

## GROUPS WITH SOME CENTRAL AUTOMORPHISMS FIXING THE CENTRAL KERNEL QUOTIENT

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Article type: Research Article

(Received: 16 March 2022, Received in revised form 26 July 2022)

(Accepted: 17 September 2022, Published Online: 17 September 2022)

**ABSTRACT.** Let  $G$  be a group. An automorphism  $\alpha$  of the group  $G$  is called a central automorphism, if  $x^{-1}x^\alpha \in Z(G)$  for all  $x \in G$ . Let  $L_c(G)$  be the central kernel of  $G$ , that is, the set of elements of  $G$  fixed by all central automorphisms of  $G$  and  $\text{Aut}_{L_c}(G)$  denote the group of all central automorphisms of  $G$  fixing  $G/L_c(G)$  element-wise. In the present paper, we investigate the properties of such automorphisms. Moreover, a full classification of  $p$ -groups  $G$  of order at most  $p^5$  where  $\text{Aut}_{L_c}(G) = \text{Inn}(G)$  is also given.

*Keywords:* Automorphism group; Central kernel, Central autocommutator.

*2020 MSC:* Primary 20D45; Secondary 20D25.

### 1. Introduction

Let  $G$  be a group and  $p$  a prime number. We denote by  $G'$ ,  $Z(G)$ ,  $\text{Inn}(G)$  and  $\text{Aut}(G)$ , the commutator subgroup, the center, the group of all inner automorphisms and the group of all automorphisms of  $G$ , respectively. For each  $x \in G$  and  $\alpha \in \text{Aut}(G)$ , the element  $[x, \alpha] = x^{-1}x^\alpha$  is called the autocommutator of  $x$  and  $\alpha$ , in which  $x^\alpha$  is the image of  $x$  under  $\alpha$ . An automorphism  $\alpha$  of  $G$  is called a central automorphism if  $[x, \alpha] \in Z(G)$  for all  $x \in G$ . An automorphism  $\alpha$  of  $G$  is called an IA-automorphism if  $[x, \alpha] \in G'$  for all  $x \in G$ . Also, an automorphism  $\alpha$  of  $G$  is called a class preserving automorphism if  $x^\alpha \in x^G$  for all  $x \in G$ , where  $x^G$  is the conjugacy class of  $x$  in  $G$ . Let  $\text{Autcent}(G)$ ,  $\text{IA}(G)$  and  $\text{Aut}_c(G)$ , denote the group of all central automorphisms, IA-automorphisms and class preserving automorphisms of  $G$ , respectively. In 1994, Hegarty [7] introduced the concept of absolute center subgroup of a group  $G$ , as follows:

$$L(G) = \{x \in G \mid [x, \alpha] = 1, \forall \alpha \in \text{Aut}(G)\}.$$

It is easy to check that the absolute center of  $G$  is a characteristic subgroup contained in the center of  $G$ . Haimo [5] introduced the following subgroup of a given group  $G$ , which we call similar to [2], the central kernel of  $G$  and denote

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DOI: 10.22103/jmmr.2022.19047.1220

Publisher: Shahid Bahonar University of Kerman

How to cite: R. Soleimani, *Groups with some central automorphisms fixing the central kernel quotient*, J. Mahani Math. Res. 2023; 12(2): 165-177.



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by  $L_c(G)$ , as

$$L_c(G) = \{x \in G \mid [x, \alpha] = 1, \forall \alpha \in \text{Autcent}(G)\}.$$

Since the central automorphisms of  $G$  fixing  $G'$  element-wise, we conclude that  $G' \leq L_c(G)$ , and so  $G/L_c(G)$  is abelian. Also

$$K_c(G) = \langle [x, \alpha] \mid x \in G, \alpha \in \text{Autcent}(G) \rangle,$$

is said the central autocommutator subgroup of  $G$  (see [2]). One can easily check that  $L_c(G)$  is a characteristic subgroup of  $G$  contains  $L(G)$  and  $K_c(G)$  is a central characteristic subgroup of  $G$ . Now, we call  $\alpha \in \text{Autcent}(G)$  to be central kernel automorphism, when  $[x, \alpha] \in L_c(G)$ , for all  $x \in G$ . According to [2], let  $\text{Aut}_{L_c}(G)$  denote the group of all central kernel automorphisms of  $G$ . Clearly,  $\text{Aut}_{L_c}(G)$  is a normal subgroup of  $\text{Autcent}(G)$  and acts trivially on the central kernel of  $G$ . Davoudirad et al. ([2], [3]) for an arbitrary group  $G$ , investigate some properties of  $\text{Aut}_{L_c}(G)$  and the central kernel subgroup of  $G$ .

In this paper, first we give some necessary and sufficient conditions on a finite  $p$ -group  $G$  such that  $\text{Aut}_{L_c}(G)$  is equal to  $C_{\text{Aut}_{L_c}(G)}(Z(G))$ ,  $\text{Inn}(G)$ ,  $\text{IA}(G)$  and  $\text{Aut}_c(G)$ , respectively. Finally, we classify all  $p$ -groups  $G$  of order at most  $p^5$  such that  $\text{Aut}_{L_c}(G) = \text{Inn}(G)$ .

## 2. Preliminaries

For a finite group  $G$ ,  $\exp(G)$ ,  $d(G)$ ,  $\Omega_i(G)$ ,  $\text{cl}(G)$  and  $o(x)$ , denote the exponent of  $G$ , minimal number of generators of  $G$ , the subgroup of  $G$  generated by its elements of order dividing  $p^i$ , the nilpotency class of  $G$  and the order of  $x$ , respectively. For a finite  $p$ -group  $G$ , if  $A$  is a normal subgroup of  $\text{Aut}(G)$ , then we use  $C_A(Z(G))$  to denote the group of all automorphisms of  $A$  which centralizing  $Z(G)$  element-wise. Moreover, let us denote by  $C_n$  the cyclic group of order  $n$ , where  $n \geq 1$  and  $C_n^k$  be the direct product of  $k$  copies of  $C_n$ ,  $D_8$  the dihedral group,  $Q_8$  the quaternion group of order 8, respectively. Recall that an abelian finite  $p$ -group  $A$  has invariants or is of type  $(n_1, n_2, \dots, n_k)$  if it is the direct product of cyclic subgroups of orders  $p^{n_1}, p^{n_2}, \dots, p^{n_k}$ , where  $n_1 \geq n_2 \geq \dots \geq n_k > 0$ . We use the notation  $\text{Hom}(G, A)$  to denote the group of homomorphisms of  $G$  into an abelian group  $A$ . Finally, recall that a group  $G$  is called a central product of its subgroups  $A, B$  if  $A$  and  $B$  commute element-wise and together generate  $G$ . In this situation, we write  $G = A * B$ .

The following lemma is a well-known result and will be used in the sequel.

**Lemma 2.1.** *Let  $A, B$  and  $C$  be finite abelian groups. Then*

- (i)  $\text{Hom}(A \times B, C) \cong \text{Hom}(A, C) \times \text{Hom}(B, C)$ .
- (ii)  $\text{Hom}(A, B \times C) \cong \text{Hom}(A, B) \times \text{Hom}(A, C)$ .
- (iii)  $\text{Hom}(C_m, C_n) \cong C_d$ , where  $d$  is the greatest common divisor of  $m$  and  $n$ .

**Corollary 2.2.** *Let  $A, B$  and  $C$  be finite abelian  $p$ -groups,  $\exp(C) = p^t$  and  $A \leq B$ . Then  $\text{Hom}(A, C) \cong \text{Hom}(B, C)$  if and only if  $A \cong H \times A_1, B \cong H \times B_1$  where all invariants of  $A_1, B_1$  are at least  $t$ ,  $d(A_1) = d(B_1)$  and  $\exp(H) < p^t$ .*

*Proof.* It can be proved by using Lemma 2.1 and induction on  $|C|$ . □

### 3. Main results

In this section, we provide some results concerning the group of all central kernel automorphisms of  $G$ . First, we define two subgroups of  $\text{Autcent}(G)$  and  $G$  as follows:

$$C_{\text{Autcent}(G)}(\text{Aut}_{L_c}(G)) = \{\alpha \in \text{Autcent}(G) \mid \alpha\beta = \beta\alpha, \forall \beta \in \text{Aut}_{L_c}(G)\},$$

and

$$E_{L_c}(G) = [G, C_{\text{Autcent}(G)}(\text{Aut}_{L_c}(G))].$$

Obviously,  $E_{L_c}(G)$  is a characteristic subgroup in  $G$ , which is contained in  $K_c(G)$ . Also, if  $G/Z(G)$  is abelian, then

$$G' = [G, \text{Inn}(G)] \leq [G, C_{\text{Autcent}(G)}(\text{Aut}_{L_c}(G))] \leq E_{L_c}(G).$$

The following lemma states the useful property of  $E_{L_c}(G)$ , which will be needed for our further investigation.

**Lemma 3.1.** *If  $G$  is an arbitrary group, then  $\text{Aut}_{L_c}(G)$  acts trivially on the subgroup  $E_{L_c}(G)$  of  $G$ .*

*Proof.* Let  $\alpha \in \text{Aut}_{L_c}(G)$ . Then  $g^{-1}g^\alpha \in L_c(G)$  for all  $g \in G$  and so  $g^\alpha = gt_g$ , for some  $t_g \in L_c(G)$ . By taking an automorphism  $\beta \in C_{\text{Autcent}(G)}(\text{Aut}_{L_c}(G))$ , we have

$$\begin{aligned} [g, \beta]^\alpha &= (g^{-1}g^\beta)^\alpha = (g^{-1})^\alpha (g^\beta)^\alpha = (g^{-1})^\alpha (g^\alpha)^\beta \\ &= t_g^{-1} g^{-1} g^\beta t_g^\beta = g^{-1} g^\beta t_g^{-1} t_g = [g, \beta], \end{aligned}$$

which completes the proof. □

**Lemma 3.2.** *Let  $G$  be a group. Then*

- (i)  $\text{Aut}_{L_c}(G) \cong \text{Hom}(G/E_{L_c}(G)L_c(G), L_c(G) \cap Z(G))$ .
- (ii)  $C_{\text{Aut}_{L_c}(G)}(Z(G)) \cong \text{Hom}(G/Z(G)L_c(G), L_c(G) \cap Z(G))$ .

*Proof.* (i) Take an automorphism  $\theta \in \text{Aut}_{L_c}(G)$ . Then we see that  $f_\theta : gE_{L_c}(G)L_c(G) \mapsto g^{-1}g^\theta$ , defines a homomorphism from  $G/E_{L_c}(G)L_c(G)$  to  $L_c(G) \cap Z(G)$  and the map  $\varphi$  sending  $\theta$  to  $f_\theta$  defines a monomorphism from  $\text{Aut}_{L_c}(G)$  to the group  $\text{Hom}(G/E_{L_c}(G)L_c(G), L_c(G) \cap Z(G))$ . Also, let  $f \in \text{Hom}(G/E_{L_c}(G)L_c(G), L_c(G) \cap Z(G))$ . Then the map  $\theta = \theta_f$  defined by  $x^\theta = xf(xE_{L_c}(G)L_c(G))$ , for all  $x \in G$ , is a central kernel automorphism of  $G$  and  $\varphi(\theta) = \varphi(\theta_f) = f$ . Hence  $\varphi$  is onto and the proof is complete.

(ii) It is sufficient to observe that for each  $\theta \in C_{\text{Aut}_{L_c}(G)}(Z(G))$ , the map

$$f_\theta : G/Z(G)L_c(G) \rightarrow L_c(G) \cap Z(G)$$

$$gZ(G)L_c(G) \mapsto g^{-1}g^\theta$$

defines a homomorphism and  $\theta \mapsto f_\theta$  is an isomorphism from  $C_{\text{Aut}_{L_c}(G)}(Z(G))$  to  $\text{Hom}(G/Z(G)L_c(G), L_c(G) \cap Z(G))$ .  $\square$

**Theorem 3.3.** *Let  $G$  be a finite  $p$ -group and  $G/E_{L_c}(G)L_c(G)$ ,  $G/Z(G)L_c(G)$  and  $L_c(G) \cap Z(G)$  are of types  $(a_1, a_2, \dots, a_k)$ ,  $(b_1, b_2, \dots, b_m)$  and  $(c_1, c_2, \dots, c_n)$ , respectively. Then*

$$\text{Aut}_{L_c}(G) = C_{\text{Aut}_{L_c}(G)}(Z(G))$$

*if and only if  $Z(G) \leq E_{L_c}(G)L_c(G)$  or  $d(G/E_{L_c}(G)L_c(G)) = d(G/Z(G)L_c(G))$  and  $b_{l+1} < c_1 \leq b_l$ , where  $l$  is the largest integer between 1 and  $m$  such that  $b_l < a_l$ .*

*Proof.* Let  $G$  be a finite  $p$ -group such that  $\text{Aut}_{L_c}(G) = C_{\text{Aut}_{L_c}(G)}(Z(G))$  and  $Z(G) \not\leq E_{L_c}(G)L_c(G)$ . We claim that  $Z(G) \leq \Phi(G)$ ; otherwise, let  $M$  be a maximal subgroup of  $G$  such that  $Z(G) \not\leq M$ . We write  $G = M\langle z \rangle$  where  $z \in Z(G) \setminus M$  and choose an element  $u \in \Omega_1(Z(G) \cap L_c(G))$ . Then the map  $\alpha : hz^i \mapsto h(zu)^i$ , where  $h \in M$  and  $0 \leq i < p$ , defines an automorphism of  $G$  which is in  $\text{Aut}_{L_c}(G)$ . So that  $\alpha$  is an automorphism of  $G$  fixes  $Z(G)$  element-wise, whence  $u = 1$  which is impossible. Therefore  $Z(G) \leq \Phi(G)$  and so  $k = d(G/E_{L_c}(G)L_c(G)) = d(G/Z(G)L_c(G)) = m$ . Since  $E_{L_c}(G)L_c(G) < Z(G)L_c(G)$ , we have  $G/Z(G)L_c(G)$  is a proper quotient group of  $G/E_{L_c}(G)L_c(G)$ . Since  $\text{Hom}(G/E_{L_c}(G)L_c(G), L_c(G) \cap Z(G)) \cong \text{Hom}(G/Z(G)L_c(G), L_c(G) \cap Z(G))$ , by using Corollary 2.2,  $G/E_{L_c}(G)L_c(G) \cong X \times Y$ ,  $G/Z(G)L_c(G) \cong H \times Y$ , where  $X, H$  are of types  $(a_1, \dots, a_l)$  and  $(b_1, \dots, b_l)$ , respectively, in which  $d(X) = d(H) = l$ . Hence  $l$  is the largest integer between 1 and  $m$  such that  $b_l < a_l$  and by Corollary 2.2,  $b_{l+1} < c_1 \leq b_l$ , as required.

Conversely, if  $Z(G) \leq E_{L_c}(G)L_c(G)$ , then  $\text{Aut}_{L_c}(G) = C_{\text{Aut}_{L_c}(G)}(Z(G))$ . Next, assume that  $E_{L_c}(G)L_c(G) < Z(G)L_c(G)$ ,  $k = m$  and  $l$  is the largest integer between 1 and  $m$  such that  $b_l < a_l$  and  $b_{l+1} < c_1 \leq b_l$ . Let  $G/E_{L_c}(G)L_c(G) = X \times Y$ , where  $X, Y$  are of types  $(a_1, \dots, a_l)$  and  $(a_{l+1}, \dots, a_m)$ . Moreover,  $G/Z(G)L_c(G) = H \times K$ , where  $H, K$  have invariants  $(b_1, \dots, b_l)$  and  $(b_{l+1}, \dots, b_m)$ . Since  $a_i = b_i$  for  $l+1 \leq i \leq m$ , we have  $K = Y$ . Therefore by Corollary 2.2,

$$\text{Hom}(G/E_{L_c}(G)L_c(G), L_c(G) \cap Z(G)) \cong \text{Hom}(G/Z(G)L_c(G), L_c(G) \cap Z(G)),$$

and hence  $\text{Aut}_{L_c}(G) = C_{\text{Aut}_{L_c}(G)}(Z(G))$ , which completes the proof.  $\square$

Let  $G$  be a finite  $p$ -group of class 2. Since  $\text{Aut}_{\text{cent}}(G)$  acts trivially on the central kernel of  $G$ , we have  $L_c(G) \leq Z(G)$ . Let  $G/E_{L_c}(G)L_c(G)$ ,  $G/Z(G)$  and  $L_c(G)$  are of types  $(a_1, a_2, \dots, a_k)$ ,  $(b_1, b_2, \dots, b_m)$  and  $(c_1, c_2, \dots, c_n)$ , respectively. Also let  $t$  be the largest integer between 1 and  $m$  such that  $b_1 = b_2 = \dots = b_t$ . It is shown [10, Lemma 0.4] that,  $t \geq 2$ . Set  $\bar{A} = A/Z(G)$  is of type  $(b_1, b_2, \dots, b_t)$

and  $\bar{A}$  is isomorphic to a subgroup of  $\bar{B} = B/E_{L_c}(G)L_c(G)$  which is of type  $(a_1, a_2, \dots, a_t)$ .

By keeping the above notation, in the following theorem, we give a necessary and sufficient condition on a fixed finite  $p$ -group  $G$  of class 2 such that each automorphism of  $\text{Aut}_{L_c}(G)$  fixes the center of  $G$  element-wise.

**Theorem 3.4.** *Let  $G$  be a finite  $p$ -group of class 2. Then*

$$\text{Aut}_{L_c}(G) = C_{\text{Aut}_{L_c}(G)}(Z(G))$$

*if and only if one of the following conditions holds:*

- (i)  $E_{L_c}(G)L_c(G) = Z(G)$  or
- (ii)  $E_{L_c}(G)L_c(G) < Z(G)$ ,  $k = m$ ,  $(G/Z(G))/\bar{A} \cong (G/E_{L_c}(G)L_c(G))/\bar{B}$  and  $\exp(G') = \exp(L_c(G))$ .

*Proof.* First assume that  $\text{Aut}_{L_c}(G) = C_{\text{Aut}_{L_c}(G)}(Z(G))$ . Since  $\text{cl}(G) = 2$ , it follows that  $L_c(G) \leq Z(G)$ . We may suppose that  $E_{L_c}(G)L_c(G) < Z(G)$ . By Theorem 3.3,  $k = d(G/E_{L_c}(G)L_c(G)) = d(G/Z(G)) = m$ . Since  $G/Z(G)$  is a proper quotient group of  $G/E_{L_c}(G)L_c(G)$ , there exists some  $1 \leq j \leq m$  such that  $b_j < a_j$ . Let  $l$  be the largest integer between 1 and  $m$  such that  $b_l < a_l$ . We claim that  $\exp(G') = \exp(L_c(G))$ . To do this, we observe that by Theorem 3.3,  $\exp(L_c(G)) \leq p^{b_l} \leq p^{b_1} = \exp(G/Z(G))$ . It follows that

$$\exp(G') \leq \exp(L_c(G)) \leq \exp(G/Z(G)) = \exp(G'),$$

by [10, Lemma 0.4], because  $G/L_c(G)$  is abelian. So we conclude that  $\exp(G') = \exp(L_c(G))$ , as desired. Next,  $b_1 = c_1 \leq b_l$  shows that  $c_1 = b_1 = b_2 = \dots = b_l$  and hence  $l \leq t$ . Set  $\bar{A} = A/Z(G)$  is of type  $(b_1, b_2, \dots, b_t)$ ,  $\bar{B} = B/E_{L_c}(G)L_c(G)$  which is of type  $(a_1, a_2, \dots, a_t)$  and  $U$  and  $V$  are of types  $(a_{t+1}, a_{t+2}, \dots, a_k)$  and  $(b_{t+1}, b_{t+2}, \dots, b_k)$ . Since  $a_i = b_i$  for all  $l+1 \leq i \leq m$ , then  $U \cong V$  and therefore  $(G/Z(G))/\bar{A} \cong V \cong U \cong (G/E_{L_c}(G)L_c(G))/\bar{B}$ , as required.

Conversely, if  $E_{L_c}(G)L_c(G) = Z(G)$ , then it is clear that  $\text{Aut}_{L_c}(G) = C_{\text{Aut}_{L_c}(G)}(Z(G))$ . Next, suppose that  $E_{L_c}(G)L_c(G) < Z(G)$ ,  $k = m$ ,  $\exp(G') = \exp(L_c(G))$  and  $(G/Z(G))/\bar{A} \cong (G/E_{L_c}(G)L_c(G))/\bar{B}$ . Since  $G/Z(G)$  is a proper quotient group of  $G/E_{L_c}(G)L_c(G)$ , let  $l$  be the largest integer between 1 and  $m$  such that  $b_l < a_l$ . Hence  $l \leq t$ , because of  $b_i = a_i$  for  $t+1 \leq i \leq m$ . Now  $p^{c_1} = \exp(L_c(G)) = \exp(G') = \exp(G/Z(G)) = p^{b_1}$  and so  $c_1 = b_1 = b_2 = \dots = b_l$ , which together with Theorem 3.3, gives the proof.  $\square$

**Lemma 3.5.** *Let  $G$  be a finite non-abelian  $p$ -group. Then  $C_{\text{Aut}_{L_c}(G)}(Z(G)) = \text{Inn}(G)$  if and only if  $L_c(G) \leq Z(G)$  and  $L_c(G)$  is cyclic.*

*Proof.* Suppose that  $L_c(G)$  is cyclic and  $L_c(G) \leq Z(G)$ . Hence  $\exp(G/Z(G)) = \exp(G')$ , since  $G' \leq L_c(G)$ . This implies that  $\exp(G/Z(G))$  divides  $\exp(L_c(G))$ . Then by Lemma 3.2,

$$C_{\text{Aut}_{L_c}(G)}(Z(G)) \cong \text{Hom}(G/Z(G), L_c(G)) \cong G/Z(G)$$

and so  $C_{\text{Aut}_{L_c}(G)}(Z(G)) = \text{Inn}(G)$ , as required.

Conversely, assume that  $C_{\text{Aut}_{L_c}(G)}(Z(G)) = \text{Inn}(G)$ . It follows that  $L_c(G) \leq Z(G)$ , which together with Lemma 3.2 and the fact that

$$\text{Hom}(G/Z(G), L_c(G)) \cong G/Z(G),$$

completes the proof.  $\square$

In the following result, we give some properties of finite non-abelian  $p$ -groups  $G$  such that  $\text{Aut}_{L_c}(G) = \text{Inn}(G)$ . Let  $G$  be a finite non-abelian  $p$ -group and  $G/E_{L_c}(G)L_c(G)$  is of type  $(a_1, a_2, \dots, a_k)$ . Also if  $G/Z(G)$  is abelian, then it has invariants  $(b_1, b_2, \dots, b_m)$ .

By fixing the above notation, we have the following result:

**Theorem 3.6.** *Let  $G$  be a finite non-abelian  $p$ -group. Then  $\text{Aut}_{L_c}(G) = \text{Inn}(G)$  if and only if  $L_c(G)$  is cyclic,  $L_c(G) \leq Z(G)$ ,  $m = k$  and one of the following conditions holds:*

- (i)  $E_{L_c}(G)L_c(G) = Z(G)$  or
- (ii)  $b_t = r$  and  $a_s = b_s$  for  $s = t + 1, \dots, k$ , where  $\exp(L_c(G)) = p^r$  and  $t$  is the largest integer between 1 and  $k$  such that  $a_t > r$ .

*Proof.* Suppose that  $\text{Aut}_{L_c}(G) = \text{Inn}(G)$ . By Lemma 3.5, we deduce that  $L_c(G) \leq Z(G)$  and  $L_c(G)$  is cyclic, because  $C_{\text{Aut}_{L_c}(G)}(Z(G)) = \text{Inn}(G)$ . Now by Lemma 3.2, we have

$$\begin{aligned} d(G/Z(G)) &= d(\text{Hom}(G/E_{L_c}(G)L_c(G), L_c(G))) \\ &= d(G/E_{L_c}(G)L_c(G))d(L_c(G)) = d(G/E_{L_c}(G)L_c(G)), \end{aligned}$$

and so  $m = k$ . If  $\exp(G/E_{L_c}(G)L_c(G)) \leq \exp(L_c(G))$ , then

$$G/Z(G) \cong \text{Aut}_{L_c}(G) \cong \text{Hom}(G/E_{L_c}(G)L_c(G), L_c(G)) \cong G/E_{L_c}(G)L_c(G),$$

because  $L_c(G)$  is cyclic. Therefore  $E_{L_c}(G)L_c(G) = Z(G)$ .

Next, let  $\exp(G/E_{L_c}(G)L_c(G)) > \exp(L_c(G))$  and  $t$  is the largest integer such that  $a_t > r$ , where  $\exp(L_c(G)) = p^r$ . Then we observe that

$$\text{Hom}(G/E_{L_c}(G)L_c(G), L_c(G)) \cong C_{p^r} \times C_{p^r} \times \dots \times C_{p^r} \times C_{p^{a_{t+1}}} \times \dots \times C_{p^{a_k}}.$$

Now, since  $\text{Hom}(G/E_{L_c}(G)L_c(G), L_c(G)) \cong G/Z(G) = C_{p^{b_1}} \times C_{p^{b_2}} \times \dots \times C_{p^{b_k}}$ , it follows that  $b_1 = b_2 = \dots = b_t = r$  and  $a_i = b_i$  for  $t + 1 \leq i \leq k$ , as required.

Conversely, if  $E_{L_c}(G)L_c(G) = Z(G)$ , then

$$\text{Aut}_{L_c}(G) \cong \text{Hom}(G/E_{L_c}(G)L_c(G), L_c(G)) = \text{Hom}(G/Z(G), L_c(G)) \cong G/Z(G),$$

because  $L_c(G)$  is cyclic and  $G' \leq L_c(G)$ . Hence  $\text{Aut}_{L_c}(G) = \text{Inn}(G)$ . Next assume that  $E_{L_c}(G)L_c(G) < Z(G)$ ,  $b_t = r$  and  $a_s = b_s$  for  $s = t + 1, \dots, k$ , where  $\exp(L_c(G)) = p^r$  and  $t$  is the largest integer between 1 and  $k$  such that  $a_t > r$ . As  $G$  is of class 2 and  $G/L_c(G)$  is abelian, so

$$p^{b_1} = \exp(G/Z(G)) = \exp(G') | \exp(L_c(G)) = p^r.$$

Therefore  $r \geq b_1 \geq b_2 \geq \dots \geq b_t = r$ , which shows that  $b_1 = b_2 = \dots = b_t = r$ . Since  $a_s = b_s$  for  $s = t + 1, \dots, k$ , we have

$$\begin{aligned} \text{Hom}(G/E_{L_c}(G)L_c(G), L_c(G)) &\cong C_{p^r} \times \dots \times C_{p^r} \times C_{p^{a_{t+1}}} \times \dots \times C_{p^{a_k}} \\ &= C_{p^{b_1}} \times \dots \times C_{p^{b_t}} \times C_{p^{b_{t+1}}} \times \dots \times C_{p^{b_k}} \\ &= G/Z(G). \end{aligned}$$

Therefore by Lemma 3.2,  $\text{Aut}_{L_c}(G) = \text{Inn}(G)$ . This completes the proof.  $\square$

**Lemma 3.7.** *Let  $G$  be a finite group such that  $Z(G/E_{L_c}(G)) = H/E_{L_c}(G)$ . Then*

- (i)  $Z(\text{Inn}(G)) \leq \text{Aut}_{L_c}(G)$  and  $H = Z_2(G)$ .
- (ii)  $\text{Aut}_{L_c}(G) = Z(\text{Inn}(G))$  if and only if  $\text{Hom}(G/E_{L_c}(G)L_c(G), L_c(G) \cap Z(G)) \cong H/Z(G)$ .

*Proof.* (i) Let  $i_t \in Z(\text{Inn}(G))$ , where  $i_t$  is an inner automorphism of  $G$  induced by the element  $t$  in  $G$ . Then  $i_t \in \text{Autcent}(G)$  and for all  $g \in G$ ,  $[g, i_t] = [g, t] \in L_c(G)$  since  $G/L_c(G)$  is abelian. Thus  $i_t \in \text{Aut}_{L_c}(G)$ . Next we show that  $H = Z_2(G)$ . Let  $t \in H$ . Then for all  $g \in G$ ,  $[g, t] \in E_{L_c}(G) \leq Z(G)$ . Thus  $t \in Z_2(G)$ . On the other hand, assume that  $t \in Z_2(G)$  and  $\alpha = i_t$ . Then for all  $g \in G$ ,  $[g, \alpha] = [g, t] \in Z(G)$  which shows that  $\alpha \in \text{Autcent}(G)$ . As  $\alpha \in C_{\text{Autcent}(G)}(\text{Aut}_{L_c}(G))$ , it follows that  $[g, t] = [g, \alpha] \in E_{L_c}(G)$ , for all  $g \in G$  and hence  $t \in H$ .

(ii) Suppose that  $\text{Aut}_{L_c}(G) = Z(\text{Inn}(G))$ . By (i) and Lemma 3.2 we have

$$\text{Hom}(G/E_{L_c}(G)L_c(G), L_c(G) \cap Z(G)) \cong H/Z(G).$$

Conversely, suppose that  $\text{Hom}(G/E_{L_c}(G)L_c(G), L_c(G) \cap Z(G)) \cong H/Z(G)$ . Then, by Lemma 3.2,

$$\text{Aut}_{L_c}(G) \cong H/Z(G) = Z_2(G)/Z(G) = Z(\text{Inn}(G)).$$

Therefore  $|\text{Aut}_{L_c}(G)| = |Z(\text{Inn}(G))|$ , which together with (i),  $\text{Aut}_{L_c}(G) = Z(\text{Inn}(G))$ , as required.  $\square$

**Corollary 3.8.** *Let  $G$  be an extra-special  $p$ -group. Then  $\text{Aut}_{L_c}(G) = \text{Inn}(G)$ .*

*Proof.* The proof follows at once from the fact that  $G' = E_{L_c}(G) = L_c(G) = Z(G) \cong C_p$  and Theorem 3.6.  $\square$

**Lemma 3.9.** *Let  $G$  be a finite non-abelian  $p$ -group such that  $\text{Aut}_{L_c}(G) = \text{IA}(G)$ . Then*

- (i)  $L_c(G) \leq Z(G)$  and  $E_{L_c}(G)L_c(G) \leq \Phi(G)$ .
- (ii)  $\text{Aut}_{L_c}(G) \cong \text{Hom}(G/G', L_c(G)) \cong \text{Hom}(G/L_c(G), G')$ .

*Proof.* (i) Since  $\text{Aut}_{L_c}(G) = \text{IA}(G)$ , it is easy to see that  $L_c(G) \leq Z(G)$ . We prove that  $E_{L_c}(G)L_c(G) \leq \Phi(G)$ . Suppose on the contrary, that there exists a maximal subgroup  $M$  of  $G$  such that  $E_{L_c}(G)L_c(G) \not\leq M$ . Then  $G = M\langle l \rangle$ , for some  $l$  in  $E_{L_c}(G)L_c(G) \setminus M$ . Choose an element  $u$  in  $\Omega_1(G')$ . We observe that the map  $\alpha : hl^i \mapsto hl^i u^i$ , where  $h \in M$  and  $0 \leq i < p$ , is an IA-automorphism

of  $G$ , which is a central kernel automorphism of  $G$ . Hence  $[l, \alpha] = 1$  and so  $u = 1$ , a contradiction. Therefore  $E_{L_c}(G)L_c(G) \leq \Phi(G)$ .

(ii) Let  $\alpha \in \text{Aut}_{L_c}(G)$ . Then  $f_\alpha : G \rightarrow L_c(G)$  given by  $f_\alpha(x) = x^{-1}x^\alpha$  defines a homomorphism from  $G$  to  $L_c(G)$ , and  $\alpha \mapsto f_\alpha$  is an injective map from  $\text{Aut}_{L_c}(G)$  to  $\text{Hom}(G, L_c(G))$ . Conversely, if  $f \in \text{Hom}(G, L_c(G))$ , then the map  $\alpha = \alpha_f$  defined by  $\alpha(x) = xf(x)$  for all  $x \in G$  is an endomorphism of  $G$ . Since by (i),  $x^{-1}\alpha(x) \in L_c(G) \leq E_{L_c}(G)L_c(G) \leq \Phi(G)$  for all  $x \in G$ , we may write  $G$  as the product of the image of  $\alpha$  and the Frattini subgroup of  $G$  and so the image of  $\alpha$  must be  $G$  itself. Thus  $\alpha$  is an automorphism of  $G$ . Consequently  $\alpha = \alpha_f \in \text{Aut}_{L_c}(G)$ ,  $f_{\alpha_f} = f$  and so  $|\text{Aut}_{L_c}(G)| = |\text{Hom}(G, L_c(G))|$ . Finally, suppose that  $\beta, \gamma \in \text{Aut}_{L_c}(G)$ . Then for any  $x \in G$ ,

$$f_{\beta\gamma}(x) = x^{-1}x^{\gamma\beta} = x^{-1}(xx^{-1}x^\gamma)^\beta = x^{-1}x^\beta x^{-1}x^\gamma,$$

since  $x^{-1}x^\gamma \in L_c(G)$ . Thus  $f_{\beta\gamma}(x) = f_\beta(x)f_\gamma(x)$  and so  $\alpha \mapsto f_\alpha$  is a homomorphism, as desired. Next we show that  $\text{Aut}_{L_c}(G) \cong \text{Hom}(G/L_c(G), G')$ . For any  $\alpha \in \text{Aut}_{L_c}(G)$ , the map  $f_\alpha : G/L_c(G) \rightarrow G'$  given by  $f_\alpha(xL_c(G)) = x^{-1}x^\alpha$  defines a homomorphism from  $G/L_c(G)$  to  $G'$  and the map  $f$  sending  $\alpha$  to  $f_\alpha$  is a monomorphism of the group  $\text{Aut}_{L_c}(G)$  to  $\text{Hom}(G/L_c(G), G')$ .

Conversely, for any  $f \in \text{Hom}(G/L_c(G), G')$ , the map  $\theta = \theta_f : G \rightarrow G$  defined by  $g^\theta = gf(gL_c(G))$ , where  $g \in G$ , is a central kernel automorphism and  $f(\theta) = f_\theta = f$ . Thus  $f$  is onto and hence  $\text{Aut}_{L_c}(G) \cong \text{Hom}(G/L_c(G), G')$ . Now the proof is complete.  $\square$

**Corollary 3.10.** *Let  $G$  be a finite non-abelian  $p$ -group. Then  $\text{Aut}_{L_c}(G) = \text{IA}(G)$  if and only if  $G' = L_c(G) \leq Z(G)$ .*

*Proof.* First suppose that  $\text{Aut}_{L_c}(G) = \text{IA}(G)$ . By Lemma 3.9,  $L_c(G) \leq Z(G)$  and  $\text{Aut}_{L_c}(G) \cong \text{Hom}(G/L_c(G), G') \cong \text{Hom}(G/G', L_c(G))$ . We claim that  $G' = L_c(G)$ . Suppose on the contrary, that  $G' < L_c(G)$ . Then  $G/L_c(G)$  is a proper quotient subgroup of  $G/G'$  and

$$|G/G'|/|G/L_c(G)| = |L_c(G)/G'| > 1.$$

Now, it follows from [1, Lemma D] that  $\text{Hom}(G/L_c(G), G')$  is isomorphic to a proper subgroup of  $\text{Hom}(G/G', L_c(G))$ , which is a contradiction. Therefore  $G' = L_c(G)$ , as required. The converse is evident.  $\square$

**Corollary 3.11.** *Let  $G$  be a finite non-abelian  $p$ -group. Then  $\text{Aut}_{L_c}(G) = \text{Aut}_c(G)$  if and only if  $\text{Aut}_c(G) \cong \text{Hom}(G/L_c(G), G')$  and  $G' = L_c(G) \leq Z(G)$ .*

*Proof.* First suppose that  $\text{Aut}_{L_c}(G) = \text{Aut}_c(G)$ . It follows that  $L_c(G) \leq Z(G)$  and  $\text{Aut}_{L_c}(G) = \text{IA}(G)$ . Now the proof follows at once from Lemma 3.9 and Corollary 3.10.

Conversely, as  $G' = L_c(G) \leq Z(G)$  we have

$$\text{Aut}_{L_c}(G) = \text{IA}(G) \cong \text{Hom}(G/G', G') = \text{Hom}(G/L_c(G), G') \cong \text{Aut}_c(G),$$



since  $\text{IA}(G) \cong \text{Hom}(G/G', G')$  using [12, Lemma 3.1]. Now the result follows from the fact that  $\text{Aut}_c(G) \leq \text{Aut}_{L_c}(G)$ .  $\square$

In the following result, we give a sufficient condition under which the group  $\text{Aut}_{L_c}(G)$  acts trivially on  $K_c(G)$ .

**Theorem 3.12.** *Let  $G$  be a group such that  $K_c(G)$  is a torsion-free subgroup of  $G$  and  $K_c(G)/E_{L_c}(G)$  is a torsion group. Then  $\text{Aut}_{L_c}(G)$  is a torsion-free abelian group such that acts trivially on  $K_c(G)$ .*

*Proof.* Let  $\alpha \in \text{Aut}_{L_c}(G)$  and  $x$  is an element of  $K_c(G)$ . Then by hypothesis  $x^n \in E_{L_c}(G)$ , for some  $n \in \mathbb{N}$ . Since  $x^{-1}x^\alpha \in Z(G)$ , we have

$$[x, \alpha]^n = (x^{-1}x^\alpha)^n = \underbrace{x^{-1}x^\alpha \cdots x^{-1}x^\alpha}_{n\text{-times}} = x^{-n}(x^n)^\alpha = [x^n, \alpha].$$

Hence Lemma 3.1 implies that  $[x, \alpha]^n = 1$ . As  $K_c(G)$  is a torsion-free subgroup, it follows that  $[x, \alpha] = 1$  and so  $\text{Aut}_{L_c}(G)$  acts trivially on  $K_c(G)$ . Next let  $\alpha \in \text{Aut}_{L_c}(G)$  and assume that there exists  $m \in \mathbb{N}$  such that  $\alpha^m = 1$ . Since  $[x, \alpha] \in L_c(G)$ , for all  $x \in G$ , there exists  $k_x \in L_c(G)$  such that  $x^\alpha = xk_x$ . Therefore  $x^{\alpha^2} = (x^\alpha)^\alpha = (xk_x)^\alpha = xk_x^2$  and so by induction we have  $x = x^{\alpha^m} = xk_x^m$ , whence  $k_x = 1$ , since  $K_c(G)$  is a torsion-free. Thus  $\alpha = 1$ , which together with Lemma 3.2,  $\text{Aut}_{L_c}(G)$  is a torsion-free abelian group.  $\square$

**4. Classify all finite  $p$ -groups  $G$  of order  $p^n, n \leq 5$  such that  $\text{Aut}_{L_c}(G) = \text{Inn}(G)$**

Recall that a finite  $p$ -group is called a minimal non-abelian if it is a non-abelian group and all its subgroups are abelian. In this section, by the following concept, we classify all  $p$ -groups  $G$  of order at most  $p^5$  such that  $\text{Aut}_{L_c}(G) = \text{Inn}(G)$ . Since by Corollary 3.8, for a non-abelian group  $G$  of order  $p^3$ ,  $\text{Aut}_{L_c}(G) = \text{Inn}(G)$ , we may assume that  $4 \leq n \leq 5$ . First we list the following results due to Redei (see [11]).

**Lemma 4.1.** [11] *Let  $G$  be a finite minimal non-abelian  $p$ -group. Then  $G$  is one of the following groups:*

- (i)  $Q_8$ ,
- (ii)  $M_p(n, m) = \langle a, b \mid a^{p^n} = b^{p^m} = 1, a^b = a^{1+p^{n-1}} \rangle$ , where  $n \geq 2$  and  $m \geq 1$ ,
- (iii)  $M_p(n, m, 1) = \langle a, b, c \mid a^{p^n} = b^{p^m} = c^p = 1, [a, b] = c, [a, c] = [b, c] = 1 \rangle$ , where  $n \geq m \geq 1$  and if  $p = 2$ , then  $m + n > 2$ .

The following equivalent conditions about finite minimal non-abelian  $p$ -groups are always used.

**Lemma 4.2.** [11] *Let  $G$  be a finite  $p$ -group. Then the following conditions are equivalent:*

- (i)  $G$  is a minimal non-abelian  $p$ -group.

- (ii)  $d(G) = 2$  and  $G' \cong C_p$ .
- (iii)  $d(G) = 2$  and  $Z(G) = \Phi(G)$ .

The following lemma is a useful fact in proving our next results and can be verified easily.

**Lemma 4.3.** *Let  $G$  be a finite minimal non-abelian  $p$ -group. Then*

- (i)  $Z(M_p(n, m)) = \langle a^p \rangle \times \langle b^p \rangle$  and  $Z(M_p(n, m, 1)) = \langle a^p \rangle \times \langle b^p \rangle \times \langle c \rangle$ .
- (ii)  $M_p(n, m)/M_p(n, m)' \cong C_{p^{n-1}} \times C_{p^m}$  and  $M_p(n, m, 1)/M_p(n, m, 1)' \cong C_{p^n} \times C_{p^m}$ .

The following concept was introduced by Hall [6].

**Definition 4.4.** Two finite groups  $G$  and  $H$  are said to be isoclinic if there exist isomorphisms  $\psi : G/Z(G) \rightarrow H/Z(H)$  and  $\theta : G' \rightarrow H'$  such that, if  $(x_1Z(G))^\psi = y_1Z(H)$  and  $(x_2Z(G))^\psi = y_2Z(H)$ , then  $[x_1, x_2]^\theta = [y_1, y_2]$ . Notice that isoclinism is an equivalence relation among finite groups and the equivalence classes are called isoclinism families.

**Corollary 4.5.** *Let  $G$  be a non-abelian group of order  $p^4$ . Then  $\text{Aut}_{L_c}(G) = \text{Inn}(G)$  if and only if  $G$  is one of the following types:*

- (i)  $M_p(2, 2)$ , where  $p$  an odd prime,
- (ii)  $M_p(3, 1)$ ,
- (iii)  $M_p(2, 1, 1)$ .

*Proof.* Assume that  $|G| = p^4$  and  $\text{Aut}_{L_c}(G) = \text{Inn}(G)$ . By Lemma 3.5,  $L_c(G) \leq Z(G)$ . First we claim that  $|Z(G)| = p^2$ . Suppose for a contradiction, that  $|Z(G)| = p$ . Then  $G' = L_c(G) = Z(G) \cong C_p$  and so is an extra-special  $p$ -group, which is a contradiction, since the order of  $G$  is not of the form  $p^{2k+1}$ , where  $k$  is a natural number. Thus  $G/Z(G) \cong C_p^2$ , whence  $|G'| = p$  and  $\Phi(G) \leq Z(G)$ . Since  $\text{Aut}_{L_c}(G) = C_{\text{Aut}_{L_c}(G)}(Z(G))$ , by the proof of Theorem 3.3,  $Z(G) \leq \Phi(G)$  and so  $Z(G) = \Phi(G)$ . Therefore  $G$  is a minimal non-abelian  $p$ -group by Lemma 4.2. We consider two cases:

Case I.  $p$  an odd prime. It is an easy task to see that the map  $\theta$  defined by  $x^\theta = x^{1+p}$ , is a central automorphism of  $G$ . Hence for any element  $x$  of  $L_c(G)$ ,  $x = x^\theta = x^{1+p}$ , and so  $x^p = 1$ . Thus  $\exp(L_c(G)) = p$  and  $L_c(G) \cong C_p$ , by Lemma 3.5. If  $G/L_c(G) \cong C_{p^3}$ , then  $G/Z(G)$  is cyclic, a contradiction. Next, we assume that  $G/L_c(G) \cong C_{p^2} \times C_p$ . Hence  $G \cong M_p(2, 2), M_p(3, 1)$  or  $M_p(2, 1, 1)$ , by Lemma 4.3. Finally, if  $G/L_c(G) \cong C_p^3$ , then  $L_c(G) = \Phi(G) = Z(G) \cong C_p$  and so  $G$  is an extra-special  $p$ -group, a contradiction.

Case II.  $p = 2$ . Since  $G' \leq L_c(G) \leq Z(G)$ , it follows that  $|L_c(G)| = 2$  or  $4$ . If  $|L_c(G)| = 4$ , then  $L_c(G) = Z(G)$  and  $G/L_c(G) \cong C_2^2$ . Hence  $\text{Autcent}(G) = \text{Aut}_{L_c}(G) = \text{Inn}(G)$  and so  $G' = Z(G) \cong C_2$  by using the main theorem of [1], a contradiction. Next we assume that  $|L_c(G)| = 2$ , whence  $|G/L_c(G)| = 8$ . If  $G/L_c(G) \cong C_8$ , then  $G$  is cyclic, a contradiction. Moreover, as  $E_{L_c}(G) \leq Z(G)$ , if  $G/L_c(G) \cong C_2^3$ , then by Theorem 3.6,  $d(G/Z(G)) = 3$  and so  $Z(G) \cong C_2$ ,

it follows that  $G$  is an extra-special 2-group, a contradiction. Therefore we assume that  $G/L_c(G) \cong C_4 \times C_2$ . Since  $G' \cong C_2$  and  $G'$  is a characteristic subgroup of  $G$ , we observe that  $G' \leq L(G)$  and so  $G' = L(G) = L_c(G)$ . Hence  $G/L(G) \cong C_4 \times C_2$  and  $G$  is isomorphic to one of the following groups:  $M_2(3, 1)$  or  $M_2(2, 1, 1)$ , by [9, Theorem 5.1]. The converse follows at once from Lemmas 3.2, 4.2, 4.3 and Theorem 3.6.  $\square$

**Corollary 4.6.** *Let  $G$  be a non-abelian group of order  $p^5$ . Then  $\text{Aut}_{L_c}(G) = \text{Inn}(G)$  if and only if  $G$  is one of the following types:*

- (i) *The isoclinism family (5) of [8],  $M_p(2, 3)$ ,  $M_p(3, 1, 1)$ , where  $p > 2$ ,*
- (ii)  $M_p(4, 1)$ ,
- (iii)  $M_p(3, 2)$ ,
- (iv)  $M_p(2, 2, 1)$ ,
- (v)  $D_8 * D_8$ ,
- (vi)  $D_8 * Q_8$ .

*Proof.* Let  $G$  be a non-abelian group such that  $|G| = p^5$  and  $\text{Aut}_{L_c}(G) = \text{Inn}(G)$ . It follows that  $L_c(G) \leq Z(G) \leq \Phi(G)$ , by Lemma 3.5 and the proof of Theorem 3.3. We consider two cases:

Case I.  $p > 2$ . These groups lying in the isoclinism families (5), (4) or (2) of [8].

First, let  $G$  denote one of the groups in the isoclinism family (5). Hence  $G' = L_c(G) = Z(G) \cong C_p$  and  $\text{Autcent}(G) = \text{Aut}_{L_c}(G)$ . Now with the main theorem of [1],  $\text{Autcent}(G) = \text{Inn}(G)$  if and only if  $G' = Z(G)$  and  $Z(G)$  is cyclic. This happens for all groups in the isoclinism family (5).

Next, let  $G$  be one of the groups in the isoclinism family (4). Then  $G' \cong C_p^2$ , which is a contradiction, since  $G' \leq L_c(G)$  is cyclic, by Theorem 3.6.

To continue the proof, let  $G$  denote one of the groups in the isoclinism family (2). Then  $G/Z(G) \cong C_p^2$  and  $G' \cong C_p$ . Hence  $Z(G) = \Phi(G)$  and so  $d(G) = 2$ . This implies that  $G$  is a minimal non-abelian  $p$ -group, by Lemma 4.2. Moreover, by considering the automorphism  $\theta$  mentioned in Corollary 4.5,  $\exp(L_c(G)) = p$  and so  $G' = L_c(G) \cong C_p$ . If  $G/L_c(G) \cong C_{p^3} \times C_p$ , then by Lemma 4.3,  $G$  is one of the following types:  $M_p(4, 1)$ ,  $M_p(2, 3)$  or  $M_p(3, 1, 1)$ . If  $G/L_c(G) \cong C_{p^2}^2$ , then by Lemma 4.3,  $G \cong M_p(3, 2)$  or  $G \cong M_p(2, 2, 1)$ . Finally, assume that  $G/L_c(G) \cong C_{p^2} \times C_p^2$  or  $G/L_c(G) \cong C_p^4$ . In these cases,  $\text{Aut}_{L_c}(G) \neq \text{Inn}(G)$ , by Lemma 3.2.

Case II.  $p = 2$ . We can see that  $|L_c(G)| = 8, 4, 2$ . First, we assume that  $|L_c(G)| = 8$ . It follows that  $G/L_c(G) \cong C_2 \times C_2$ , which shows that  $\Phi(G) \leq L_c(G)$ . So  $L_c(G) = Z(G) = \Phi(G)$ , which implies that  $\exp(G') = 2$  and  $\text{Aut}_{L_c}(G) = \text{Autcent}(G) = \text{Inn}(G)$ . Now, by applying the main theorem of [1],  $G' = Z(G) \cong C_2$ , a contradiction. Next assume that  $|L_c(G)| = 4$ . Then  $G/L_c(G)$  is one of the groups  $C_2^3$  or  $C_4 \times C_2$ . In the first case, by a similar argument mentioned earlier,  $G' = Z(G) \cong C_2$ , which is a contradiction. Therefore  $G/L_c(G) \cong C_4 \times C_2$  and by Lemma 3.2,  $\text{Aut}_{L_c}(G) \cong \text{Hom}(C_4 \times C_2, C_4) \cong$

$C_4 \times C_2$ . Therefore  $\text{Inn}(G) \cong C_4 \times C_2$ , so  $L_c(G) = Z(G)$  and we have a contradiction  $|G| = 16$ .

Now, we may suppose that  $|L_c(G)| = 2$ . Hence  $G' = L(G) = L_c(G) \cong C_2$  and  $Z(G) = \Phi(G)$ . We discuss the following cases.

Case (1). The same as previous paragraph, if  $G/L_c(G) \cong C_2^4$ , then  $\Phi(G) \leq L_c(G)$ , so  $L_c(G) = Z(G) = \Phi(G)$ . Hence  $G$  is an extra-special 2-group, and  $G$  is one of the groups  $D_8 * D_8$  or  $D_8 * Q_8$ , by [13].

Case (2). Suppose that  $G/L_c(G) \cong C_4 \times C_2^2$ . We assume that  $G/L_c(G) = \langle \bar{x}, \bar{y}, \bar{z} \rangle$ , where  $\bar{x} = xL_c(G), \bar{y} = yL_c(G), \bar{z} = zL_c(G), o(\bar{x}) = 4$  and  $o(\bar{y}) = o(\bar{z}) = 2$ . It follows that  $G = \langle x, y, z \rangle$ . Next, by Lemma 3.2,  $\text{Aut}_{L_c}(G) \cong \text{Hom}(C_4 \times C_2^2, C_2) \cong C_2^3$ , whence  $\text{Inn}(G) \cong C_2^3$ . Since  $[x^2, y] = [x^2, z] = 1$ , we observe that  $\langle x^2 \rangle \times L_c(G) \leq Z(G)$  and so  $Z(G) \cong C_2^2$ . Now by using GAP [4], we find that there is no such group.

Case (3). If  $G/L_c(G) = G/L(G) \cong C_8 \times C_2$ , then  $G \cong M_2(4, 1)$ , by [9, Theorem 5.1].

Case (4). Suppose that  $G/L_c(G) \cong C_4^2$ . As  $L_c(G) \leq Z(G)$  and  $G/Z(G)$  is elementary abelian, we have  $G/Z(G) \cong C_2^2$ . Hence  $d(G/Z(G)) = 2$  and so  $G$  is a minimal non-abelian  $p$ -group, whence by Lemmas 4.2 and 4.3,  $G$  is isomorphic to the group  $M_2(3, 2)$  or  $M_2(2, 2, 1)$ .

The converse follows at once from Lemmas 3.2, 4.2, 4.3, Theorem 3.6 and Corollary 3.8.  $\square$

### 5. Acknowledgment

The author is grateful to the editor and the referees for their valuable comments and careful reading. Also Corollary 2.2 is due to one of the referees; who I am indebted to him/her. This research was in part supported by a grant from Payame Noor University.

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