} + ((8-4i) + (1-2i)n)H_{1-n} + ((8+3i) + (1+3i)n)H_{-n}) + (-2+2i))$.
- (b): $\sum_{k=1}^n ki^k E_{-k} = \frac{1}{3+4i}(i^n(((1-2i)n - (2-6i))E_{5-n} + ((1+3i)n - (2+7i))E_{4-n} + ((6-7i) - (4-3i)n)E_{3-n} + ((6-4i) - (4+2i)n)E_{2-n} + ((8-4i) + (1-2i)n)E_{1-n} + ((8+3i) + (1+3i)n)E_{-n}) - (24+34i))$.

Corresponding sums of the other sixth order generalized Hexanacci numbers can be calculated similarly.

7. Conclusion

Recently, there have been so many studies of the sequences of numbers in the literature and the sequences of numbers were widely used in many research areas, such as architecture, nature, art, physics and engineering, see for example [11], [12]. In this work, sum identities were proved. The method used in this paper can be used for the other linear recurrence sequences, too. We have written sum identities in terms of the generalized Hexanacci sequence, and then we have presented the formulas as special cases the corresponding identity for the Hexanacci, Hexanacci-Lucas, and other sixth-order recurrence sequences. All the listed identities in the corollaries may be proved by induction, but that

method of proof gives no clue about their discovery. We give the proofs to indicate how these identities, in general, were discovered.

Computations of the Frobenius norm, spectral norm, maximum column length norm and maximum row length norm of circulant (r-circulant, geometric circulant, semicirculant) matrices with the generalized m -step Fibonacci sequences require the sum of the numbers of the sequences. So, our results can be used to study circulant (r-circulant, geometric circulant, semicirculant) matrices with second-order linear recurrence sequences.

References

- [1] M. Akbulak, A. Öteleş, *On the sum of Pell and Jacobsthal numbers by matrix method*, Bulletin of the Iranian Mathematical Society, 40(4), (2014) 1017–1025.
- [2] E. G. Alptekin, *Pell, Pell-Lucas ve Modified Pell sayıları ile tanımlı circulant ve semicirculant matrisler* (Doctoral dissertation, Selçuk Üniversitesi Fen Bilimleri Enstitüsü), (2005).
- [3] Z. Čerin, *Formulae for sums of Jacobsthal–Lucas numbers*, Int. Math. Forum, 2(40), (2007) 1969–1984.
- [4] Z. Čerin, *Sums of Squares and Products of Jacobsthal Numbers*, Journal of Integer Sequences, 10(2), Article 07.2.5, (2007) 1–15.
- [5] L. Chen, X. Wang, *The Power Sums Involving Fibonacci Polynomials and Their Applications*, Symmetry, 11, (2019), doi.org/10.3390/sym11050635.
- [6] C. K. Cook, M. R. Bacon, *Some identities for Jacobsthal and Jacobsthal-Lucas numbers satisfying higher order recurrence relations*, Annales Mathematicae et Informaticae, 41, (2013) 27–39.
- [7] A. Coskun, N. Taskara, *On the Some Properties of Circulant Matrices with Third Order Linear Recurrent Sequences*, Mathematical Sciences and Applications E-Notes, 6(1), (2018) 12–18.
- [8] P. J. Davis, *Circulant Matrices*, JohnWiley&Sons, New York, (1979).
- [9] Ö Deveci, E. Karaduman, C.M. Campbell, *The Fibonacci-Circulant Sequences and Their Applications*, Iranian Journal of Science and Technology Transaction A-Science, 41(A4), (2007) 1033–1038.
- [10] Ö Deveci, *On The Fibonacci-Circulant p -Sequences*, Utilitas Mathematica, 108, (2018) 107–124.
- [11] Ö Deveci, *On the connections among Fibonacci, Pell, Jacobsthal and Padovan numbers*, Notes on Number Theory and Discrete Mathematics, 27(2), (2021) 111–128.
- [12] Ö Erdağ, Ö Deveci, A. G. Shannon, *Matrix Manipulations for Properties of Pell p -Numbers and their Generalizations*, Analele Stiintifice ale Universitatii Ovidius, 28(3), (2020) 89–102.
- [13] A. Esi, N. Subramanian, M. K. Ozdemir, *Chlodowsky type (λ, q) -Bernstein Stancu operators of Pascal rough triple sequences*, J. Mahani Math. Res. Cent. 2023; 12(1): 289-310.
- [14] R. Frontczak, *Sums of powers of Fibonacci and Lucas numbers: A new bottom-up approach*, Notes on Number Theory and Discrete Mathematics, 24(2), (2018) 94–103.
- [15] R. Frontczak, *Sums of Cubes Over Odd-Index Fibonacci Numbers*, Integers, 18, (2018) 1–9.
- [16] R. Frontczak, *Sums of Tribonacci and Tribonacci-Lucas Numbers*, International Journal of Mathematical Analysis, 12(1), (2018) 19–24.
- [17] A. Gnanam, B. Anitha, *Sums of Squares Jacobsthal Numbers*, IOSR Journal of Mathematics, 11(6), (2015) 62–64.

- [18] H. Gökbaş, H. Köse, *Some Sum Formulas for Products of Pell and Pell-Lucas Numbers*, Int. J. Adv. Appl. Math. and Mech. 4(4), (2017) 1–4.
- [19] R.T. Hansen., *General Identities for Linear Fibonacci and Lucas Summations*, Fibonacci Quarterly, 16(2), (1978) 121–128.
- [20] C. He, J. Ma, K. Zhang, Z. Wang, *The Upper Bound Estimation on the Spectral Norm of r -Circulant Matrices with the Fibonacci and Lucas Numbers*, J. Inequal. Appl. 2015:72, (2015).
- [21] E. Kiliç, D. Taşçi, *The Linear Algebra of The Pell Matrix*, Boletín de la Sociedad Matemática Mexicana, 3(11), (2005).
- [22] E. Kılıc, *Sums of the squares of terms of sequence $\{u_n\}$* , Proc. Indian Acad. Sci. (Math. Sci.) 118(1), (2008) 27–41.
- [23] C. Kızılateş, N. Tuglu, *On the Norms of Geometric and Symmetric Geometric Circulant Matrices with the Tribonacci Number*, Gazi University Journal of Science, 31(2), (2018) 555–567.
- [24] C. Kızılateş, N. Tuglu *On the Bounds for the Spectral Norms of Geometric Circulant Matrices*, Journal of Inequalities and Applications, 2016(1), Article ID 312, (2016). DOI 10.1186/s13660-016-1255-1
- [25] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, A Wiley-Interscience Publication, New York, (2001).
- [26] T. Koshy, *Pell and Pell-Lucas Numbers with Applications*, Springer, New York, (2014).
- [27] J. K. Merikoski, P. Haukkanen, M. Mattila, T. Tossavainen, *On the Spectral and Frobenius Norm of a Generalized Fibonacci r -Circulant Matrix*, Special Matrices, 6, (2018) 23–36.
- [28] L. R. Natividad, *On Solving Fibonacci-Like Sequences of Fourth, Fifth and Sixth Order*, International Journal of Mathematics and Computing, 3(2), (2013) 38–40.
- [29] A. Özkoç, E. Ardiyok, *Circulant and Negacyclic Matrices Via Tetranacci Numbers*, Honam Mathematical J., 38(4), (2016) 725–738.
- [30] A. Öteleş, M. Akbulak, *A Note on Generalized k -Pell Numbers and Their Determinantal Representation*, Journal of Analysis and Number Theory, 4(2), (2016) 153–158.
- [31] T. Parpar, *k 'ncı Mertebeden Rekürans Bağıntısının Özellikleri ve Bazı Uygulamaları*, Selçuk Üniversitesi, Fen Bilimleri Enstitüsü, Yüksek Lisans Tezi, (2011).
- [32] A. Pacheenburawana, W. Sintunavarat, *On the Spectral Norms of r -Circulant Matrices with the Padovan and Perrin Sequences*, Journal of Mathematical Analysis, 9(3), (2018) 110–122.
- [33] E. Polatlı, *On the Bounds for the Spectral Norms of r -Circulant Matrices with a Type of Catalan Triangle Numbers*, Journal of Science and Arts, 3(48), (2019) 575–578.
- [34] H. Prodinger, *Sums of Powers of Fibonacci Polynomials*, Proc. Indian Acad. Sci. (Math. Sci.), 119(5), (2009) 567–570.
- [35] H. Prodinger, S. J. Selkirk, *Sums of Squares of Tetranacci Numbers: A Generating Function Approach*, (2019), <http://arxiv.org/abs/1906.08336v1>.
- [36] B. Radicic, *On k -Circulant Matrices Involving the Jacobsthal Numbers*, Revista De La Union Matematica Argentina, 60(2), (2009) 431–442.
- [37] Z. Raza, M. A. Ali, *On the Norms of Circulant, r -Circulant, Semi-Circulant and Hankel Matrices with Tribonacci Sequence*, (2014) arxiv, <http://arxiv.org/abs/1407.1369v1>.
- [38] Z. Raza, M. A. Ali, *On the Norms of Some Special Matrices with Generalized Fibonacci Sequence*, J. Appl. Math. & Informatics, 33(5-6), (2015) 593–605.
- [39] Z. Raza, M. Riaz, M. A. Ali, *Some Inequalities on the Norms of Special Matrices with Generalized Tribonacci and Generalized Pell-Padovan Sequences*, arXiv, (2015), <http://arxiv.org/abs/1407.1369v2>
- [40] G. P. S. Rathore, O. Sikhwal, R. Choudhary, *Formula for finding n th Term of Fibonacci-Like Sequence of Higher Order*, International Journal of Mathematics And its Applications, 4 (2-D), (2016) 75–80.

- [41] R. Schumacher, *How to sum the squares of the Tetranacci numbers and the Fibonacci m-step numbers*, Fibonacci Quarterly, 57, (2019) 168–175.
- [42] W. Sintunavarat, *The Upper Bound Estimation for the Spectral Norm of r-Circulant and Symmetric r-Circulant Matrices with the Padovan Sequence*, J. Nonlinear Sci. Appl. 9, (2016) 92–101.
- [43] S. Shen, *On the Norms of Circulant Matrices with the (k,h)-Fibonacci and (k,h)-Lucas Numbers*, Int. J. Contemp. Math. Sciences, 6(18), (2018) 887–894.
- [44] S. Shen, *The Spectral Norms of Circulant Matrices Involving (k,h)-Fibonacci and (k,h)-Lucas Numbers*, Int. J. Contemp. Math. Sciences, 9(14), (2014) 661–665.
- [45] S. Shen, J. Cen, *On the Spectral Norms of r-Circulant Matrices with the k-Fibonacci and k-Lucas Numbers*, Int. J. Contemp. Math. Sciences, 5(12), (2010) 569–578.
- [46] S. Shen, J. Cen, *On the Bounds for the Norms of r-Circulant Matrices with the Fibonacci and Lucas Numbers*, Applied Mathematics and Computation 216, (2010) 2891–2897.
- [47] B. Shi, *The Spectral Norms of Geometric Circulant Matrices with the Generalized k-Horadam Numbers*, Journal of Inequalities and Applications, 2018:14, (2018). <https://doi.org/10.1186/s13660-017-1608-4>
- [48] S. Solak, *On the Norms of Circulant Matrices with the Fibonacci and Lucas Numbers*, Applied Mathematics and Computation, 160, (2005) 125–132.
- [49] S. Solak, *Erratum to “On the Norms of Circulant Matrices with the Fibonacci and Lucas Numbers” [Appl. Math. Comput. 160 (2005) 125–132]*, Applied Mathematics and Computation, 190, (2007) 1855–1856.
- [50] Y. Soykan, *Simson Identity of Generalized m-step Fibonacci Numbers*, Int. J. Adv. Appl. Math. and Mech. 7(2), (2019) 45–56.
- [51] Y. Soykan, *Closed Formulas for the Sums of Squares of Generalized Fibonacci Numbers*, Asian Journal of Advanced Research and Reports, 9(1), (2020) 23–39. <https://doi.org/10.9734/ajarr/2020/v9i130212>
- [52] Y. Soykan, *Closed Formulas for the Sums of Cubes of Generalized Fibonacci Numbers: Closed Formulas of and $\sum_{k=0}^n W_k^3$ and $\sum_{k=1}^n W_{-k}^3$* , Archives of Current Research International, 20(2), (2020) 58–69. DOI: 10.9734/ACRI/2020/v20i230177
- [53] Y. Soykan, *A Closed Formula for the Sums of Squares of Generalized Tribonacci numbers*, Journal of Progressive Research in Mathematics, 16(2), (2020) 2932-2941.
- [54] Y. Soykan, *A Study On Sums of Cubes of Generalized Fibonacci Numbers: Closed Formulas of $\sum_{k=0}^n x^k W_k^3$ and $\sum_{k=1}^n x^k W_{-k}^3$* , Preprints (2020), 2020040437 (doi: 10.20944/preprints202004.0437.v1).
- [55] Y. Soykan, *On Sums of Cubes of Generalized Fibonacci Numbers: Closed Formulas of $\sum_{k=0}^n kW_k^3$ and $\sum_{k=1}^n kW_{-k}^3$* , Asian Research Journal of Mathematics, 16(6), (2020) 37–52. DOI: 10.9734/ARJOM/2020/v16i630196
- [56] Y. Soykan, *On the Sums of Squares of Generalized Tribonacci Numbers: Closed Formulas of $\sum_{k=0}^n x^k W_k^2$* , Archives of Current Research International, 20(4), (2020) 22–47. DOI: 10.9734/ACRI/2020/v20i430187
- [57] Y. Soykan, *Formulae For The Sums of Squares of Generalized Tribonacci Numbers: Closed Form Formulas of $\sum_{k=0}^n kW_k^2$* , IOSR Journal of Mathematics, 16(4), (2020) 1–18. DOI: 10.9790/5728-1604010118
- [58] Y. Soykan, *A Study on Generalized Fibonacci Numbers: Sum Formulas $\sum_{k=0}^n kx^k W_k^3$ and $\sum_{k=1}^n kx^k W_{-k}^3$ for the Cubes of Terms*, Earthline Journal of Mathematical Sciences, 4(2), (2020) 297–331. <https://doi.org/10.34198/ejms.4220.297331>
- [59] Y. Soykan, *Generalized Fibonacci Numbers: Sum Formulas of the Squares of Terms*, MathLAB Journal, 5, (2020) 46–62.
- [60] Y. Soykan, *Horadam Numbers: Sum of the Squares of Terms of Sequence*, Int. J. Adv. Appl. Math. and Mech. 7(4), (2020) 34–50.
- [61] Y. Soykan, *A Study on the Sums of Squares of Generalized Fibonacci Numbers: Closed Forms of the Sum Formulas $\sum_{k=0}^n kx^k W_k^2$ and $\sum_{k=1}^n kx^k W_{-k}^2$* ,

- Asian Journal of Advanced Research and Reports, 12(1), (2020) 44–67. DOI: 10.9734/AJARR/2020/v12i130280
- [62] Y. Soykan, *On Generalized Tetranacci Numbers: Closed Form Formulas of the Sum $\sum_{k=0}^n W_k^2$ of the Squares of Terms*, International Journal of Advances in Applied Mathematics and Mechanics, 8(1), (2020) 15–26.
- [63] Y. Soykan, *On Summing Formulas For Generalized Fibonacci and Gaussian Generalized Fibonacci Numbers*, Advances in Research, 20(2), (2019) 1–15.
- [64] Y. Soykan, *Corrigendum: On Summing Formulas for Generalized Fibonacci and Gaussian Generalized Fibonacci Numbers*, Advances in Research, 21(10), (2020) 66–82. DOI: 10.9734/AIR/2020/v21i1030253
- [65] Y. Soykan, *On Summing Formulas for Horadam Numbers*, Asian Journal of Advanced Research and Reports 8(1), (2020) 45–61. DOI: 10.9734/AJARR/2020/v8i130192.
- [66] Y. Soykan, *Generalized Fibonacci Numbers: Sum Formulas*, Journal of Advances in Mathematics and Computer Science, 35(1), (2020) 89–104. DOI: 10.9734/JAMCS/2020/v35i130241.
- [67] Y. Soykan, *Generalized Tribonacci Numbers: Summing Formulas*, Int. J. Adv. Appl. Math. and Mech. 7(3), (2020) 57–76.
- [68] Y. Soykan, *Summing Formulas For Generalized Tribonacci Numbers*, Universal Journal of Mathematics and Applications, 3(1), (2020) 1–11, 2020. DOI: <https://doi.org/10.32323/ujma.637876>
- [69] Y. Soykan, *On Sum Formulas for Generalized Tribonacci Sequence*, Journal of Scientific Research & Reports, 26(7), (2020) 27–52. DOI: 10.9734/JSRR/2020/v26i730283
- [70] Y. Soykan, *Summation Formulas For Generalized Tetranacci Numbers*, Asian Journal of Advanced Research and Reports, 7(2), (2019) 1–12. doi.org/10.9734/ajarr/2019/v7i230170.
- [71] Y. Soykan, *Sum Formulas For Generalized Fifth-Order Linear Recurrence Sequences*, Journal of Advances in Mathematics and Computer Science, 34(5), (2019) 1–14. DOI: 10.9734/JAMCS/2019/v34i530224.
- [72] Y. Soykan, *Linear Summing Formulas of Generalized Pentanacci and Gaussian Generalized Pentanacci Numbers*, Journal of Advanced in Mathematics and Computer Science, 33(3), (2019) 1–14.
- [73] Y. Soykan, *On Summing Formulas of Generalized Hexanacci and Gaussian Generalized Hexanacci Numbers*, Asian Research Journal of Mathematics, 14(4), (2019) 1–14.
- [74] Y. Soykan, *A Study On Sum Formulas of Generalized Sixth-Order Linear Recurrence Sequences*, Asian Journal of Advanced Research and Reports, 14(2), (2020) 36–48. DOI:10.9734/AJARR/2020/v14i230329
- [75] Y. Soykan, *Matrix Sequences of Tribonacci and Tribonacci-Lucas Numbers*, Communications in Mathematics and Applications, 11(2), (2020) 281-295. DOI:10.26713/cma.v11i2.1102
- [76] Y. Soykan, *A Study On the Sums of Squares of Generalized Tribonacci Numbers: Closed Form Formulas of $\sum_{k=0}^n kx^k W_k^2$* , Journal of Scientific Perspectives, 5(1), (2021) 1–23. DOI:<https://doi.org/10.26900/jsp.5.1.02>
- [77] Y. Soykan, *A Study on Generalized Tetranacci Numbers: Closed Form Formulas $\sum_{k=0}^n x^k W_k^2$ of Sums of the Squares of Terms*, Asian Research Journal of Mathematics, 16(10), (2020) 109–136. DOI: 10.9734/ARJOM/2020/v16i1030234
- [78] Y. Soykan, *Generalized Fibonacci Numbers with Indices in Arithmetic Progression and Sum of Their Squares: the Sum Formula $\sum_{k=0}^n x^k W_{mk+j}^2$* , Journal of Advances in Mathematics and Computer Science, 36(6), (2021) 30–62. DOI: 10.9734/JAMCS/2021/v36i630371
- [79] Y. Soykan, *A Study On Sums of Cubes of Generalized Fibonacci Numbers: Closed Formulas of $\sum_{k=0}^n x^k W_k^3$ and $\sum_{k=1}^n x^k W_{-k}^3$* , Int. J. Adv. Appl. Math. and Mech. 8(3), (2021) 9–23.

- [80] Y. Soykan, *Sums of Cubes of Generalized Fibonacci Numbers with Indices in Arithmetic Progression: the Sum Formulas $\sum_{k=0}^n x^k W_{mk+j}^3$* , Int. J. Adv. Appl. Math. and Mech. 9(1), (2021) 6–41.
- [81] Soykan, Y., *Sum of Generalized Tribonacci Sequence: The Sum Formulas of $\sum_{k=0}^n x^k W_k$ via Generating Functions*, IOSR Journal of Mathematics (IOSR-JM), 18(1), 39-47, 2022. DOI: 10.9790/5728-1801033947
- [82] Y. Soykan, *Some Properties of Generalized Fibonacci Numbers: Identities, Recurrence Properties and Closed Forms of the Sum Formulas $\sum_{k=0}^n x^k W_{mk+j}$* , Archives of Current Research International, 21(3), (2021) 11–38. DOI: 10.9734/ACRI/2021/v21i330235
- [83] Y. Soykan, *Sum Formulas For Generalized Tetranacci Numbers: Closed Forms of the Sum Formulas $\sum_{k=0}^n x^k W_k$ and $\sum_{k=1}^n x^k W_{-k}$* , Journal of Progressive Research in Mathematics, 18(1), (2021) 24–47.
- [84] Y. Soykan, *A Study on Sum Formulas of Generalized Pentanacci Sequence: Closed Forms of the Sum Formulas $\sum_{k=0}^n x^k W_k$ and $\sum_{k=1}^n x^k W_{-k}$* , Journal of Progressive Research in Mathematics, 18(2), (2021) 20–38.
- [85] Y. Soykan, *A Study on Sum Formulas of Generalized Hexanacci Numbers: Closed Forms of the Sum Formulas $\sum_{k=0}^n x^k W_k$ and $\sum_{k=1}^n x^k W_{-k}$* , Asian Research Journal of Mathematics, 17(3), (2021) 93–118. DOI: 10.9734/ARJOM/2021/v17i330285
- [86] Y. Soykan, *Sum of the Cubes of Generalized Mersenne Numbers: the Sum Formula $\sum_{k=0}^n x^k W_{mk+j}^3$* , IOSR Journal of Mathematics, 17(5), (2021) 25–41. DOI: 10.9790/5728-1705022541
- [87] Y. Soykan, *On the Sum of the Cubes of Generalized Oresme Numbers: the Sum Formula $\sum_{k=0}^n x^k W_{mk+j}^3$* , Asian Research Journal of Current Science, 3(1), (2021) 295–308.
- [88] Y. Soykan, E. Taşdemir, C. M. Dikmen, *On the Sum of the Cubes of Generalized Balancing Numbers: the Sum Formula $\sum_{k=0}^n x^k W_{mk+j}^3$* , Open Journal of Mathematical Sciences (OMS), 6, (2022) 152-167. doi:10.30538/oms2022.0184
- [89] Y. Soykan, *A Study on the Sum of the Squares of Generalized Oresme Numbers: The Sum Formula $\sum_{k=0}^n x^k W_{mk+j}^2$* , Asian Journal of Pure and Applied Mathematics, 4(1), (2022) 16–27.
- [90] Y. Soykan, *A Study on the Sum of the Squares of Generalized p -Oresme Numbers: the Sum Formula $\sum_{k=0}^n x^k W_{mk+j}^2$* , Asian Journal of Advanced Research and Reports, 16(1), (2022) 1–24. DOI: 10.9734/AJARR/2022/v16i130444
- [91] Y. Soykan, *On the Sum of the Squares of Generalized Mersenne Numbers: the Sum Formula $\sum_{k=0}^n x^k W_{mk+j}^2$* , International Journal of Advances in Applied Mathematics and Mechanics, 9(2), (2021) 28–37.
- [92] Y. Soykan, E. Taşdemir, C. M. Dikmen, *A Study on the Sum of the Squares of Generalized Balancing Numbers: the Sum Formula $\sum_{k=0}^n x^k W_{mk+j}^2$* , Journal of Innovative Applied Mathematics and Computational Sciences, 1(1), (2021) 16–30.
- [93] Y. Soykan, *On k -circulant Matrices with the Generalized Third-Order Pell Numbers*, Notes on Number Theory and Discrete Mathematics, 27(4), (2021) 187–206. DOI:10.7546/nntdm.2021.27.4.187-206.
- [94] Y. Soykan, *Explicit Euclidean Norm, Eigenvalues, Spectral Norm and Determinant of Circulant Matrix with the Generalized Tribonacci Numbers*, Earthline Journal of Mathematical Sciences, 6(1), (2021) 131–151. <https://doi.org/10.34198/ejms.6121.131151>
- [95] Y. Soykan, *A Study On Generalized (r,s,t,u,v,y) -Numbers*, Journal of Progressive Research in Mathematics, 17(1), (2020) 54–72.
- [96] Y. Soykan, N. Özmen, *On Generalized Hexanacci and Gaussian Generalized Hexanacci Numbers*, Turk. J. Math. Comput. Sci. 13(1), (2021) 25–43. DOI:10.47000/tjmcs.787578
- [97] Y. Soykan, *On Generalized Sixth-Order Pell Sequence*, Journal of Scientific Perspectives, 4(1), (2020) 49–70. DOI: <https://doi.org/10.26900/jsp.4.005>.

- [98] Y. Soykan, E. E. Polatlı, *On Generalized Sixth-Order Jacobsthal Sequence*, Int. J. Adv. Appl. Math. and Mech. 8(3), (2021) 24–40.
- [99] Y. Soykan, *Properties of Generalized 6-primes Numbers*, Archives of Current Research International, 20(6), (2020) 12–30. DOI: 10.9734/ACRI/2020/v20i630199
- [100] Y. Soykan, *Sum Formulas of Generalized Hexanacci Sequence: Closed Forms of the Sum Formulas $\sum_{k=0}^n kW_k$ and $\sum_{k=1}^n kW_{-k}$* , International Journal of Mathematics Trends and Technology- IJM-TT, 67(4), (2021) 67–78. /doi:10.14445/22315373/IJM-TT-V67I4P510
- [101] N. Tuglu, C. Kızılateş, *On the Norms of Circulant and r-Circulant Matrices with the Hyperharmonic Fibonacci Numbers*, Journal of Inequalities and Applications, 2015, Article ID 253, (2015). <http://dx.doi.org/10.1186/s13660-015-0778-1>
- [102] R. Turkmen, H. Gökbaş, *On the Spectral Norm of r-Circulant Matrices with the Pell and Pell-Lucas Numbers*, J. Inequal. Appl. 2016:65, (2016).
- [103] Ş Uygun, *Some Bounds for the Norms of Circulant Matrices with the k-Jacobsthal and k-Jacobsthal Lucas Numbers*, Journal of Mathematics Research, 8(6), (2016) 133–138.
- [104] Ş Uygun, S. Yaşamalı, *On the Bounds for the Norms of Circulant Matrices with the Jacobsthal and Jacobsthal–Lucas Numbers*, Notes on Number Theory and Discrete Mathematics, 23(1), (2017) 91–98.
- [105] Ş Uygun, S. Yaşamalı, *On the Bounds for the Norms of r-Circulant Matrices with the Jacobsthal and Jacobsthal–Lucas Numbers*, International Journal of Pure and Applied Mathematics, 112(1), (2017) 93–102.
- [106] M. E. Waddill, *The Tetranacci Sequence and Generalizations*, Fibonacci Quarterly, 30(1), (1992) 9–20.
- [107] W. Wamiliana, S. Suharsono, P. E. Kristanto, *Counting the sum of cubes for Lucas and Gibonacci Numbers*, Science and Technology Indonesia, 4(2), (2019) 31–35.

YÜKSEL SOYKAN

ORCID NUMBER: 0000-0002-1895-211X

DEPARTMENT OF MATHEMATICS, ART AND SCIENCE FACULTY

ZONGULDAK BÜLENT ECEVİT UNIVERSITY

ZONGULDAK, TURKEY

Email address: yukse1.soykan@hotmail.com

ERKAN TAŞDEMİR

ORCID NUMBER: 0000-0002-5002-3193

PINARHISAR VOCATIONAL SCHOOL

KIRKLARELİ UNIVERSITY

KIRKLARELİ, TURKEY

Email address: erkantasdemir@hotmail.com

TÜLİN ERDOĞAN TAŞDEMİR

ORCID NUMBER: 0000-0002-7333-0217

MINISTRY OF NATIONAL EDUCATION

PINARHISAR ANATOLIAN HIGH SCHOOL

KIRKLARELİ, TURKEY

Email address: tulintasdemir@hotmail.com