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GROUPS WITH SOME CENTRAL AUTOMORPHISMS FIXING THE CENTRAL KERNEL QUOTIENT

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Abstract. Let G be a group. An automorphism α of the group G is called a central automorphism, if $x^{-1}x^{\alpha} \in Z(G)$ for all $x \in G$. Let $L_c(G)$ be the central kernel of G, that is, the set of elements of G fixed by all central automorphisms of G and $\mathrm{Aut}_{L_c}(G)$ denote the group of all central automorphisms of G fixing $G/L_c(G)$ element-wise. In the present paper, we investigate the properties of such automorphisms. Moreover, a full classification of p-groups G of order at most p^5 where $\mathrm{Aut}_{L_c}(G) = \mathrm{Inn}(G)$ is also given.

Keywords: Automorphism group; Central kernel, Central autocommutator.
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1. Introduction

Let G be a group and p a prime number. We denote by G', $Z(G)$, $\mathrm{Inn}(G)$ and $\mathrm{Aut}(G)$, the commutator subgroup, the center, the group of all inner automorphisms and the group of all automorphisms of G, respectively. For each $x \in G$ and $\alpha \in \mathrm{Aut}(G)$, the element $[x, \alpha] = x^{-1}x^{\alpha}$ is called the autocommutator of x and α, in which x^{α} is the image of x under α. An automorphism α of G is called a central automorphism if $[x, \alpha] \in Z(G)$ for all $x \in G$. An automorphism α of G is called an IA-automorphism if $[x, \alpha] \in G'$ for all $x \in G$. Also, an automorphism α of G is called a class preserving automorphism if $x^{\alpha} \in x^G$ for all $x \in G$, where x^G is the conjugacy class of x in G. Let $\mathrm{Autcent}(G)$, $\mathrm{IA}(G)$ and $\mathrm{Aut}_c(G)$, denote the group of all central automorphisms, IA-automorphisms and class preserving automorphisms of G, respectively. In 1994, Hegarty [7] introduced the concept of absolute center subgroup of a group G, as follows:

$$L(G) = \{x \in G \mid [x, \alpha] = 1, \forall \alpha \in \mathrm{Aut}(G)\}.$$

It is easy to check that the absolute center of G is a characteristic subgroup contained in the center of G. Haimo [5] introduced the following subgroup of a given group G, which we call similar to [2], the central kernel of G and denote

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by $L_c(G)$, as

$$L_c(G) = \{x \in G \mid [x, \alpha] = 1, \forall \alpha \in \mathrm{Autcent}(G)\}.$$

Since the central automorphisms of G fixing G' element-wise, we conclude that $G' \leq L_c(G)$, and so $G/L_c(G)$ is abelian. Also

$$K_c(G) = \langle [x, \alpha] \mid x \in G, \alpha \in \mathrm{Autcent}(G)\rangle,$$

is said the central autocommutator subgroup of G (see [2]). One can easily check that $L_c(G)$ is a characteristic subgroup of G contains $L(G)$ and $K_c(G)$ is a central characteristic subgroup of G. Now, we call $\alpha \in \mathrm{Autcent}(G)$ to be central kernel automorphism, when $[x, \alpha] \in L_c(G)$, for all $x \in G$. According to [2], let $\mathrm{Aut}_{L_c}(G)$ denote the group of all central kernel automorphisms of G. Clearly, $\mathrm{Aut}_{L_c}(G)$ is a normal subgroup of $\mathrm{Autcent}(G)$ and acts trivially on the central kernel of G. Davoudirad et al. ([2], [3]) for an arbitrary group G, investigate some properties of $\mathrm{Aut}_{L_c}(G)$ and the central kernel subgroup of G.

In this paper, first we give some necessary and sufficient conditions on a finite p-group G such that $\mathrm{Aut}_{L_c}(G)$ is equal to $C_{\mathrm{Aut}_{L_c}(G)}(Z(G))$, $\mathrm{Inn}(G)$, $\mathrm{IA}(G)$ and $\mathrm{Aut}_c(G)$, respectively. Finally, we classify all p-groups G of order at most p^5 such that $\mathrm{Aut}_{L_c}(G) = \mathrm{Inn}(G)$.

2. Preliminaries

For a finite group G, $\exp(G), d(G)$, $\Omega_i(G)$, $\mathrm{cl}(G)$ and $o(x)$, denote the exponent of G, minimal number of generators of G, the subgroup of G generated by its elements of order dividing p^i, the nilpotency class of G and the order of x, respectively. For a finite p-group G, if A is a normal subgroup of $\mathrm{Aut}(G)$, then we use $C_A(Z(G))$ to denote the group of all automorphisms of A which centralizing $Z(G)$ element-wise. Moreover, let us denote by C_n the cyclic group of order n, where $n \geq 1$ and C_n^k be the direct product of k copies of C_n, D_8 the dihedral group, Q_8 the quaternion group of order 8, respectively. Recall that an abelian finite p-group A has invariants or is of type $(n_1, n_2, ..., n_k)$ if it is the direct product of cyclic subgroups of orders $p^{n_1}, p^{n_2}, ..., p^{n_k}$, where $n_1 \geq n_2 \geq \cdots \geq n_k > 0$. We use the notation $\mathrm{Hom}(G, A)$ to denote the group of homomorphisms of G into an abelian group A. Finally, recall that a group G is called a central product of its subgroups A, B if A and B commute element-wise and together generate G. In this situation, we write $G = A * B$.

The following lemma is a well-known result and will be used in the sequel.

Lemma 2.1. *Let A, B and C be finite abelian groups. Then*

(i) $\mathrm{Hom}(A \times B, C) \cong \mathrm{Hom}(A, C) \times \mathrm{Hom}(B, C)$.
(ii) $\mathrm{Hom}(A, B \times C) \cong \mathrm{Hom}(A, B) \times \mathrm{Hom}(A, C)$.
(iii) $\mathrm{Hom}(C_m, C_n) \cong C_d$, *where d is the greatest common divisor of m and n.*

Corollary 2.2. *Let A, B and C be finite abelian p-groups, $\exp(C) = p^t$ and $A \leq B$. Then $\mathrm{Hom}(A, C) \cong \mathrm{Hom}(B, C)$ if and only if $A \cong H \times A_1, B \cong H \times B_1$ where all invariants of A_1, B_1 are at least t, $d(A_1) = d(B_1)$ and $\exp(H) < p^t$.*

Proof. It can be proved by using Lemma 2.1 and induction on $|C|$. □

3. Main results

In this section, we provide some results concerning the group of all central kernel automorphisms of G. First, we define two subgroups of $\mathrm{Autcent}(G)$ and G as follows:

$$C_{\mathrm{Autcent}(G)}(\mathrm{Aut}_{L_c}(G)) = \{\alpha \in \mathrm{Autcent}(G) \mid \alpha\beta = \beta\alpha,\ \forall \beta \in \mathrm{Aut}_{L_c}(G)\},$$

and

$$E_{L_c}(G) = [G, C_{\mathrm{Autcent}(G)}(\mathrm{Aut}_{L_c}(G))].$$

Obviously, $E_{L_c}(G)$ is a characteristic subgroup in G, which is contained in $K_c(G)$. Also, if $G/Z(G)$ is abelian, then

$$G' = [G, \mathrm{Inn}(G)] \leq [G, C_{\mathrm{Autcent}(G)}(\mathrm{Aut}_{L_c}(G))] \leq E_{L_c}(G).$$

The following lemma states the useful property of $E_{L_c}(G)$, which will be needed for our further investigation.

Lemma 3.1. *If G is an arbitrary group, then $\mathrm{Aut}_{L_c}(G)$ acts trivially on the subgroup $E_{L_c}(G)$ of G.*

Proof. Let $\alpha \in \mathrm{Aut}_{L_c}(G)$. Then $g^{-1}g^{\alpha} \in L_c(G)$ for all $g \in G$ and so $g^{\alpha} = gt_g$, for some $t_g \in L_c(G)$. By taking an automorphism $\beta \in C_{\mathrm{Autcent}(G)}(\mathrm{Aut}_{L_c}(G))$, we have

$$\begin{aligned}[g,\beta]^{\alpha} &= (g^{-1}g^{\beta})^{\alpha} = (g^{-1})^{\alpha}(g^{\beta})^{\alpha} = (g^{-1})^{\alpha}(g^{\alpha})^{\beta} \\ &= t_g^{-1}g^{-1}g^{\beta}t_g^{\beta} = g^{-1}g^{\beta}t_g^{-1}t_g = [g,\beta],\end{aligned}$$

which completes the proof. □

Lemma 3.2. *Let G be a group. Then*

(i) $\mathrm{Aut}_{L_c}(G) \cong \mathrm{Hom}(G/E_{L_c}(G)L_c(G), L_c(G) \cap Z(G))$.
(ii) $C_{\mathrm{Aut}_{L_c}(G)}(Z(G)) \cong \mathrm{Hom}(G/Z(G)L_c(G), L_c(G) \cap Z(G))$.

Proof. (i) Take an automorphism $\theta \in \mathrm{Aut}_{L_c}(G)$. Then we see that $f_\theta : gE_{L_c}(G)L_c(G) \mapsto g^{-1}g^{\theta}$, defines a homomorphism from $G/E_{L_c}(G)L_c(G)$ to $L_c(G) \cap Z(G)$ and the map φ sending θ to f_θ defines a monomorphism from $\mathrm{Aut}_{L_c}(G)$ to the group $\mathrm{Hom}(G/E_{L_c}(G)L_c(G), L_c(G) \cap Z(G))$. Also, let $f \in \mathrm{Hom}(G/E_{L_c}(G)L_c(G), L_c(G) \cap Z(G))$. Then the map $\theta = \theta_f$ defined by $x^{\theta} = xf(xE_{L_c}(G)L_c(G))$, for all $x \in G$, is a central kernel automorphism of G and $\varphi(\theta) = \varphi(\theta_f) = f$. Hence φ is onto and the proof is complete.

(ii) It is sufficient to observe that for each $\theta \in C_{\mathrm{Aut}_{L_c}(G)}(Z(G))$, the map

$$f_\theta : G/Z(G)L_c(G) \to L_c(G) \cap Z(G)$$

$$gZ(G)L_c(G) \mapsto g^{-1}g^{\theta}$$

defines a homomorphism and $\theta \mapsto f_\theta$ is an isomorphism from $C_{\mathrm{Aut}_{L_c}(G)}(Z(G))$ to $\mathrm{Hom}(G/Z(G)L_c(G), L_c(G) \cap Z(G))$. $\square$

Theorem 3.3. *Let G be a finite p-group and $G/E_{L_c}(G)L_c(G)$, $G/Z(G)L_c(G)$ and $L_c(G) \cap Z(G)$ are of types $(a_1, a_2, ..., a_k)$, $(b_1, b_2, ..., b_m)$ and $(c_1, c_2, ..., c_n)$, respectively. Then*

$$\mathrm{Aut}_{L_c}(G) = C_{\mathrm{Aut}_{L_c}(G)}(Z(G))$$

if and only if $Z(G) \leq E_{L_c}(G)L_c(G)$ or $d(G/E_{L_c}(G)L_c(G)) = d(G/Z(G)L_c(G))$ and $b_{l+1} < c_1 \leq b_l$, where l is the largest integer between 1 and m such that $b_l < a_l$.

Proof. Let G be a finite p-group such that $\mathrm{Aut}_{L_c}(G) = C_{\mathrm{Aut}_{L_c}(G)}(Z(G))$ and $Z(G) \nleq E_{L_c}(G)L_c(G)$. We claim that $Z(G) \leq \Phi(G)$; otherwise, let M be a maximal subgroup of G such that $Z(G) \nleq M$. We write $G = M\langle z \rangle$ where $z \in Z(G) \setminus M$ and choose an element $u \in \Omega_1(Z(G) \cap L_c(G))$. Then the map $\alpha : hz^i \mapsto h(zu)^i$, where $h \in M$ and $0 \leq i < p$, defines an automorphism of G which is in $\mathrm{Aut}_{L_c}(G)$. So that α is an automorphism of G fixes $Z(G)$ element-wise, whence $u = 1$ which is impossible. Therefore $Z(G) \leq \Phi(G)$ and so $k = d(G/E_{L_c}(G)L_c(G)) = d(G/Z(G)L_c(G)) = m$. Since $E_{L_c}(G)L_c(G) < Z(G)L_c(G)$, we have $G/Z(G)L_c(G)$ is a proper quotient group of $G/E_{L_c}(G)L_c(G)$. Since $\mathrm{Hom}(G/E_{L_c}(G)L_c(G), L_c(G) \cap Z(G)) \cong \mathrm{Hom}(G/Z(G)L_c(G), L_c(G) \cap Z(G))$, by using Corollary 2.2, $G/E_{L_c}(G)L_c(G) \cong X \times Y$, $G/Z(G)L_c(G) \cong H \times Y$, where X, H are of types $(a_1, ..., a_l)$ and $(b_1, ..., b_l)$, respectively, in which $d(X) = d(H) = l$. Hence l is the largest integer between 1 and m such that $b_l < a_l$ and by Corollary 2.2, $b_{l+1} < c_1 \leq b_l$, as required.

Conversely, if $Z(G) \leq E_{L_c}(G)L_c(G)$, then $\mathrm{Aut}_{L_c}(G) = C_{\mathrm{Aut}_{L_c}(G)}(Z(G))$. Next, assume that $E_{L_c}(G)L_c(G) < Z(G)L_c(G)$, $k = m$ and l is the largest integer between 1 and m such that $b_l < a_l$ and $b_{l+1} < c_1 \leq b_l$. Let $G/E_{L_c}(G)L_c(G) = X \times Y$, where X, Y are of types $(a_1, ..., a_l)$ and $(a_{l+1}, ..., a_m)$. Moreover, $G/Z(G)L_c(G) = H \times K$, where H, K have invariants $(b_1, ..., b_l)$ and $(b_{l+1}, ..., b_m)$. Since $a_i = b_i$ for $l + 1 \leq i \leq m$, we have $K = Y$. Therefore by Corollary 2.2,

$$\mathrm{Hom}(G/E_{L_c}(G)L_c(G), L_c(G) \cap Z(G)) \cong \mathrm{Hom}(G/Z(G)L_c(G), L_c(G) \cap Z(G)),$$

and hence $\mathrm{Aut}_{L_c}(G) = C_{\mathrm{Aut}_{L_c}(G)}(Z(G))$, which completes the proof. $\square$

Let G be a finite p-group of class 2. Since $\mathrm{Autcent}(G)$ acts trivially on the central kernel of G, we have $L_c(G) \leq Z(G)$. Let $G/E_{L_c}(G)L_c(G)$, $G/Z(G)$ and $L_c(G)$ are of types $(a_1, a_2, ..., a_k)$, $(b_1, b_2, ..., b_m)$ and $(c_1, c_2, ..., c_n)$, respectively. Also let t be the largest integer between 1 and m such that $b_1 = b_2 = \cdots = b_t$. It is shown [10, Lemma 0.4] that, $t \geq 2$. Set $\bar{A} = A/Z(G)$ is of type $(b_1, b_2, ..., b_t)$

and $\bar{A}$ is isomorphic to a subgroup of $\bar{B} = B/E_{L_c}(G)L_c(G)$ which is of type $(a_1, a_2, ..., a_t)$.

By keeping the above notation, in the following theorem, we give a necessary and sufficient condition on a fixed finite p-group G of class 2 such that each automorphism of $\mathrm{Aut}_{L_c}(G)$ fixes the center of G element-wise.

Theorem 3.4. *Let G be a finite p-group of class 2. Then*

$$\mathrm{Aut}_{L_c}(G) = C_{\mathrm{Aut}_{L_c}(G)}(Z(G))$$

if and only if one of the following conditions holds:

(i) $E_{L_c}(G)L_c(G) = Z(G)$ *or*

(ii) $E_{L_c}(G)L_c(G) < Z(G)$, $k = m$, $(G/Z(G))/\bar{A} \cong (G/E_{L_c}(G)L_c(G))/\bar{B}$ *and* $\exp(G') = \exp(L_c(G))$.

Proof. First assume that $\mathrm{Aut}_{L_c}(G) = C_{\mathrm{Aut}_{L_c}(G)}(Z(G))$. Since $\mathrm{cl}(G) = 2$, it follows that $L_c(G) \leq Z(G)$. We may suppose that $E_{L_c}(G)L_c(G) < Z(G)$. By Theorem 3.3, $k = d(G/E_{L_c}(G)L_c(G)) = d(G/Z(G)) = m$. Since $G/Z(G)$ is a proper quotient group of $G/E_{L_c}(G)L_c(G)$, there exists some $1 \leq j \leq m$ such that $b_j < a_j$. Let l be the largest integer between 1 and m such that $b_l < a_l$. We claim that $\exp(G') = \exp(L_c(G))$. To do this, we observe that by Theorem 3.3, $\exp(L_c(G)) \leq p^{b_l} \leq p^{b_1} = \exp(G/Z(G))$. It follows that

$$\exp(G') \leq \exp(L_c(G)) \leq \exp(G/Z(G)) = \exp(G'),$$

by [10, Lemma 0.4], because $G/L_c(G)$ is abelian. So we conclude that $\exp(G') = \exp(L_c(G))$, as desired. Next, $b_1 = c_1 \leq b_l$ shows that $c_1 = b_1 = b_2 = \cdots = b_l$ and hence $l \leq t$. Set $\bar{A} = A/Z(G)$ is of type $(b_1, b_2, ..., b_t)$, $\bar{B} = B/E_{L_c}(G)L_c(G)$ which is of type $(a_1, a_2, ..., a_t)$ and U and V are of types $(a_{t+1}, a_{t+2}, ..., a_k)$ and $(b_{t+1}, b_{t+2}, ..., b_k)$. Since $a_i = b_i$ for all $l+1 \leq i \leq m$, then $U \cong V$ and therefore $(G/Z(G))/\bar{A} \cong V \cong U \cong (G/E_{L_c}(G)L_c(G))/\bar{B}$, as required.

Conversely, if $E_{L_c}(G)L_c(G) = Z(G)$, then it is clear that $\mathrm{Aut}_{L_c}(G) = C_{\mathrm{Aut}_{L_c}(G)}(Z(G))$. Next, suppose that $E_{L_c}(G)L_c(G) < Z(G)$, $k = m$, $\exp(G') = \exp(L_c(G))$ and $(G/Z(G))/\bar{A} \cong (G/E_{L_c}(G)L_c(G))/\bar{B}$. Since $G/Z(G)$ is a proper quotient group of $G/E_{L_c}(G)L_c(G)$, let l be the largest integer between 1 and m such that $b_l < a_l$. Hence $l \leq t$, because of $b_i = a_i$ for $t+1 \leq i \leq m$. Now $p^{c_1} = \exp(L_c(G)) = \exp(G') = \exp(G/Z(G)) = p^{b_1}$ and so $c_1 = b_1 = b_2 = \cdots = b_l$, which together with Theorem 3.3, gives the proof. □

Lemma 3.5. *Let G be a finite non-abelian p-group. Then $C_{\mathrm{Aut}_{L_c}(G)}(Z(G)) = \mathrm{Inn}(G)$ if and only if $L_c(G) \leq Z(G)$ and $L_c(G)$ is cyclic.*

Proof. Suppose that $L_c(G)$ is cyclic and $L_c(G) \leq Z(G)$. Hence $\exp(G/Z(G)) = \exp(G')$, since $G' \leq L_c(G)$. This implies that $\exp(G/Z(G))$ divides $\exp(L_c(G))$. Then by Lemma 3.2,

$$C_{\mathrm{Aut}_{L_c}(G)}(Z(G)) \cong \mathrm{Hom}(G/Z(G), L_c(G)) \cong G/Z(G)$$

and so $C_{\mathrm{Aut}_{L_c}(G)}(Z(G)) = \mathrm{Inn}(G)$, as required.

Conversely, assume that $C_{\mathrm{Aut}_{L_c}(G)}(Z(G)) = \mathrm{Inn}(G)$. It follows that $L_c(G) \leq Z(G)$, which together with Lemma 3.2 and the fact that

$$\mathrm{Hom}(G/Z(G), L_c(G)) \cong G/Z(G),$$

completes the proof. □

In the following result, we give some properties of finite non-abelian p-groups G such that $\mathrm{Aut}_{L_c}(G) = \mathrm{Inn}(G)$. Let G be a finite non-abelian p-group and $G/E_{L_c}(G)L_c(G)$ is of type $(a_1, a_2, ..., a_k)$. Also if $G/Z(G)$ is abelian, then it has invariants $(b_1, b_2, ..., b_m)$.

By fixing the above notation, we have the following result:

Theorem 3.6. *Let G be a finite non-abelian p-group. Then $\mathrm{Aut}_{L_c}(G) = \mathrm{Inn}(G)$ if and only if $L_c(G)$ is cyclic, $L_c(G) \leq Z(G)$, $m = k$ and one of the following conditions holds:*

(i) *$E_{L_c}(G)L_c(G) = Z(G)$ or*

(ii) *$b_t = r$ and $a_s = b_s$ for $s = t+1, ..., k$, where $\exp(L_c(G)) = p^r$ and t is the largest integer between 1 and k such that $a_t > r$.*

Proof. Suppose that $\mathrm{Aut}_{L_c}(G) = \mathrm{Inn}(G)$. By Lemma 3.5, we deduce that $L_c(G) \leq Z(G)$ and $L_c(G)$ is cyclic, because $C_{\mathrm{Aut}_{L_c}(G)}(Z(G)) = \mathrm{Inn}(G)$. Now by Lemma 3.2, we have

$$\begin{aligned} d(G/Z(G)) &= d(\mathrm{Hom}(G/E_{L_c}(G)L_c(G), L_c(G))) \\ &= d(G/E_{L_c}(G)L_c(G))d(L_c(G)) = d(G/E_{L_c}(G)L_c(G)), \end{aligned}$$

and so $m = k$. If $\exp(G/E_{L_c}(G)L_c(G)) \leq \exp(L_c(G))$, then

$$G/Z(G) \cong \mathrm{Aut}_{L_c}(G) \cong \mathrm{Hom}(G/E_{L_c}(G)L_c(G), L_c(G)) \cong G/E_{L_c}(G)L_c(G),$$

because $L_c(G)$ is cyclic. Therefore $E_{L_c}(G)L_c(G) = Z(G)$.

Next, let $\exp(G/E_{L_c}(G)L_c(G)) > \exp(L_c(G))$ and t is the largest integer such that $a_t > r$, where $\exp(L_c(G)) = p^r$. Then we observe that

$$\mathrm{Hom}(G/E_{L_c}(G)L_c(G), L_c(G)) \cong C_{p^r} \times C_{p^r} \times \cdots \times C_{p^r} \times C_{p^{a_{t+1}}} \times \cdots \times C_{p^{a_k}}.$$

Now, since $\mathrm{Hom}(G/E_{L_c}(G)L_c(G), L_c(G)) \cong G/Z(G) = C_{p^{b_1}} \times C_{p^{b_2}} \times \cdots \times C_{p^{b_k}}$, it follows that $b_1 = b_2 = \cdots = b_t = r$ and $a_i = b_i$ for $t+1 \leq i \leq k$, as required.

Conversely, if $E_{L_c}(G)L_c(G) = Z(G)$, then

$$\mathrm{Aut}_{L_c}(G) \cong \mathrm{Hom}(G/E_{L_c}(G)L_c(G), L_c(G)) = \mathrm{Hom}(G/Z(G), L_c(G)) \cong G/Z(G),$$

because $L_c(G)$ is cyclic and $G' \leq L_c(G)$. Hence $\mathrm{Aut}_{L_c}(G) = \mathrm{Inn}(G)$. Next assume that $E_{L_c}(G)L_c(G) < Z(G)$, $b_t = r$ and $a_s = b_s$ for $s = t+1, ..., k$, where $\exp(L_c(G)) = p^r$ and t is the largest integer between 1 and k such that $a_t > r$. As G is of class 2 and $G/L_c(G)$ is abelian, so

$$p^{b_1} = \exp(G/Z(G)) = \exp(G') | \exp(L_c(G)) = p^r.$$

Therefore $r \geq b_1 \geq b_2 \geq \cdots \geq b_t = r$, which shows that $b_1 = b_2 = \cdots = b_t = r$. Since $a_s = b_s$ for $s = t+1, ..., k$, we have

$$\begin{aligned} \mathrm{Hom}(G/E_{L_c}(G)L_c(G), L_c(G)) &\cong C_{p^r} \times \cdots \times C_{p^r} \times C_{p^{a_{t+1}}} \times \cdots \times C_{p^{a_k}} \\ &= C_{p^{b_1}} \times \cdots \times C_{p^{b_t}} \times C_{p^{b_{t+1}}} \times \cdots \times C_{p^{b_k}} \\ &= G/Z(G). \end{aligned}$$

Therefore by Lemma 3.2, $\mathrm{Aut}_{L_c}(G) = \mathrm{Inn}(G)$. This completes the proof. □

Lemma 3.7. *Let G be a finite group such that $Z(G/E_{L_c}(G)) = H/E_{L_c}(G)$. Then*

(i) $Z(\mathrm{Inn}(G)) \leq \mathrm{Aut}_{L_c}(G)$ *and* $H = Z_2(G)$.
(ii) $\mathrm{Aut}_{L_c}(G) = Z(\mathrm{Inn}(G))$ *if and only if* $\mathrm{Hom}(G/E_{L_c}(G)L_c(G), L_c(G) \cap Z(G)) \cong H/Z(G)$.

Proof. (i) Let $i_t \in Z(\mathrm{Inn}(G))$, where i_t is an inner automorphism of G induced by the element t in G. Then $i_t \in \mathrm{Autcent}(G)$ and for all $g \in G$, $[g, i_t] = [g, t] \in L_c(G)$ since $G/L_c(G)$ is abelian. Thus $i_t \in \mathrm{Aut}_{L_c}(G)$. Next we show that $H = Z_2(G)$. Let $t \in H$. Then for all $g \in G$, $[g, t] \in E_{L_c}(G) \leq Z(G)$. Thus $t \in Z_2(G)$. On the other hand, assume that $t \in Z_2(G)$ and $\alpha = i_t$. Then for all $g \in G$, $[g, \alpha] = [g, t] \in Z(G)$ which shows that $\alpha \in \mathrm{Autcent}(G)$. As $\alpha \in C_{\mathrm{Autcent}(G)}(\mathrm{Aut}_{L_c}(G))$, it follows that $[g, t] = [g, \alpha] \in E_{L_c}(G)$, for all $g \in G$ and hence $t \in H$.
(ii) Suppose that $\mathrm{Aut}_{L_c}(G) = Z(\mathrm{Inn}(G))$. By (i) and Lemma 3.2 we have

$$\mathrm{Hom}(G/E_{L_c}(G)L_c(G), L_c(G) \cap Z(G)) \cong H/Z(G).$$

Conversely, suppose that $\mathrm{Hom}(G/E_{L_c}(G)L_c(G), L_c(G) \cap Z(G)) \cong H/Z(G)$. Then, by Lemma 3.2,

$$\mathrm{Aut}_{L_c}(G) \cong H/Z(G) = Z_2(G)/Z(G) = Z(\mathrm{Inn}(G)).$$

Therefore $|\mathrm{Aut}_{L_c}(G)| = |Z(\mathrm{Inn}(G))|$, which together with (i), $\mathrm{Aut}_{L_c}(G) = Z(\mathrm{Inn}(G))$, as required. □

Corollary 3.8. *Let G be an extra-special p-group. Then* $\mathrm{Aut}_{L_c}(G) = \mathrm{Inn}(G)$.

Proof. The proof follows at once from the fact that $G' = E_{L_c}(G) = L_c(G) = Z(G) \cong C_p$ and Theorem 3.6. □

Lemma 3.9. *Let G be a finite non-abelian p-group such that* $\mathrm{Aut}_{L_c}(G) = \mathrm{IA}(G)$. *Then*

(i) $L_c(G) \leq Z(G)$ *and* $E_{L_c}(G)L_c(G) \leq \Phi(G)$.
(ii) $\mathrm{Aut}_{L_c}(G) \cong \mathrm{Hom}(G/G', L_c(G)) \cong \mathrm{Hom}(G/L_c(G), G')$.

Proof. (i) Since $\mathrm{Aut}_{L_c}(G) = \mathrm{IA}(G)$, it is easy to see that $L_c(G) \leq Z(G)$. We prove that $E_{L_c}(G)L_c(G) \leq \Phi(G)$. Suppose on the contrary, that there exists a maximal subgroup M of G such that $E_{L_c}(G)L_c(G) \nleq M$. Then $G = M\langle l \rangle$, for some l in $E_{L_c}(G)L_c(G) \backslash M$. Choose an element u in $\Omega_1(G')$. We observe that the map $\alpha : hl^i \mapsto hl^i u^i$, where $h \in M$ and $0 \leq i < p$, is an IA-automorphism of G, which is a central kernel automorphism of G. Hence $[l,\alpha]=1$ and so $u=1$, a contradiction. Therefore $E_{L_c}(G)L_c(G)\leq\Phi(G)$.

(ii) Let $\alpha\in\mathrm{Aut}_{L_c}(G)$. Then $f_\alpha:G\to L_c(G)$ given by $f_\alpha(x)=x^{-1}x^\alpha$ defines a homomorphism from G to $L_c(G)$, and $\alpha\mapsto f_\alpha$ is an injective map from $\mathrm{Aut}_{L_c}(G)$ to $\mathrm{Hom}(G,L_c(G))$. Conversely, if $f\in\mathrm{Hom}(G,L_c(G))$, then the map $\alpha=\alpha_f$ defined by $\alpha(x)=xf(x)$ for all $x\in G$ is an endomorphism of G. Since by (i), $x^{-1}\alpha(x)\in L_c(G)\leq E_{L_c}(G)L_c(G)\leq\Phi(G)$ for all $x\in G$, we may write G as the product of the image of α and the Frattini subgroup of G and so the image of α must be G itself. Thus α is an automorphism of G. Consequently $\alpha=\alpha_f\in\mathrm{Aut}_{L_c}(G)$, $f_{\alpha_f}=f$ and so $|\mathrm{Aut}_{L_c}(G)|=|\mathrm{Hom}(G,L_c(G))|$. Finally, suppose that $\beta,\gamma\in\mathrm{Aut}_{L_c}(G)$. Then for any $x\in G$,

$$f_{\beta\gamma}(x)=x^{-1}x^{\gamma\beta}=x^{-1}(xx^{-1}x^{\gamma})^{\beta}=x^{-1}x^{\beta}x^{-1}x^{\gamma},$$

since $x^{-1}x^\gamma\in L_c(G)$. Thus $f_{\beta\gamma}(x)=f_\beta(x)f_\gamma(x)$ and so $\alpha\mapsto f_\alpha$ is a homomorphism, as desired. Next we show that $\mathrm{Aut}_{L_c}(G)\cong\mathrm{Hom}(G/L_c(G),G')$. For any $\alpha\in\mathrm{Aut}_{L_c}(G)$, the map $f_\alpha:G/L_c(G)\to G'$ given by $f_\alpha(xL_c(G))=x^{-1}x^\alpha$ defines a homomorphism from $G/L_c(G)$ to G' and the map f sending α to f_α is a monomorphism of the group $\mathrm{Aut}_{L_c}(G)$ to $\mathrm{Hom}(G/L_c(G),G')$.

Conversely, for any $f\in\mathrm{Hom}(G/L_c(G),G')$, the map $\theta=\theta_f:G\to G$ defined by $g^\theta=gf(gL_c(G))$, where $g\in G$, is a central kernel automorphism and $f(\theta)=f_\theta=f$. Thus f is onto and hence $\mathrm{Aut}_{L_c}(G)\cong\mathrm{Hom}(G/L_c(G),G')$. Now the proof is complete. □

Corollary 3.10. *Let G be a finite non-abelian p-group. Then* $\mathrm{Aut}_{L_c}(G)=\mathrm{IA}(G)$ *if and only if* $G'=L_c(G)\leq Z(G)$.

Proof. First suppose that $\mathrm{Aut}_{L_c}(G)=\mathrm{IA}(G)$. By Lemma 3.9, $L_c(G)\leq Z(G)$ and $\mathrm{Aut}_{L_c}(G)\cong\mathrm{Hom}(G/L_c(G),G')\cong\mathrm{Hom}(G/G',L_c(G))$. We claim that $G'=L_c(G)$. Suppose on the contrary, that $G'<L_c(G)$. Then $G/L_c(G)$ is a proper quotient subgroup of G/G' and

$$|G/G'|/|G/L_c(G)|=|L_c(G)/G'|>1.$$

Now, it follows from [1, Lemma D] that $\mathrm{Hom}(G/L_c(G),G')$ is isomorphic to a proper subgroup of $\mathrm{Hom}(G/G',L_c(G))$, which is a contradiction. Therefore $G'=L_c(G)$, as required. The converse is evident. □

Corollary 3.11. *Let G be a finite non-abelian p-group. Then* $\mathrm{Aut}_{L_c}(G)=\mathrm{Aut}_c(G)$ *if and only if* $\mathrm{Aut}_c(G)\cong\mathrm{Hom}(G/L_c(G),G')$ *and* $G'=L_c(G)\leq Z(G)$.

Proof. First suppose that $\mathrm{Aut}_{L_c}(G)=\mathrm{Aut}_c(G)$. It follows that $L_c(G)\leq Z(G)$ and $\mathrm{Aut}_{L_c}(G)=\mathrm{IA}(G)$. Now the proof follows at once from Lemma 3.9 and Corollary 3.10.

Conversely, as $G'=L_c(G)\leq Z(G)$ we have

$$\mathrm{Aut}_{L_c}(G)=\mathrm{IA}(G)\cong\mathrm{Hom}(G/G',G')=\mathrm{Hom}(G/L_c(G),G')\cong\mathrm{Aut}_c(G),$$

since $\mathrm{IA}(G) \cong \mathrm{Hom}(G/G', G')$ using [12, Lemma 3.1]. Now the result follows from the fact that $\mathrm{Aut}_c(G) \leq \mathrm{Aut}_{L_c}(G)$. □

In the following result, we give a sufficient condition under which the group $\mathrm{Aut}_{L_c}(G)$ acts trivially on $K_c(G)$.

Theorem 3.12. *Let G be a group such that $K_c(G)$ is a torsion-free subgroup of G and $K_c(G)/E_{L_c}(G)$ is a torsion group. Then $\mathrm{Aut}_{L_c}(G)$ is a torsion-free abelian group such that acts trivially on $K_c(G)$.*

Proof. Let $\alpha \in \mathrm{Aut}_{L_c}(G)$ and x is an element of $K_c(G)$. Then by hypothesis $x^n \in E_{L_c}(G)$, for some $n \in \mathbb{N}$. Since $x^{-1}x^\alpha \in Z(G)$, we have

$$[x, \alpha]^n = (x^{-1}x^\alpha)^n = \underbrace{x^{-1}x^\alpha \cdots x^{-1}x^\alpha}_{n-times} = x^{-n}(x^n)^\alpha = [x^n, \alpha].$$

Hence Lemma 3.1 implies that $[x, \alpha]^n = 1$. As $K_c(G)$ is a torsion-free subgroup, it follows that $[x, \alpha] = 1$ and so $\mathrm{Aut}_{L_c}(G)$ acts trivially on $K_c(G)$. Next let $\alpha \in \mathrm{Aut}_{L_c}(G)$ and assume that there exists $m \in \mathbb{N}$ such that $\alpha^m = 1$. Since $[x, \alpha] \in L_c(G)$, for all $x \in G$, there exists $k_x \in L_c(G)$ such that $x^\alpha = xk_x$. Therefore $x^{\alpha^2} = (x^\alpha)^\alpha = (xk_x)^\alpha = x{k_x}^2$ and so by induction we have $x = x^{\alpha^m} = x{k_x}^m$, whence $k_x = 1$, since $K_c(G)$ is a torsion-free. Thus $\alpha = 1$, which together with Lemma 3.2, $\mathrm{Aut}_{L_c}(G)$ is a torsion-free abelian group. □

4. Classify all finite p-groups G of order $p^n, n \leq 5$ such that $\mathrm{Aut}_{L_c}(G) = \mathrm{Inn}(G)$

Recall that a finite p-group is called a minimal non-abelian if it is a non-abelian group and all its subgroups are abelian. In this section, by the following concept, we classify all p-groups G of order at most p^5 such that $\mathrm{Aut}_{L_c}(G) = \mathrm{Inn}(G)$. Since by Corollary 3.8, for a non-abelian group G of order p^3, $\mathrm{Aut}_{L_c}(G) = \mathrm{Inn}(G)$, we may assume that $4 \leq n \leq 5$. First we list the following results due to Redei (see [11]).

Lemma 4.1. *[11] Let G be a finite minimal non-abelian p-group. Then G is one of the following groups:*

(i) Q_8,
(ii) $M_p(n, m) = \langle a, b \mid a^{p^n} = b^{p^m} = 1, a^b = a^{1+p^{n-1}} \rangle$, *where* $n \geq 2$ *and* $m \geq 1$,
(iii) $M_p(n, m, 1) = \langle a, b, c \mid a^{p^n} = b^{p^m} = c^p = 1, [a, b] = c, [a, c] = [b, c] = 1 \rangle$, *where* $n \geq m \geq 1$ *and if* $p = 2$, *then* $m + n > 2$.

The following equivalent conditions about finite minimal non-abelian p-groups are always used.

Lemma 4.2. *[11] Let G be a finite p-group. Then the following conditions are equivalent:*

(i) *G is a minimal non-abelian p-group.*

(ii) $d(G) = 2$ *and* $G' \cong C_p$.
(iii) $d(G) = 2$ *and* $Z(G) = \Phi(G)$.

The following lemma is a useful fact in proving our next results and can be verified easily.

Lemma 4.3. *Let G be a finite minimal non-abelian p-group. Then*

(i) $Z(M_p(n,m)) = \langle a^p \rangle \times \langle b^p \rangle$ *and* $Z(M_p(n,m,1)) = \langle a^p \rangle \times \langle b^p \rangle \times \langle c \rangle$.
(ii) $M_p(n,m)/M_p(n,m)' \cong C_{p^{n-1}} \times C_{p^m}$ *and* $M_p(n,m,1)/M_p(n,m,1)' \cong C_{p^n} \times C_{p^m}$.

The following concept was introduced by Hall [6].

Definition 4.4. Two finite groups G and H are said to be isoclinic if there exist isomorphisms $\psi : G/Z(G) \to H/Z(H)$ and $\theta : G' \to H'$ such that, if $(x_1Z(G))^\psi = y_1Z(H)$ and $(x_2Z(G))^\psi = y_2Z(H)$, then $[x_1,x_2]^\theta = [y_1,y_2]$. Notice that isoclinism is an equivalence relation among finite groups and the equivalence classes are called isoclinism families.

Corollary 4.5. *Let G be a non-abelian group of order p^4. Then* $\mathrm{Aut}_{L_c}(G) = \mathrm{Inn}(G)$ *if and only if G is one of the following types:*

(i) $M_p(2,2)$, *where p an odd prime,*
(ii) $M_p(3,1)$,
(iii) $M_p(2,1,1)$.

Proof. Assume that $|G| = p^4$ and $\mathrm{Aut}_{L_c}(G) = \mathrm{Inn}(G)$. By Lemma 3.5, $L_c(G) \leq Z(G)$. First we claim that $|Z(G)| = p^2$. Suppose for a contradiction, that $|Z(G)| = p$. Then $G' = L_c(G) = Z(G) \cong C_p$ and so is an extra-special p-group, which is a contradiction, since the order of G is not of the form p^{2k+1}, where k is a natural number. Thus $G/Z(G) \cong C_p^2$, whence $|G'| = p$ and $\Phi(G) \leq Z(G)$. Since $\mathrm{Aut}_{L_c}(G) = C_{\mathrm{Aut}_{L_c}(G)}(Z(G))$, by the proof of Theorem 3.3, $Z(G) \leq \Phi(G)$ and so $Z(G) = \Phi(G)$. Therefore G is a minimal non-abelian p-group by Lemma 4.2. We consider two cases:

Case I. p an odd prime. It is an easy task to see that the map θ defined by $x^\theta = x^{1+p}$, is a central automorphism of G. Hence for any element x of $L_c(G)$, $x = x^\theta = x^{1+p}$, and so $x^p = 1$. Thus $\exp(L_c(G)) = p$ and $L_c(G) \cong C_p$, by Lemma 3.5. If $G/L_c(G) \cong C_{p^3}$, then $G/Z(G)$ is cyclic, a contradiction. Next, we assume that $G/L_c(G) \cong C_{p^2} \times C_p$. Hence $G \cong M_p(2,2), M_p(3,1)$ or $M_p(2,1,1)$, by Lemma 4.3. Finally, if $G/L_c(G) \cong C_p^3$, then $L_c(G) = \Phi(G) = Z(G) \cong C_p$ and so G is an extra-special p-group, a contradiction.

Case II. $p = 2$. Since $G' \leq L_c(G) \leq Z(G)$, it follows that $|L_c(G)| = 2$ or 4. If $|L_c(G)| = 4$, then $L_c(G) = Z(G)$ and $G/L_c(G) \cong C_2^2$. Hence $\mathrm{Autcent}(G) = \mathrm{Aut}_{L_c}(G) = \mathrm{Inn}(G)$ and so $G' = Z(G) \cong C_2$ by using the main theorem of [1], a contradiction. Next we assume that $|L_c(G)| = 2$, whence $|G/L_c(G)| = 8$. If $G/L_c(G) \cong C_8$, then G is cyclic, a contradiction. Moreover, as $E_{L_c}(G) \leq Z(G)$, if $G/L_c(G) \cong C_2^3$, then by Theorem 3.6, $d(G/Z(G)) = 3$ and so $Z(G) \cong C_2$, it follows that G is an extra-special 2-group, a contradiction. Therefore we assume that $G/L_c(G) \cong C_4 \times C_2$. Since $G' \cong C_2$ and G' is a characteristic subgroup of G, we observe that $G' \leq L(G)$ and so $G' = L(G) = L_c(G)$. Hence $G/L(G) \cong C_4 \times C_2$ and G is isomorphic to one of the following groups: $M_2(3,1)$ or $M_2(2,1,1)$, by [9, Theorem 5.1]. The converse follows at once from Lemmas 3.2, 4.2, 4.3 and Theorem 3.6. □

Corollary 4.6. *Let G be a non-abelian group of order p^5. Then $\mathrm{Aut}_{L_c}(G) = \mathrm{Inn}(G)$ if and only if G is one of the following types:*

(i) *The isoclinism family (5) of [8], $M_p(2,3)$, $M_p(3,1,1)$, where $p > 2$,*
(ii) $M_p(4,1)$,
(iii) $M_p(3,2)$,
(iv) $M_p(2,2,1)$,
(v) $D_8 * D_8$,
(vi) $D_8 * Q_8$.

Proof. Let G be a non-abelian group such that $|G| = p^5$ and $\mathrm{Aut}_{L_c}(G) = \mathrm{Inn}(G)$. It follows that $L_c(G) \leq Z(G) \leq \Phi(G)$, by Lemma 3.5 and the proof of Theorem 3.3. We consider two cases:

Case I. $p > 2$. These groups lying in the isoclinism families (5), (4) or (2) of [8].

First, let G denote one of the groups in the isoclinism family (5). Hence $G' = L_c(G) = Z(G) \cong C_p$ and $\mathrm{Autcent}(G) = \mathrm{Aut}_{L_c}(G)$. Now with the main theorem of [1], $\mathrm{Autcent}(G) = \mathrm{Inn}(G)$ if and only if $G' = Z(G)$ and $Z(G)$ is cyclic. This happens for all groups in the isoclinism family (5).

Next, let G be one of the groups in the isoclinism family (4). Then $G' \cong C_p^2$, which is a contradiction, since $G' \leq L_c(G)$ is cyclic, by Theorem 3.6.

To continue the proof, let G denote one of the groups in the isoclinism family (2). Then $G/Z(G) \cong C_p^2$ and $G' \cong C_p$. Hence $Z(G) = \Phi(G)$ and so $d(G) = 2$. This implies that G is a minimal non-abelian p-group, by Lemma 4.2. Moreover, by considering the automorphism θ mentioned in Corollary 4.5, $\exp(L_c(G)) = p$ and so $G' = L_c(G) \cong C_p$. If $G/L_c(G) \cong C_{p^3} \times C_p$, then by Lemma 4.3, G is one of the following types: $M_p(4,1), M_p(2,3)$ or $M_p(3,1,1)$. If $G/L_c(G) \cong C_{p^2}^2$, then by Lemma 4.3, $G \cong M_p(3,2)$ or $G \cong M_p(2,2,1)$. Finally, assume that $G/L_c(G) \cong C_{p^2} \times C_p^2$ or $G/L_c(G) \cong C_p^4$. In these cases, $\mathrm{Aut}_{L_c}(G) \neq \mathrm{Inn}(G)$, by Lemma 3.2.

Case II. $p = 2$. We can see that $|L_c(G)| = 8, 4, 2$. First, we assume that $|L_c(G)| = 8$. It follows that $G/L_c(G) \cong C_2 \times C_2$, which shows that $\Phi(G) \leq L_c(G)$. So $L_c(G) = Z(G) = \Phi(G)$, which implies that $\exp(G') = 2$ and $\mathrm{Aut}_{L_c}(G) = \mathrm{Autcent}(G) = \mathrm{Inn}(G)$. Now, by applying the main theorem of [1], $G' = Z(G) \cong C_2$, a contradiction. Next assume that $|L_c(G)| = 4$. Then $G/L_c(G)$ is one of the groups C_2^3 or $C_4 \times C_2$. In the first case, by a similar argument mentioned earlier, $G' = Z(G) \cong C_2$, which is a contradiction. Therefore $G/L_c(G) \cong C_4 \times C_2$ and by Lemma 3.2, $\mathrm{Aut}_{L_c}(G) \cong \mathrm{Hom}(C_4 \times C_2, C_4) \cong$

$C_4 \times C_2$. Therefore $\mathrm{Inn}(G) \cong C_4 \times C_2$, so $L_c(G) = Z(G)$ and we have a contradiction $|G| = 16$.

Now, we may suppose that $|L_c(G)| = 2$. Hence $G' = L(G) = L_c(G) \cong C_2$ and $Z(G) = \Phi(G)$. We discuss the following cases.

Case (1). The same as previous paragraph, if $G/L_c(G) \cong C_2^4$, then $\Phi(G) \leq L_c(G)$, so $L_c(G) = Z(G) = \Phi(G)$. Hence G is an extra-special 2-group, and G is one of the groups $D_8 * D_8$ or $D_8 * Q_8$, by [13].

Case (2). Suppose that $G/L_c(G) \cong C_4 \times C_2^2$. We assume that $G/L_c(G) = \langle \bar{x}, \bar{y}, \bar{z} \rangle$, where $\bar{x} = xL_c(G), \bar{y} = yL_c(G)$, $\bar{z} = zL_c(G)$, $o(\bar{x}) = 4$ and $o(\bar{y}) = o(\bar{z}) = 2$. It follows that $G = \langle x, y, z \rangle$. Next, by Lemma 3.2, $\mathrm{Aut}_{L_c}(G) \cong \mathrm{Hom}(C_4 \times C_2^2, C_2) \cong C_2^3$, whence $\mathrm{Inn}(G) \cong C_2^3$. Since $[x^2, y] = [x^2, z] = 1$, we observe that $\langle x^2 \rangle \times L_c(G) \leq Z(G)$ and so $Z(G) \cong C_2^2$. Now by using GAP [4], we find that there is no such group.

Case (3). If $G/L_c(G) = G/L(G) \cong C_8 \times C_2$, then $G \cong M_2(4,1)$, by [9, Theorem 5.1].

Case (4). Suppose that $G/L_c(G) \cong C_4^2$. As $L_c(G) \leq Z(G)$ and $G/Z(G)$ is elementary abelian, we have $G/Z(G) \cong C_2^2$. Hence $d(G/Z(G)) = 2$ and so G is a minimal non-abelian p-group, whence by Lemmas 4.2 and 4.3, G is isomorphic to the group $M_2(3,2)$ or $M_2(2,2,1)$.

The converse follows at once from Lemmas 3.2, 4.2, 4.3, Theorem 3.6 and Corollary 3.8. □

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