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Propagation of inhomogeneous waves in unsaturated visco-poroelastic layered media

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Abstract

In this study, we present an analytical model developed to describe broadband inhomogeneous wave propagation in an unsaturated visco-poroelastic layered medium which can be applied in geomechanics issues as hydrocarbon reservoirs. By taking into account the effect of the tortuosity parameter on the movement of pore fluids, the proposed formulation is capable of describing the wave behavior at high as well as mid and low frequencies. The boundary conditions proposed in this study, account for the connection between the surface pores, along with the slip that occurs between the two media at their interface. This enables us to model the layered medium in a more realistic way, where, the pore fluids are able to pass through the layers and the layers are able to move relative to each other. Finally, a sensitivity analysis is carried out and the effect of the various parameters on wave propagation inside the layered medium is investigated.

1. Introduction

Scientists and researchers have always been interested in finding the properties of waves and their utilization in various fields of science and technology. Amongst which of particular interest is the science of earth and the possibility of tapping into its resources and understanding its geological structures and inner workings. In this regard, there have been many efforts by the scientific communities around the world to develop mathematical models to understand the complexities of wave propagation in unsaturated soils.

The first serious study of wave propagation in porous media can be credited to Biot [1, 2]. Wherein, he developed a model for wave propagation in a saturated porous medium. He showed for the first time that three waves can propagate in such a medium, of which two belong to the compressional waves and the third one is related to the sole shear wave. Since then, others have tried to use, verify and develop the equations and results which are developed by Biot [3-7]. In this regard, Plona [5] and Berryman [3] have shown the validity of the Biot's theory and the propagation of the second compressional wave in a saturated porous medium.

Concerning the extension of the Biot's formalism, using the concepts of mixture theory, Brutsaert [8] has developed a model capable of predicting wave behavior in an unsaturated porous medium, where the pores are filled by two immiscible fluids. In the following years Santos et al. [9, 10] have also extended the theory of Biot and developed a method for obtaining the elastic constants related to an unsaturated porous medium. Tuncay and Corapcioglo [11] have used a Lagrangian viewpoint and the volume averaging technique to develop a model for wave propagation in an unsaturated porous medium, showing the existence of three compressional waves and one shear wave. Also Lo et al. [12] have used an Eulerian framework and developed a model to describe wave behavior in such a medium.

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Even though these models are all capable of predicting wave propagation behavior at low frequencies, they all fall short at considering the inertial coupling effects and hence are not suitable to analyze wave behavior at high frequencies, where the tortuosity parameter plays an important role. therefore, by using the relations proposed by Fredlund and Morgenstern [13] and Fredlund and Rehardjo [14], and by the constitutive relations of Conte et al. [15] and Smeulders [16], Ghasemzadeh and Abounouri [17] have developed a model capable of predicting wave velocity at all frequencies, in a single-layered medium.

It seems that Geertsma and Smith [18] were the first ones to develop a mathematical description for wave propagation in a layered porous medium. It should be noted that in their study it was assumed that the wave incidence is normal to the interface between the two media. In order to successfully describe the wave propagation phenomenon in a layered medium, one needs to know how the wave travels inside each one of those media, and what happens when these waves reach the interface between the two layers. Or in other words, one needs to have two models, one for describing wave propagation in a single medium and the other for describing wave behavior at the interface of every two adjoining media. Therefore, a need for boundary conditions feels at the interface. For instance, Yang et al. [19] had used the traction-free boundary condition for the expression of the scattering wave.

Deresiewicz and Skalak [20] and Murty [21] had come up with boundary conditions that have been used by researchers in the ensuing years. In this regard Deresiewicz and Skalak [20] came up with the general boundary condition for the case that the pores at the interface of the two porous media are either fully connected or are not connected at all. Murty [21] has also introduced a more general boundary condition which has been named as loose contact, where a portion of the incident energy is dissipated and the rest is partitioned between that of the reflected and refracted waves. Also, Vollmann et al. [22] has shown that the border between the two half-spaced graded in a way leads to a frequency and wavelength which depend on refraction and reflection behavior of waves.

Later, researchers including but not limited to Hajra and Mukhopadhyay [23], Wu et al. [24], vashisth et al. [25], Sharma and Saini [26] and Lin et al. [27] have used these boundary conditions to model the phenomenon of reflection and refraction of the incident waves in saturated porous media.

Tomar and Arora [28] started the first attempts at modeling wave propagation phenomenon in a three-phase layered porous medium, where the pores are filled by two immiscible fluids. In order to reach that goal they used the formalism developed by Tuncay and Corapcioglo [11] and ignored the viscosity of the pore fluids. They also assumed that one of the two media is a porous elastic half-space consisting of three phases and the other one is an elastic solid consisting of only the solid phase. Later Arora and Tomar [29] extended their 2006 paper to a layered medium consisting of a three-phase elastic porous half-space for the upper and the lower media.

Yeh et al. [30] used the model of Lo et al. [12] and the normal coordinates were developed by Lo et al. [31] to derive the energy and amplitude ratios of the reflected and refracted waves at the interface of two unsaturated porous media. The novelty of their approach compared to Arora and Tomar [29] was the use of normal coordinates to obtain the relations, between the amplitude ratios of the propagating waves in different phases of the porous medium.

Sharma and Kumar [32] have derived the amplitude and energy ratios of the reflected waves at the surface of an unsaturated porous medium and air. They based their model on that of Tuncay and Corapcioglo [11] and used the concept of Christoffel equations (Reference?). One of the distinct features of their study is the use of the concept of inhomogeneous waves in a way as defined by Borcherdt [33, 34], where it is proven that for wave propagation in a viscoelastic medium, the angle between the planes with equal amplitude and the planes with equal phase is not equal. Kumar and Majhi [35] have studied on propagation of shear horizontal (SH) plane waves in inhomogeneous soil using visco-elastic coefficient as function of depth. Furthermore, Amirkhizi et al. [36] have investigated the scattering of oblique SH waves off finite periodic media, consisting of elastic and viscoelastic layers.

Kumar and Sharma [37] have extended the model developed by Lo et al. [12] to layered porous media. In their formulation, they have used the concept of inhomogeneous wave propagation. They have also utilized the boundary conditions obtained by Sharma [38] for the case of a welded contact in which surface pores at the interface of the two media are partially connected.

Based on the brief account that has been given for the phenomenon of wave propagation in saturated and unsaturated layered media, it is clear that the mathematical foundations are now fully established for tackling such problems even deeper. Therefore in this paper we will build upon the collective findings of all the researchers mentioned in the previous paragraphs, and by utilizing the constitutive relations obtained by Ghasemzadeh and Abounouri [17] and [39], we will develop a formalism for inhomogeneous wave propagation in a layered three-phase unsaturated viscoelastic porous medium.

The novelty of the formalism derived in this paper is the inclusion of inertial coupling effects at high frequencies caused by the tortuosity of the porous medium and consequently the credibility of the results at high-frequency values. Another feature of the model developed here is that the concept of loose boundary condition is utilized in tackling the problem of a layered three-phase unsaturated porous medium, and the effect of the bonding parameter Ψ is seen on the amplitude and energy ratios of the reflected and refracted waves. Finally, by taking into account the connection in the surface pores of the two media at their interface, the boundary conditions proposed, enable us to provide a more realistic mathematical interpretation of the situation that exists in the field.

2. Model parameters and the constitutive relations

2.1. Dispersion relations

Based on Ghasemzadeh and Abounouri [17] the following governing equations can be given for wave propagation in a three-phase unsaturated porous medium:Captions for Tables should appear *above* the table. All other captions should appear *below* the illustration (figures, graphs, photos).

$$\zeta\, \nabla\nabla\cdot\vec{u}_s - G(\nabla\nabla\cdot\vec{u}_s - \nabla^2\vec{u}_s) + \varsigma\, \nabla\nabla\cdot\vec{u}_w + \xi\, \nabla\nabla\cdot\vec{u}_a - (\tau_w-1)\phi_w\rho_w\partial_t^2(\vec{u}_s-\vec{u}_w) - (\tau_a-1)\phi_a\rho_a\partial_t^2(\vec{u}_s-\vec{u}_a) - b_w\partial_t(\vec{u}_s-\vec{u}_w) - b_a\partial_t(\vec{u}_s-\vec{u}_a) - \rho_s(1-\phi)\partial_t^2\vec{u}_s = 0 \quad (1)$$

$$-\phi_w W\nabla\nabla\cdot\vec{u}_s + \phi_w^2 L\nabla\nabla\cdot\vec{u}_w + \phi_a\phi_w C\nabla\nabla\cdot\vec{u}_a + (\tau_w-1)\phi_w\rho_w\partial_t^2(\vec{u}_s-\vec{u}_w) + b_w\partial_t(\vec{u}_s-\vec{u}_w) - \phi_w\rho_w\partial_t^2\vec{u}_w = 0 \quad (2)$$

$$-\phi_a M\nabla\nabla\cdot\vec{u}_s + \phi_a\phi_w C\nabla\nabla\cdot\vec{u}_w + \phi_a^2 N\nabla\nabla\cdot\vec{u}_a + (\tau_a-1)\phi_a\rho_a\partial_t^2(\vec{u}_s-\vec{u}_a) + b_a\partial_t(\vec{u}_s-\vec{u}_a) - \phi_a\rho_a\partial_t^2\vec{u}_a = 0 \quad (3)$$

where $\vec{u}_s$, $\vec{u}_w$, $\vec{u}_a$ are defined as the displacement fields of the solid phase, the wetting pore fluid phase (as water) and the non-wetting pore fluid phase (as air or oil), respectively. ϕ_w and ϕ_a denote volume fractions of the wetting and the non-wetting pore fluid phases. τ_w and τ_a denote the effective tortuosity for the wetting and the non-wetting pore fluid phases. The parameters ρ_s, ρ_w and ρ_a are solid, water and air density. The parameters G and χ are defined as the shear modulus of the solid skeleton, and effective stress parameter, respectively. The other parameters used in equations (1)-(3) can be seen in Appendix (A).

In order to solve the system of partial differential equations (1)-(3), they used the Helmholtz decomposition of the displacement field and obtained the uncoupled system of partial differential equations for the compressional and shear waves propagating in a three-phase unsaturated porous medium as shown below:

For the scalar part:

$$\zeta\nabla^2\Phi_s + \varsigma\nabla^2\Phi_w + \xi\nabla^2\Phi_a - (\tau_w-1)\phi_w\rho_w\partial_t^2\Phi_s + (\tau_w-1)\phi_w\rho_w\partial_t^2\Phi_w - (\tau_a-1)\phi_a\rho_a\partial_t^2\Phi_s + (\tau_a-1)\phi_a\rho_a\partial_t^2\Phi_a - b_w\partial_t\Phi_s + b_w\partial_t\Phi_w - b_a\partial_t\Phi_s + b_a\partial_t\Phi_a - \rho_s(1-\phi)\partial_t^2\Phi_s = 0 \quad (4)$$

$$-\phi_w W\nabla^2\Phi_s + (\phi_w)^2 L\nabla^2\Phi_w + \phi_w\phi_a C\nabla^2\Phi_a + (\tau_w-1)\phi_w\rho_w\partial_t^2\Phi_s - (\tau_w-1)\phi_w\rho_w\partial_t^2\Phi_w + b_w\partial_t\Phi_s - b_w\partial_t\Phi_w - \phi_w\rho_w\partial_t^2\Phi_s = 0 \quad (5)$$

$$-\phi_a M\nabla^2\Phi_s + (\phi_a)^2 N\nabla^2\Phi_a + \phi_w\phi_a C\nabla^2\Phi_w + (\tau_a-1)\phi_a\rho_a\partial_t^2\Phi_s - (\tau_a-1)\phi_a\rho_a\partial_t^2\Phi_a + b_a\partial_t\Phi_s - b_a\partial_t\Phi_a - \phi_a\rho_a\partial_t^2\Phi_s = 0 \quad (6)$$

and for the vector part:

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$$G\nabla^2\psi_s-(\tau_w-1)\phi_w\rho_w\partial_t^2\psi_s+(\tau_w-1)\phi_w\rho_w\partial_t^2\psi_w-(\tau_a-1)\phi_a\rho_a\partial_t^2\psi_s+(\tau_a-1)\phi_a\rho_a\partial_t^2\psi_a-b_w\partial_t\psi_s+b_w\partial_t\psi_w-b_a\partial_t\psi_s+b_a\partial_t\psi_a-\rho_s(1-\phi)\partial_t^2\psi_s=0 \quad (7)$$

$$(\tau_w-1)\phi_w\rho_w\partial_t^2\psi_s-(\tau_w-1)\phi_w\rho_w\partial_t^2\psi_w+b_w\partial_t\psi_s-b_w\partial_t\psi_w-\phi_w\rho_w\partial_t^2\psi_s=0 \quad (8)$$

$$(\tau_a-1)\phi_a\rho_a\partial_t^2\psi_s-(\tau_a-1)\phi_a\rho_a\partial_t^2\psi_a+b_a\partial_t\psi_s-b_a\partial_t\psi_a-\phi_a\rho_a\partial_t^2\psi_s=0 \quad (9)$$

In the above equations, $\Phi_i(i=s,w,a)$ and $\Psi_j(j=s,w,a)$ denote the scalar and vector potentials, respectively for the solid and fluid phases.

By transforming equations (4)-(6) from the space-time domain to the frequency-wavenumber domain, and using some algebraic manipulations, Ghasemzadeh, and Abonouri [17] have obtained the following dispersion relations for compressional wave propagation in a three-phase unsaturated porous medium where the pores are filled by two immiscible fluids:

$$A_1+A_2k_p^2+A_3\left(k_p^2\right)^2+A_4\left(k_p^2\right)^3=0 \quad (10)$$

where kP stands for compression wave number and A1, A2, A3, and A4 are introduced in Appendix (A).

The dispersion relation (10) is a cubic polynomial in k_p^2 with three complex roots. This shows that in an unsaturated three-phase porous medium, three waves propagate which based on the descending value of their real parts; they respectively denote P1, P2, and P3 waves. The complex roots only occur when the medium is dispersive; i.e., when the pore fluids are viscous. It should be noted that the solid phase in the formalism developed here is elastic and therefore, intrinsic attenuation only occurs if the pore fluids are viscous. If the medium is dispersion less; i.e., if the pore fluids are not viscous then the roots are real numbers and hence no attenuation occurs. It should also be noted that the P1 and P2 waves are similar to the fast and slow Biot waves and propagate regardless of the existence of the second pore fluid. The P3 wave is created as a result of the existence of the second pore fluid and generally has the lowest velocity and the highest attenuation .

Similarly, by transforming the relations (7)-(9) from the space-time domain to the wavenumber-frequency domain and doing some algebraic manipulation similar to that of Ghasemzadeh and Abonouri [17] have done, the following dispersion relation for shear wave propagation can be obtained:

$$b_{22}b_{33}Gk_s^2+b_{13}b_{22}b_{31}+b_{12}b_{21}b_{33}-b_{11}b_{22}b_{33}=0 \quad (11)$$

The dispersion relation (11) is a bilinear equation showing that two roots exist, of which only one is physically possible. This shows that in an unsaturated three-phase porous medium only one shear wave propagates. Similar to the case of compressional waves, if the medium is dispersive; then, the roots are complex numbers and if the medium is non-dispersive; then, the roots are real numbers. The existence of only one shear wave also tells us that the addition of the second pore fluid does not have any effect on the number of shear waves propagating in the porous medium.

In order to model wave propagation in a layered three-phase unsaturated porous medium, it is of importance to find some relations amongst the amplitude ratios of waves in all of the three phases. For that matter, based on Lo et al. [31] and by using the relations (4)-(6) or their counterparts in wavenumber-frequency domain and some algebraic manipulations, the following relations can be obtained:

$$A_{wi}=\Gamma_i A_{si}\ ;(i=1,2,3) \quad (12)$$

$$A_{ai}=\Pi_i A_{si}\ ;(i=1,2,3) \quad (13)$$

where, A_{si}, A_{wi} and A_{ai} denote the amplitude ratios for the solid, wetting pore fluid and the non-wetting pore fluid phases for compressional wave propagation. Here the subscript $i=1,2,3$ denotes that the relations belong to the case of the P1, P2 and the P3 wave propagation. Other parameters used are defined below:

$$\Gamma_i=-\left(Y_{11}^iY_{23}^i-Y_{13}^iY_{21}^i\right)/\left(Y_{12}^iY_{23}^i-Y_{13}^iY_{22}^i\right);\ (i=1,2,3) \quad (14)$$

$$\Pi_i=\left(Y_{11}^iY_{22}^i-Y_{12}^iY_{21}^i\right)/\left(Y_{12}^iY_{23}^i-Y_{13}^iY_{22}^i\right);\ (i=1,2,3) \quad (15)$$

where the following relations hold:

$$Y_{jk}^i=B_{jk}-\lambda_i^2a_{jk},\quad \lambda_i=1/v_i;\quad \lambda_i=1/v_i;\quad (i,k=1,2,3),(j=1,2) \quad (10)$$

$$
\begin{aligned}
&B_{11}=(1/\omega)((\rho_a(\tau_a-1)\phi_a+\rho_a+\rho_w(\tau_w-1)\phi_w-\rho_s(\phi-1))\omega-ib_a-ib_w);\ \left(i=\sqrt{-1}\right)\\
&B_{12}=B_{21}=\left(-\phi_w\left(\tau_w-1\right)\rho_w\omega+ib_w\right)/\omega,\\
&B_{13}=\left(-\phi_a\left(\tau_a-1\right)\rho_a\omega+ib_a\right)/\omega;\ \left(i=\sqrt{-1}\right)\\
&B_{22}=\phi_w\rho_w\tau_w-\left(ib_w\right)/\omega;\ \left(i=\sqrt{-1}\right),\qquad B_{23}=0
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
&a_{11}=\zeta,\qquad a_{12}=\varsigma,\qquad a_{13}=\xi\\
&a_{21}=-\left(W\phi_w\right),\qquad a_{22}=L\phi_w^2,\qquad a_{23}=\phi_w\phi_a C
\end{aligned}
\tag{12}
$$

Also in a similar manner, by using the relations (8)-(9) or their counterpart in the wavenumber-frequency domain and some simple algebraic manipulations such as that outlined in Tomar and Arora [28], the following relations can be obtained for shear wave amplitude in different phases of the porous media:

$$
B_w=J_1B_s,\qquad B_a=J_2B_s \tag{19}
$$

where, B_s, B_w and B_a denote the shear wave amplitudes for the solid, wetting pore fluid and the non-wetting pore fluid phases. The parameters J_1 and J_2 are defined below:

$$
J_1=\left(\omega\left(\tau_w-1\right)^2\phi_w\rho_w-ib_w\omega\right)\Big/\left(\omega\left(\tau_w-1\right)^2\phi_w\rho_w+\omega^2\phi_w\rho_w-ib_w\omega\right);\ \left(i=\sqrt{-1}\right) \tag{13}
$$

$$
J_2=\left(-\phi_a\rho_a\left(\tau_a-1\right)\omega+ib_a\right)\Big/\left(-\omega\phi_a\rho_a\tau_a+ib_a\right);\ \left(i=\sqrt{-1}\right) \tag{14}
$$

It can be deduced from dispersion relations that the interference of non-variable parameters of the environment affects wave propagation. This shows the fact that the non-homogeneity of materials affects the speed of the waves.

2.1. Amplitude and energy ratio coefficients

In order to obtain wave energy ratios, first, we need to define particle velocities and surface tractions at the boundary of the two layers. To this end, we will define vector and displacement potentials so that we can use them in obtaining the particle displacement vectors. It should be noted that for simplicity we assume that the waves only propagate inside the x-z plane. Similar to figure 1, by assuming the propagation of plane waves and using potential functions given by Borcherdt [40], we have the following relations:

$$
r=x\bar{e}_x+z\bar{e}_z \tag{15}
$$

$$
K_{p0}^u=k\bar{e}_x-d_{p0}^u\bar{e}_z \tag{16}
$$

$$
K_{pi}^u=k\bar{e}_x+d_{pi}^u\bar{e}_z\ (i=1,2,3),K_{s1}^u=k\bar{e}_x+d_s^u\bar{e}_z \tag{17}
$$

$$
K_{pi}=k\bar{e}_x-d_{pi}\bar{e}_z\ (i=1,2,3),K_{s1}=k\bar{e}_x-d_s\bar{e}_z \tag{18}
$$

where:

$$
d_{pi}^u=\sqrt{k_{pi}^{u2}-k^2}\ (i=0,1,2,3),d_{s1}^u=\sqrt{k_{s1}^{u2}-k^2} \tag{19}
$$

$$
d_{pi}=\sqrt{k_{pi}^2-k^2}\ (i=1,2,3),\quad d_{s1}=\sqrt{k_{s1}^2-k^2} \tag{20}
$$

$$
|P|=\left(\left(\left(\mathrm{Re}\left(k_{p0}^{u2}\right)\right)+\sqrt{\left(\mathrm{Re}\left(k_{p0}^{u2}\right)\right)^2+\frac{1}{\left(\cos\gamma_0^u\right)^2}\left(\mathrm{Im}\left(k_{p0}^{u2}\right)\right)^2}\right)\Big/2\right)^{1/2} \tag{21}
$$

$$
|A|=-\left(1/2|P|\cos\gamma_0^u\right)\mathrm{Im}\left(k_{p0}^{u2}\right) \tag{22}
$$

$$
k=|P|\sin\left(\theta_0^u\right)-i|A|\sin\left(\theta_0^u-\gamma_0^u\right) \tag{23}
$$

$$
k_{pi}^u=\omega/v_i^u;\ (i=1,2,3),k_{s1}^u=\omega/v_4^u \tag{24}
$$

$$
k_{pi}=\omega/v_i;\ (i=1,2,3),k_{s1}=\omega/v_4 \tag{25}
$$

In the above relations, subscripts 0, 1, 2, and

3 denote the incident, P1, P2 and P3 waves, respectively. p_i denotes parameters defined for compressional waves and $S1$ is for the shear wave. It should also be noted that the superscript u denotes the parameters pertaining to the upper medium, whereas the non-existence of u means that the parameters are defined for the lower medium. Parameters, d^u_{pi}, $d_{pi,}$ and, d^u_{s1}, d_{s1} account for the compressional and shear wave attenuation when propagating in a dissipative medium. The parameter k defined in (23) is the wave vector and to ensure the validity of the Snell`s law at the boundary of the two layers, it is the same for the upper and lower medium. Parameters, k^u_{pi}, $k^u_{s1,}$ and, k_{pi}, k_{s1} are the wave vectors for the compressional and shear waves in the upper and lower medium.

Also similar to that shown in figure 1, $\gamma^u_i; (i = 0,1,2,3,4)$ denotes the angle between the propagation and attenuation vectors for the incident, reflected P1, P2, P3, and the shear wave and $\gamma_i; (i = 1,2,3,4)$ denotes the angle between the propagation and attenuation vectors for the refracted P1, P2, P3 and the shear wave. $\theta^u_i; (i = 0,1,2,3,4)$ denotes the angle between the propagation vectors of the incident, P1, P2, P3, and the shear wave with that of the Z axis for the upper medium and $\theta_i; (i = 1,2,3,4)$ denotes its counterpart for the P1, P2, P3 and the shear wave in the lower medium. Similarly $A^u_i; (i = 0,1,2,3,4)$ denotes the attenuation vectors for the incident, P1, P2, P3 and the shear wave in the upper medium and $A_i; (i = 1,2,3,4)$ denote that for the P1, P2, P3 and the shear wave in the lower medium. Finally, $P^u_i; (i = 0,1,2,3,4)$ and $P_i; (i = 1,2,3,4)$ denote the propagation vectors for the incident, P1, P2, P3 and the shear wave in upper and lower media.

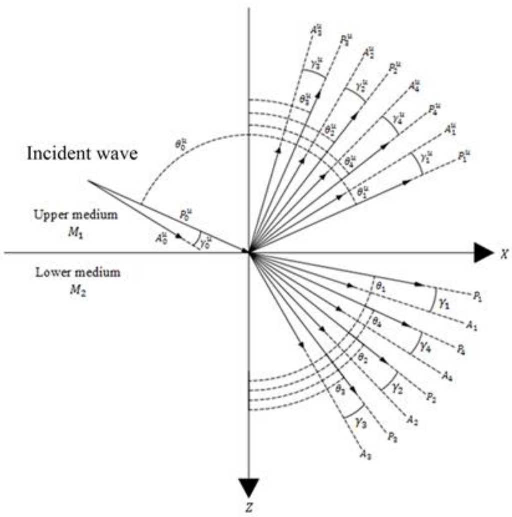

Figure 1. Schematics of the propagation and the attenuation vectors of the incident, reflected and refracted waves in unsaturated Visco-pore elastic layered media

It should also be noted that in order to define the relations (15)-(25) we only need the propagation and attenuation angles, i.e. θ^u_0, and γ^u_0;, for the incident wave. The other angles are related to θ^u_0 and γ^u_0 due to the validity of the Snell``s law at the interface, as stated by Borcerdt et al. [41] and established by equating the parameter k from equation (23) for the upper and the lower media.

Regarding the displacement potentials in the upper medium we have:

$$\Phi^u_s = \sum_{j=0}^{3} A^u_{sj} \exp\left(i\left(\omega t - K^u_{\phi j}\cdot\vec{r}\right)\right),\quad \Phi^u_w = \sum_{j=0}^{3} \Gamma^u_j A^u_{sj} \exp\left(i\left(\omega t - K^u_{\phi j}\cdot\vec{r}\right)\right),\quad \Phi^u_a = \sum_{j=0}^{3} \Pi^u_j A^u_{sj} \exp\left(i\left(\omega t - K^u_{\phi j}\cdot\vec{r}\right)\right) \tag{26}$$

$$\Psi^u_s = B^u_{s1} \exp\left(i\left(\omega t - K^u_{\psi 1}\cdot\vec{r}\right)\right),\quad \Psi^u_w = J^u_1 B^u_{s1} \exp\left(i\left(\omega t - K^u_{\psi 1}\cdot\vec{r}\right)\right),\quad \Psi^u_a = J^u_2 B^u_{s1} \exp\left(i\left(\omega t - K^u_{\psi 1}\cdot\vec{r}\right)\right) \tag{27}$$

and for the lower medium:

$$\Phi_s = \sum_{j=1}^{3} A_{sj}\, exp\left(i(\omega t - K_{\phi j}\cdot\vec{r})\right),\quad \Phi_w = \sum_{j=1}^{3} \Gamma_j A_{sj}\, exp\left(i(\omega t - K_{\phi j}\cdot\vec{r})\right), \Phi_a = \sum_{j=1}^{3} \Pi_j A_{sj}\, exp\left(i(\omega t - K_{\phi j}\cdot\vec{r})\right) \tag{28}$$

$$\Psi_s = B_{s1} \exp\left(i\left(\omega t - K_{\psi 1}\cdot\vec{r}\right)\right),\quad \Psi_w = J_1 B_{s1} \exp\left(i\left(\omega t - K_{\psi 1}\cdot\vec{r}\right)\right),\quad \Psi_a = J_2 B_{s1} \exp\left(i\left(\omega t - K_{\psi 1}\cdot\vec{r}\right)\right) \tag{29}$$

In equations (26)-(29), ; Φ_s, Φ_w and Φ_a denote the scalar displacement potentials for wave motion in the solid, wetting and the non-wetting fluid phases. Similarly, Ψ_s , Ψ_w and Ψ_a denote vector displacement potentials for wave propagation in the solid, wetting and the non-wetting fluid phases. A_{sj}; $(j = 1,2,3)$ Denote wave amplitudes for the P1, P2 and P3 waves. Also B_{s1} is the wave amplitude for the sole shear wave. The same goes for the upper medium where the parameters are denoted by the superscript u in their definition. The subscript 0 which is used here for the upper medium is related to that of the incident wave, which can be any one of the three compressional waves or the shear wave propagating in the porous medium. Parameters Γ_j, Π_j, J_1, and J_2 were respectively defined in **Error! Reference source not found.**, **Error! Reference source not found.**, (13), and (14).

Using Helmholtz decomposition of the displacement field and by assuming the validity of plane strain conditions in the x-z plane, the particle displacements are obtained as:

$$\begin{aligned} u_{sx}^u &= \partial\Phi_s^u/\partial x - \partial\Psi_s^u/\partial z; \quad u_{wx}^u = \partial\Phi_w^u/\partial x - \partial\Psi_w^u/\partial z; \quad u_{ax}^u = \partial\Phi_a^u/\partial x - \partial\Psi_a^u/\partial z \\ u_{sz}^u &= \partial\Phi_s^u/\partial z + \partial\Psi_s^u/\partial x; \quad u_{wz}^u = \partial\Phi_w^u/\partial z + \partial\Psi_w^u/\partial x; \quad u_{az}^u = \partial\Phi_a^u/\partial z + \partial\Psi_a^u/\partial x \end{aligned} \tag{30}$$

$$\begin{aligned} u_{sx} &= \partial\Phi_s/\partial x - \partial\Psi_s/\partial z; \quad u_{wx} = \partial\Phi_w/\partial x - \partial\Psi_w/\partial z; \quad u_{ax} = \partial\Phi_a/\partial x - \partial\Psi_a/\partial z \\ u_{sz} &= \partial\Phi_s/\partial z + \partial\Psi_s/\partial x; \quad u_{wz} = \partial\Phi_w/\partial z + \partial\Psi_w/\partial x; \quad u_{az} = \partial\Phi_a/\partial z + \partial\Psi_a/\partial x \end{aligned} \tag{31}$$

Based on Conte et al. [15] and Ghasemzadeh and Abounouri [17], the following constitutive relations governing stress-strain conditions in the medium are defined:

$$\sigma = G\left(\nabla\vec{u}_s + \nabla^T\vec{u}_s\right) + HI\nabla\cdot\vec{u}_s + \phi_w\left[\chi L + (1-\chi)C\right]I\nabla\cdot\vec{u}_w + \phi_a\left[\chi C + (1-\chi)N\right]I\nabla\cdot\vec{u}_a \tag{32}$$

$$p_w = W\nabla\cdot\vec{u}_s - \phi_w L\nabla\cdot\vec{u}_w - \phi_a C\nabla\cdot\vec{u}_a, \quad p_a = M\nabla\cdot\vec{u}_s - \phi_w C\nabla\cdot\vec{u}_w - \phi_a N\nabla\cdot\vec{u}_a \tag{33}$$

In the above equations σ denotes the total stress tensor. p_w and p_a denote the pore-water and pore-air pressures. I stands for the identity tensor and the rest of the parameters were previously defined.

From equations (30)-(33), we can now obtain the particle velocities and the surface tractions at the interface of the two layers. Then by using the boundary conditions given in relation (34) which are valid at the boundaries of the two porous media, we can obtain an inhomogeneous system of equations comprising of eight equations and eight unknowns. Solving this system of equations numerically using the LU method gives us the amplitude ratios of the reflected and refracted waves with respect to that of the incident wave, which then enables us to calculate the normalized energy ratios.

Boundary conditions for the loosely-bonded interface are defined as:

$$\begin{aligned} &\sigma_{zz}^u = \sigma_{zz} && p_w^u - p_w = \frac{Z}{Z-1}\left(\dot{u}_{sz}^u - \dot{u}_{wz}^u\right) \\ &\sigma_{zx}^u = \sigma_{zx} && p_a^u - p_a = \frac{Z}{Z-1}\left(\dot{u}_{sz}^u - \dot{u}_{az}^u\right) \\ &\dot{u}_{sz}^u = \dot{u}_{sz} && \dot{u}_{sz}^u - \dot{u}_{wz}^u = \dot{u}_{sz} - \dot{u}_{wz} \\ &\sigma_{zx}^u = \frac{\Psi}{\Psi-1}T\left(\dot{u}_{Sx} - \dot{u}_{Sx}^u\right) && \dot{u}_{sz}^u - \dot{u}_{az}^u = \dot{u}_{sz} - \dot{u}_{az} \end{aligned} \tag{34}$$

σ_{zz} and σ_{zx} denote the normal and tangential stresses at the interface and the Parameters $\dot{u}_{sz}$ and $\dot{u}_{sx}$ define particle velocities in the z and x directions. The superscript u shows that the parameters are defined for the upper medium. The parameter T denotes the surface flow impedance parameter which is a nonzero finite positive value representing the resistance to the free discharge of pore fluids at the interface.

In the boundary conditions defined above, it is assumed that the surface pores at the interface of the two layers can be either totally connected or disconnected or they can be in a state between these two extremes. This state of connection between the pores of the two media is defined by the parameter Z. Here when $\mathrm{Z} = 0$;, the fluid pressures between the upper and lower media are equal. This, in turn, means that the fluids are in contact with each other and therefore we have open pores. On the other hand, when Z=1, the relative particle velocities in each of the media equal zero. This means that the pores are totally closed. Finally, when Z is between 0 and 1, the pores are partially connected.

It is also assumed that as a result of the imposed tangential stress a slip might occur at the interface. In this regard, the parameter Ψ is termed the bonding parameter and takes the value of 0 and 1 and all the values in between. In other words, the bonding parameter shows how close the boundary condition is to the smooth and welded contact. The value 0 for the bonding parameter shows that the two layers would slip with respect to each other;, even, if the tangential stresses are trivial. And the value 1 for the bonding parameter which denotes the welded contact shows that the two layers would never slip with respect to each other, even if the tangential stresses are big.

Using equations (30)-(34), we can obtain the following non-homogeneous system of equations involving eight equations and eight unknowns:

$$\sum_{J=1}^{8} R_J H_{LJ} = q_L; \quad (L=1,2,3,4,5,6,7,8) \tag{35}$$

In the above system of equations, R_J denotes the amplitude ratios of the reflected (J =1, 2, 3, 4) and refracted (J = 5, 6, 7, 8) waves with respect to that of the incident wave. H_{LJ}, and q_L are some coefficients defined in Appendix (B).

After obtaining the amplitude ratios, we can use the energy flux relations as defined by Achenbach [41] to obtain energy ratios of the reflected and refracted waves with respect to that of the incident wave. These relations are defined as below:

$$P_{ij} = \sigma_{zxi}\dot{u}_{sxj} + \sigma_{zzi}\dot{u}_{szj} + p_{wi}\left(\dot{u}_{szj} - \dot{u}_{wzj}\right) + p_{ai}\left(\dot{u}_{szj} - \dot{u}_{azj}\right) \quad (i,j=1,2,3,4) \tag{36}$$

$$P^u_{ij} = \sigma^u_{zxi}\dot{u}^u_{sxj} + \sigma^u_{zzi}\dot{u}^u_{szj} + p^u_{wi}\left(\dot{u}^u_{szj} - \dot{u}^u_{wzj}\right) + p^u_{ai}\left(\dot{u}^u_{szj} - \dot{u}^u_{azj}\right) \quad (i,j=0,1,2,3,4) \tag{37}$$

Equations (36) and (37) denote the average energy flux for the lower and the upper media respectively. In other words, they define the average rate of energy transmission across the interface between the two porous half-spaces. By using these equations, we can obtain the relations given in Appendix (C) for the energy flux of the reflected and transmitted waves and their interaction.

After obtaining the average energy flux relations as in ((C.1)-((C.30), we can now obtain the normalized energy matrices as given below [37]:

$$E^u = \begin{bmatrix} E^u_{00} & E^u_{01} & E^u_{02} & E^u_{03} & E^u_{04} \\ E^u_{10} & E^u_{11} & E^u_{12} & E^u_{13} & E^u_{14} \\ E^u_{20} & E^u_{21} & E^u_{22} & E^u_{23} & E^u_{24} \\ E^u_{30} & E^u_{31} & E^u_{32} & E^u_{33} & E^u_{34} \\ E^u_{40} & E^u_{41} & E^u_{42} & E^u_{43} & E^u_{44} \end{bmatrix}, \quad E^u_{ij} = P^u_{ij} \big/ P^u_{00}; \quad (i,j=0,1,2,3,4) \tag{38}$$

$$E = \begin{bmatrix} E_{11} & E_{12} & E_{13} & E_{14} \\ E_{21} & E_{22} & E_{23} & E_{24} \\ E_{31} & E_{32} & E_{33} & E_{34} \\ E_{41} & E_{42} & E_{43} & E_{44} \end{bmatrix}, \quad E_{ij} = P_{ij} \big/ P^u{}_{00}; \quad (i,j=1,2,3,4) \tag{39}$$

In equations (38) and (39), the diagonal elements represent the energy ratios of the reflected and refracted waves with respect to that of the incident wave. The rest of the elements represent interaction energy ratios of the reflected and refracted waves with that of the incident wave and one another.

The following relations hold for the elements of the energy matrices given above:

$$E_I^u = \sum_{j=0}^{4}\sum_{i=0}^{4} E_{ij}^u - \sum_{i=0}^{4} E_{ii}^u \quad (47)$$

$$E_I = \sum_{j=1}^{4}\sum_{i=1}^{4} E_{ij} - \sum_{i=1}^{4} E_{ii} \quad (40)$$

$$E_T = E_I^u + E_I + \sum_{i=1}^{4} E_{ii}^u + \sum_{i=1}^{4} E_{ii} = 1 \quad (49)$$

In the above equations; E_I^u, E_I and E_T denote the total interaction energy ratio in the upper medium, the total interaction energy in the lower medium and the total energy ratio, respectively. It should be noted that E_{00} should not be taken into account when calculating the total energy ratio. Also according to the law of the preservation of energy, as shown in equation **Error! Reference source not found.** the total energy ratio should equal unity.

3. Sensitivity analysis and results

Due to a large number of parameters used in the analytical model developed in the last section, it is almost impossible to clearly see the effect of each parameter on wave behavior at the boundary of the two porous media. Hence, it is useful that a numerical analysis is carried out to see the effect of important parameters involved in the model.

The soil parameters used in this study are shown in table 1. As can be seen from this table, an unsaturated silty loam (Fredlund and Xing [42]) is used for the upper medium and an unsaturated sand (Ghasemzadeh and Abonouri [17]) is used for the lower medium. The parameters a, n, m are related to the soil water characteristic curve which is defined for unsaturated soils. A detailed account of the soil water characteristic curve and the methods used to obtain the desired parameters can be found in Fredlund et al. [43].The pore fluid parameters are also shown in table 2.

Table 1. Soil properties (Ghasemzadeh and Abonouri [17] & Fredlund and Xing [42])

Parameter	Unit	Symbol	Upper medium M1 (Silty Loam [42])	Lower medium M2 (sand [17])
Solid density	$kg.m^{-3}$	ρ_s	2798	2650
Soil Poisson ratio	–	ν	0.3	0.3
Soil shear modulus	MPa	G	45	50
Porosity	–	ϕ	0.48	0.3
Intrinsic permeability	m^2	k_0	6×10^{-13}	10^{-11}
Soil parameter	–	a	67.32	10
Soil parameter	–	m	0.499	1
Soil parameter	–	n	7.32	2
Residual suction	kPa	ψ_r	200	2000
Residual saturation	–	S_r	0.270	0.05
Pore-size index	–	λ	1.82	2.2

Table 2. Pore fluid properties

Parameter	Unit	Symbol	Value
Air density	$kg.m^{-3}$	ρ_a	1.1
Air bulk modulus	MPa	K_a	0.11
Air viscosity	$Pa.s$	η_a	18×10^{-6}
Water density	$kg.m^{-3}$	ρ_w	1000
Water bulk modulus	GPa	K_w	2.25
Water viscosity	$Pa.s$	η_w	1×10^{-3}

In the following, we will first start by analyzing the effect of frequency and the degree of saturation of the wetting pore fluid phase on wave propagation velocity in the two adjoining porous media. The wave numbers obtained from the dispersion relations **Error! Reference source not found.** and **Error! Reference source not found.** are complex numbers, implying that the waves undergo intrinsic attenuation in the Visco-poroelastic medium. In order to obtain wave phase velocity (v) and intrinsic attenuation coefficient (c), the following relations are used:

$$V = \omega / \mathrm{Re}(k) \qquad (41)$$

$$C = Q^{-1} = 2\left|\mathrm{Im}(k)/\mathrm{Re}(k)\right| \qquad (42)$$

where, k stands for the complex wave number and is defined as $k = \mathrm{Re}(k) + \mathrm{Im}(k)i$

Figure 2 shows the wave velocity variations with respect to a change in frequency and the degree of water saturation for the upper and lower media. Here similar to the notation used in figure 1, M1 denotes the upper medium and M2 denotes the lower medium. Also S_w^u denotes the degree of water saturation in the upper medium (M1) and S_w denotes that for the lower medium (M2).

Using these figures it can be seen that an increase in the frequency leads to an increase in wave velocities of the compressional and shear waves both in the upper and lower media. But, there are differences in the sensitivity of these waves to frequency variations. Where, for the P2 and P3 waves, we have the highest variations with regards to a change in frequency and for the P1 and shear waves the variations are comparatively much smaller.

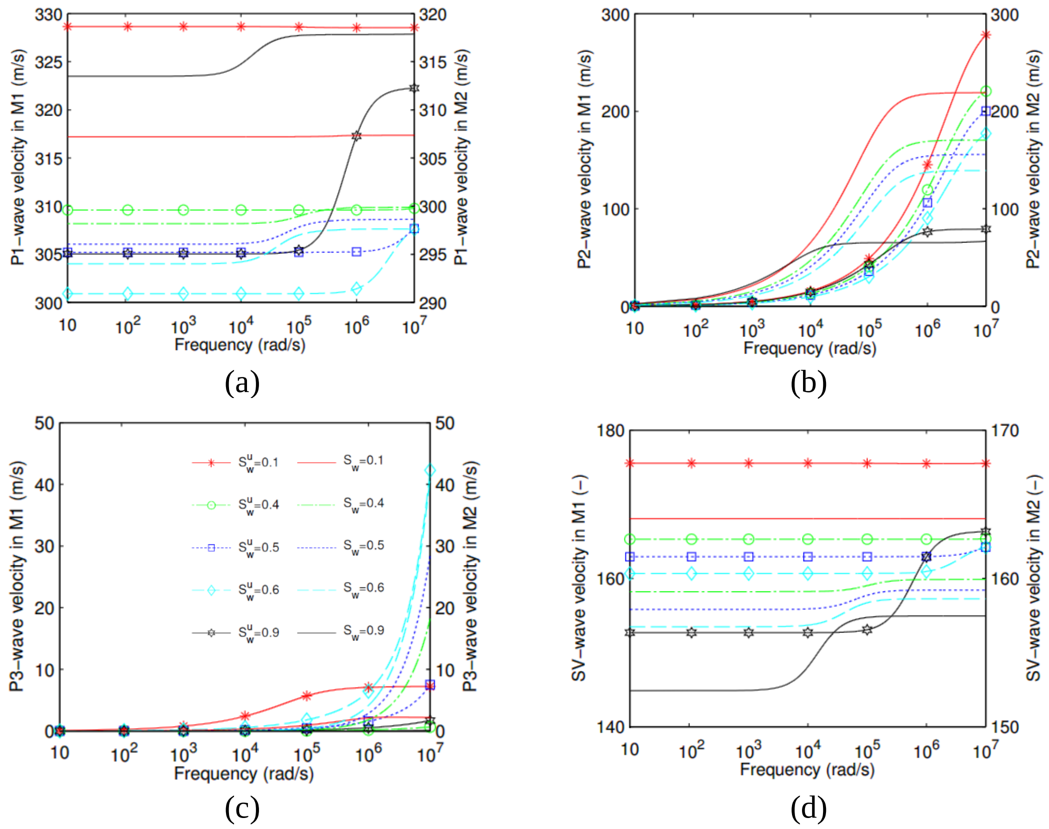

Figure 2. P1 (a), P2 (b), P3 (c) and SV (d) wave velocities with respect to ω and S_w for the upper and lower media.

Based on these figures, it can also be observed that for some of the water saturation degrees considered, the velocity variations are not uniform with respect to changes in frequency. This means that for some of the water saturation degrees we get sudden jumps in wave velocity;

whereas, for others, we do not. Based on this, it can be deduced that the water saturation degree plays an important role in determining both the velocity of the waves and their sensitivity to certain frequency range variations.

In terms of the medium`s effect on the velocity variations, we can see that the P1 and the shear waves travel at higher velocities when they are propagating through the upper medium. This behavior is expected, as the silty loam has a higher solid density than the sand. Interestingly, when we observe the behavior of the P2 and P3 waves, we see that they do not always travel faster in the upper medium. Knowing that the P2 and P3 waves only exist in a porous medium with various fluid phases, one can deduce that their behavior is also hugely affected by the parameters of the porous medium other than the solid density.

Figure 3 shows variations in the intrinsic attenuation with respect to variations in frequency and the degree of water saturation for the upper and lower media. Based on these figures it can be observed that for the P1 and the shear waves, a bell-shaped curve is present both for the upper and lower media, with the ones for the upper medium being shifted towards higher frequencies. Other distinct patterns that can be seen with respect to wave attenuation versus frequency belong to that of the P2 and P3 waves. For the P2 wave it can be seen that the highest attenuations occur at lower frequencies and as the frequency increases, the attenuation diminishes. This shows that P2 waves can travel further at high frequencies when compared to that of lower frequencies.

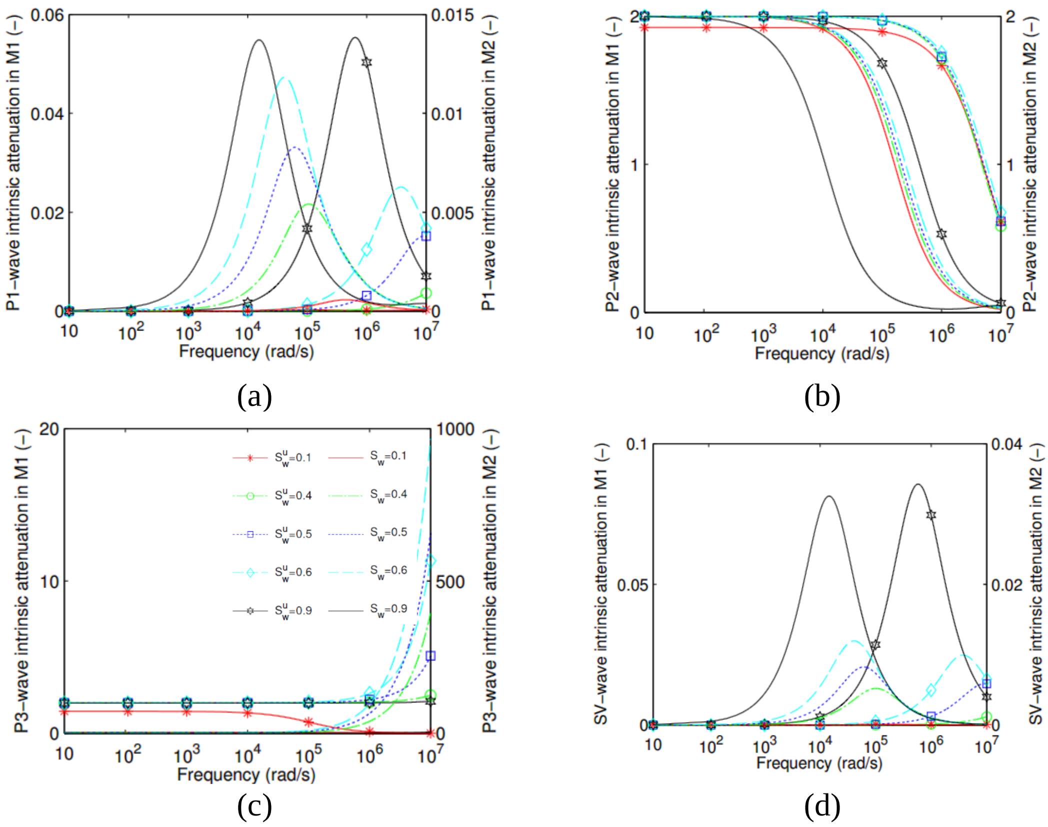

Figure 3. P1 (a), P2 (b), P3 (c) and SV (d) wave intrinsic attenuations with respect to ω and S_w for the upper and lower media.

Unlike that of the P2 wave, the P3 wave intrinsic attenuation increases with frequency with the exception of the case for $S_w = 0.1$. Similar to that of the velocity variations, this reduction or increase in intrinsic attenuation with frequency is not uniform. We can see that depending on the type of the wave and the degree of water saturation, the reduction or increase in attenuation starts at certain frequencies and most of it occurs at specific frequency ranges.

Also relevant to the topic is the effect of the medium type on attenuation variations. It can be seen that for the P1, P2 and the shear waves, not only the attenuation curves shift to higher frequencies for the upper medium, but they also have higher values. This shows that for most of the frequency range considered, the ratio of the imaginary to the real part of the wave number parameter is bigger for the upper medium than for the lower medium. This trend however, does not hold true for the P3 wave. It can be seen that the amount of intrinsic attenuation is much higher for the P3 wave in the lower medium than in the upper medium. This along with the case of velocity variations discussed before shows that the mechanisms controlling the P3 wave behavior are different than those of the other waves.

Figure 4 shows the change in the normalized energy ratios for the reflected and refracted P1, P2, P3 and shear waves at the surface along with their interaction energy ratios in the lower and upper media, with respect to a change in the angle of incidence and frequency. Here ω^u denotes frequencies for the energy ratios of the reflected waves and ω denotes that of the refracted waves. Apart from the parameters mentioned in tables 1 and 2, the values of S_w=0.6, Z=1, γ_0^u=45°, Ψ=1 and T=1×10^6 MPa.s/m are also used in obtaining the energy ratios shown in these figures.

Using these figures, it can be seen that for normal angles of incidence, the energy of the incident P1 wave is partitioned to the refracted P1 wave mostly and for low grazing angles most of it, is given to the reflected one. It can also be observed that the energy ratio curves for the reflected and refracted P1 waves resemble that of an exponential curve in the sense that the energy ratio value is constant for most of the incident angles and then most variations occur for the final 15 degrees. From a mathematical viewpoint, this mostly happens as the potential functions defined in equations (26)-(29) were of exponential forms.

Another thing that can be deduced from the P1 energy ratio curves is the fact that the reflected and refracted waves are out of phase with each other. Considering the effects of frequency on the energy ratio of the P1 wave, no noticeable variations are observed. This means that the P1 wave is not affected by changes in frequency and its value stays the same regardless of its propagation frequency.

Comparing the values of the energy ratios for the P2, P3 and shear waves with that of the P1 wave, it is clear that the P1 wave is dominant both in the reflected and refracted cases. This is expected, as the type of the incident wave is also of the P1 type. If we had considered the incidence of other wave types at the interface, more energy would have been partitioned to other waves. It is also worth noting that in a real physical situation, only the P1 and shear waves have a chance of reaching the interface of the two layers, and the P2 and P3 waves would probably totally dissipate long before they even reach the interface due to high intrinsic attenuation.

When considering the behavior of other waves other than the P1, we can see that unlike the P1 wave, where we had distinct patterns with respect to θ; these waves show erratic behaviors, in the sense that their behavior changes based on the propagation frequency. This shows that these waves are more affected by changes in frequency. The same behavior was also seen in the velocity and intrinsic attenuation parameters for the P2 and P3 waves.

Figure 5 shows the change in the normalized energy ratios for the reflected and refracted P1, P2, P3, and shear waves at the surface along with their interaction energy ratios in the upper and lower media, with respect to a change in the angle of incidence and the degree of saturation of the water phase. Here S_w^u denotes water saturation degree for the energy ratio of the reflected wave and S_w denotes that of the refracted one. Apart from the parameters mentioned in tables 1 and 2, the values of $\omega = 100\ rad/s$, $\gamma_0^u = 45°$, $Z = 1$, $\Psi = 1$ and $T = 1 \times 10^6\ MPa.s/m$ are used in obtaining the energy ratios shown in these figures.

Using these figures, it can be seen that changing the water saturation degree does not affect the overall trend in the energy ratio variations with respect to a change in the incident angle. But it is clear that the P1 wave has more sensitivity to S_w variations than ω. Here we can see that when the water saturation degree is increased, the curves for the reflected and refracted P1 wave energy ratios are shifted to the right. Also, similar to the case of frequency variations, we can see that the reflected and refracted energy ratios are out of phase with each other.

Considering the energy ratio variations for the P2, P3 and the shear waves, we can see that the P2 and the P3 waves are more sensitive to frequency variations; whereas, the shear wave is more sensitive to S_w variations. This behavior is in line with that we saw in velocity and intrinsic attenuation variations of these waves with respect to frequency and water saturation degree variations.

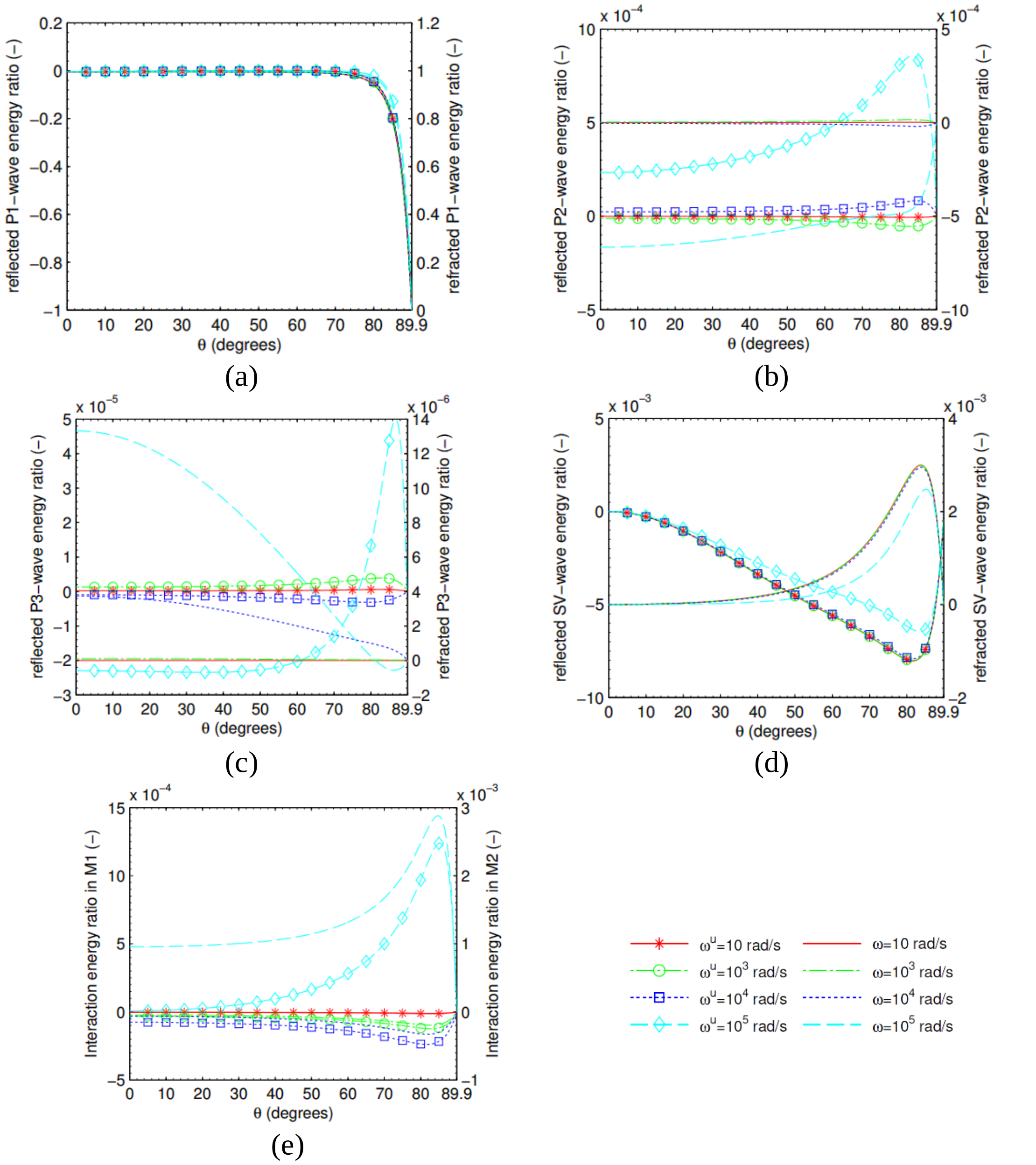

Figure 4. Wave energy ratio in layered media, (a) P1, (b) P2, (c) P3, (d) SV and (e) interaction energy ratios with respect to (θ) and (ω)

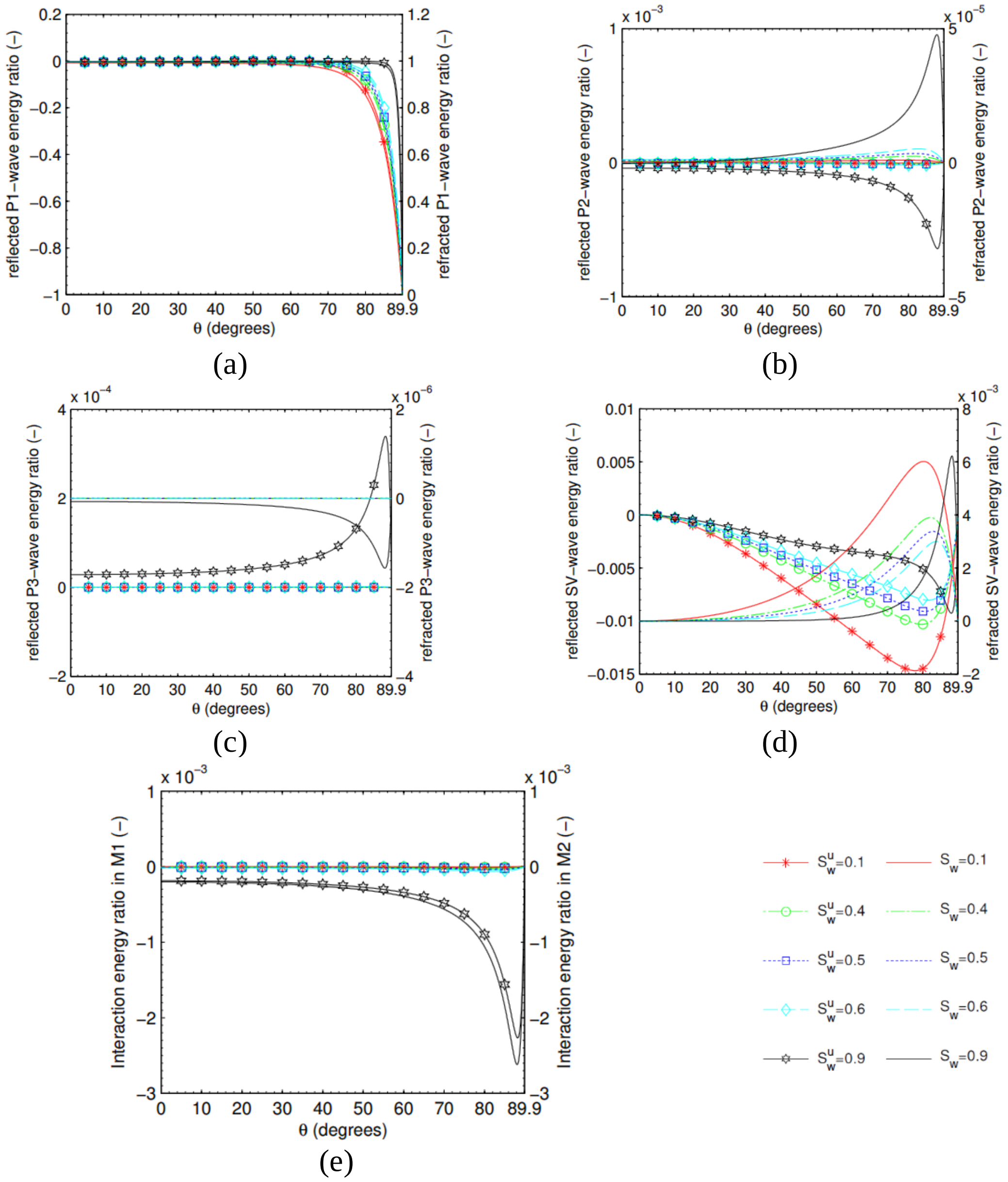

Figure 5. Normalized wave energy ratio in layered media, (a) P1, (b) P2, (c) P3, (d) SV and (e) interaction energy ratios with respect to (θ) and (S_w)

Figure 6 shows the energy ratios of the reflected and refracted P1, P2, P3 and shear waves along with their interaction energy ratios in the lower and upper media, with respect to a change in the angle of incidence and the bonding parameter. Here Ψ^u denotes the bonding parameter for the energy ratio of the reflected wave and Ψ denotes that of the refracted wave. Apart from the parameters mentioned in tables 1 and 2, the values of $\omega = 100\ rad/s$, $\gamma_0^u = 45°$, $Z = 1$, $S_w = 1$ and $T = 1 \times 10^6\ MPa.s/m$ are used in obtaining the energy ratios shown in these figures.

The most important issue to note here is the effect of Ψ on the total energy ratio. It was mentioned before that this parameter shows how much slip occurs at the interface of the two porous media. Where $\Psi = 1$ denotes the case for the welded contact and $\Psi = 0$ denotes the case for smooth contact. The former means that no slip is allowed to happen at the interface, while the latter implies that the two media slip pass each other with no resistance.

Here again, we see that changing the bonding parameter does not affect the overall variation behavior of the reflected and refracted waves, particularly for the P1 wave. Nonetheless, changing this parameter can create subtle variations, which can be easily observed. For example it can be seen that for some of the values of the Ψ parameter; namely, the $\Psi = 0$ and $\Psi = 0.1$, the P1 wave energy ratio has dips and peaks; whereas, for the case of the S_w and ω variations we do not see such a behavior. We can also see that, the symmetry we observed in figures 4 and 5 is more pronounced here for P2, P3 and shear modes. This shows that when the frequency and the water saturation degree are constant, the reflected and refracted waves have more or less the same variation patterns with respect to the incident angle but in the opposite phases.

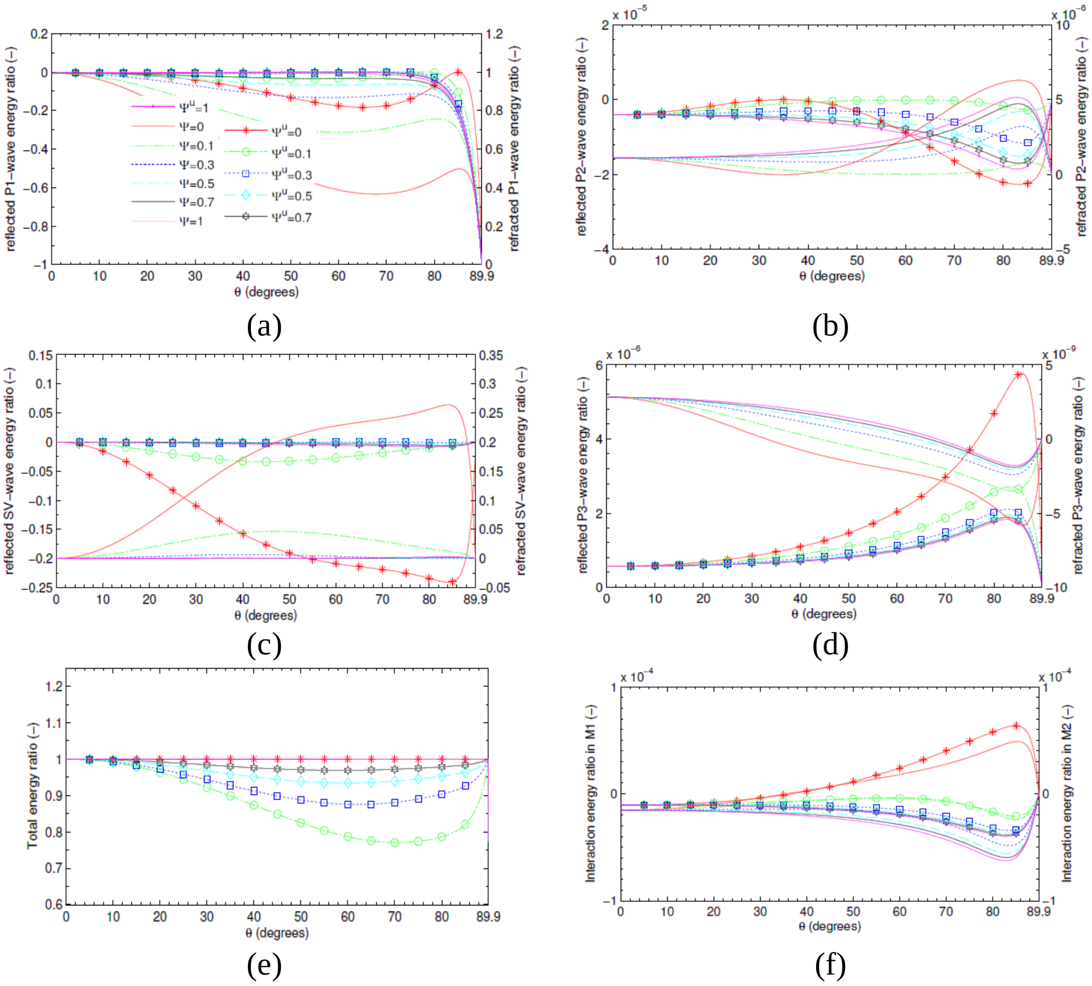

Figure 6. Wave energy ratio with respect to incident angle (θ) and bonding parameter (Ψ) in layered media, (a) P1, (b) P2, (c) P3, (d) SV, (e) total energy ration and (f) interaction energy ratios

Also based on the figure 6 (e), we can see that most of the energy dissipation belongs to $\Psi = 0.1$ and the higher or lower than we go towards the smooth and welded contact i.e., $\Psi = 0$ and $\Psi = 1$, where the amount of energy dissipation decreases, until at the limiting values of the bonding parameter no dissipation occurs and the total energy ratio equals unity. This final figure shows how introducing the notion of slip at the interface into the model can affect the total energy dissipation.

Finally, Due to the classic definition of the critical angle used in waveform tomography, there should be an incident angle at which no refraction occurs. Whereas, based on figures 4, 5 and 6 it can be observed that no such angle exists. This along with the fact that equating the energy ratio of the refracted matrix in equation (46) to zero, does not give a solution for the incident angle proves that no critical angle exists at the layer of the two viscoelastic media investigated in this study. In the continuation of this research, regarding the significant advancement in investigating phenomena on a micro-scale [44-46], the wave propagation equations can be studied at nano and micro scales.

4. Conclusions

- In the present study, an analytical model is developed capable of predicting broadband wave behavior in layered unsaturated visco-poroelastic media. Using the concept of the Helmholtz decomposition of a vector field and by transforming the uncoupled system of partial differential equations governing wave motion from the time-space domain to the wavenumber-frequency domain, dispersion relations were obtained for the compressional and shear waves that consider the tortuosity in the pore-space of the unsaturated medium and hence unlike similar models can account for the inertial coupling effects. These dispersion relations show that in an unsaturated three phase porous medium 4 wave modes can propagate, where three of them are of compressional modes and one is the shear mode.
- The other important issue considered in this study is the use of inhomogeneous waves and the loose boundary condition to model wave behavior at the interface of the two porous media. It was shown that through the bonding parameter, we can account for the slip that occurs at the interface and the subsequent energy loss that follows such a phenomenon. This along with the fact that the surface pore connection was seen in the boundary connections, allowed us to propose a more realistic mathematical view of the actual field condition that exists at the interface of the two porous media.
- We have carried out sensitivity analyses on wave velocity and energy ratios of the propagating waves and have shown that the P1 and the shear waves are not much affected by variations in frequency and the degree of water saturation, whereas the P2 and particularly the P3 waves are hugely influenced by such variations. This can explain from a mathematical point of view how increasing frequency or saturation degree can affect different wave modes and the maximum amplitude drop in waveforms with respect to offset from the source as seen in actual field seismograms.
- The other important result that can be obtained from this study is the invariability in the trend of the P1-wave energy ratio variations with respect to the incident P1 wave when the rest of the parameters were altered. This shows that energy partitioning at the interface of soil layers or inhomogeneities in the real media that leads to scattering is mostly dependent on ray incidence angle and not the physical source or medium parameters.
- It was shown that for the incident P1 wave, most of the energy is partitioned to the reflected and refracted P1 waves, and the other waves take only a trivial amount of the incident wave energy. From this, it can be deduced that the scattering phenomena in real media which account for part of the wave energy attenuation can mostly be attributed to reflection and refraction of the same wave mode at inhomogeneities inside the medium and to a less extent to mode conversion.
- Finally, it was mentioned that no solution exists for the case of total reflection and hence it can be concluded that no critical angle exists for the unsaturated Visco-poroelastic media considered in this study. This can have implications in improving refraction tomography techniques in future studies.

5. Acknowledgements

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Appendix A

The other parameters used in (1)-(3) are defined as:

$$\zeta = 2G + H + \phi_w W + \phi_a M \tag{A.1}$$

$$\varsigma = \phi_w \left[\chi L + (1-\chi) C \right] - \phi_w^2 L - \phi_a \phi_w C \tag{A.2}$$

$$\xi = \phi_a \left[\chi C + (1-\chi) N \right] - \phi_a \phi_w C - \phi_a^2 N \tag{A.3}$$

$$\phi_i = \phi S_i ;\ i = a, w ;\ S_a = 1 - S_w \tag{A.4}$$

$$a_s = \nu / 1 - 2\nu \tag{A.5}$$

$$H = 2a_s G - \chi W - (1-\chi) M \tag{A.6}$$

$$W = -K_w A / D + \phi_w L + \phi_a C \tag{A.7}$$

$$M = -K_a B / D + \phi_w C + \phi_a N \tag{A.8}$$

$$L = K_w \left[\phi_a m_1^s + K_a \left(m_1^s m_2^w - m_2^s m_1^w \right) \right] / D \tag{A.9}$$

$$N = K_a \left[\phi_w m_1^s + K_w \left(m_1^s m_2^w - m_2^s m_1^w \right) \right] / D \tag{A.10}$$

$$C = K_w K_a \left(m_1^s m_2^w - m_2^s m_1^w \right) / D \tag{A.11}$$

$$B = \phi_w m_1^a + K_w \left(m_1^s m_2^w - m_2^s m_1^w \right) \tag{A.12}$$

$$A = \phi_a m_1^w + K_a \left(m_1^s m_2^w - m_2^s m_1^w \right) \tag{A.13}$$

$$D = \left(m_1^s m_2^w - m_2^s m_1^w \right) \left(\phi_a K_w + \phi_w K_a \right) + \phi_a \phi_w m_1^s \tag{A.14}$$

$$b_i = \phi_i^2 \eta_i / k_i ;\ i = a, w \tag{A.15}$$

$$\tau_w = \left(\frac{1 - S_r}{S_w - S_r}\right)\left(1 - 0.5\left(1 - \frac{1}{\phi}\right)\right) \tag{A.16}$$

$$\tau_a = \left(\frac{1 - S_r - S_e}{1 - S_w - S_e}\right)\left(1 - 0.5\left(1 - \frac{1}{\phi}\right)\right) \tag{A.17}$$

$$k_w = k_0 \left(\frac{S_w - S_r}{1 - S_r}\right)^{(2+3\lambda)/\lambda} \tag{A.18}$$

$$k_a = k_0 \left(1 - \frac{S_w - S_r}{1 - S_r}\right)^2 \left(1 - \left(\frac{S_w - S_r}{1 - S_r}\right)^{(2+\lambda)/\lambda}\right) \tag{A.19}$$

$$m_1^s = \frac{3(1 - 2\upsilon)}{2(1+\upsilon) G} \tag{A.20}$$

$$m_2^w = -\left[\frac{\left(\frac{1}{\psi + \psi_r}\right)}{\ln\left(1 + \frac{10^6}{\psi_r}\right)} \mathrm{x} \frac{\theta_s}{\left(\ln\left(e + \left(\frac{\psi}{a}\right)^n\right)\right)^m} \right] + \left[\left(1 - \frac{\ln\left(1 + \frac{\psi}{\psi_r}\right)}{\ln\left(1 + \frac{10^6}{\psi_r}\right)}\right) \mathrm{x} \left(-mn \frac{\theta_s}{\left(\ln\left(e + \left(\frac{\psi}{a}\right)\right)\right)^{m+1}}\right) \mathrm{x} \left(\frac{\psi^{n-1}}{ea^n + \psi^n}\right) \right] \tag{A.21}$$

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$$m_1^w = m_2^s \tag{A.22}$$

$$S_a = 1 - S_w(\psi) \tag{A.23}$$

where the parameter ϕ is the porosity and the parameters K_w and K_a are defined as the bulk modulus of the wetting and the non-wetting pore fluid phases. η_w and η_a are the viscosity of the wetting pore fluid and the non-wetting pore fluid phases. k_w and k_a denote the effective permeability of the wetting and the non-wetting pore fluid phases, m_1^s , m_2^s , m_1^w and m_2^w are defined as the volume change coefficient of the solid phase with respect to net normal stress, volume change coefficient of the solid phase with respect to matric suction, volume change coefficient of the wetting pore fluid phase with respect to net normal stress and the volume change coefficient of the wetting pore fluid phase with respect to the matric suction, respectively. Also, the volume change coefficients of the non-wetting phase are related to m_1^s, m_2^s, m_1^w and m_2^w using the following equations:

$$m_1^a = m_1^s - m_1^w \tag{A.24}$$

$$m_2^a = m_2^s - m_2^w \tag{A.25}$$

Knowing the volume change coefficients for the solid phase m_2^s and m_1^s , one can define the effective stress parameter χ as:

$$\chi = m_2^s \big/ m_1^s \tag{A.26}$$

The parameters of dispersion equation of compressional waves in equation **Error! Reference source not found.** are defined as:

$$A_1 = -a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} + a_{11}a_{22}a_{33} \tag{A.27}$$

$$\begin{aligned} A_2 = & -a_{11}a_{33}L\left(\phi_w\right)^2 - a_{11}a_{22}N\left(\phi_a\right)^2 + a_{13}\left(a_{21}a_{23} + a_{31}L\left(\phi_w\right)^2 - a_{22}M\phi_a\right) \\ & + a_{12}\left(a_{23}a_{31} + a_{21}N\left(\phi_a\right)^2 - a_{33}W\phi_w\right) - a_{22}a_{33}\zeta + a_{22}a_{31}\xi + a_{21}a_{33}\zeta \end{aligned} \tag{A.28}$$

$$\begin{aligned} A_3 = & a_{13}LM\phi_w^2\phi_a + a_{11}\left(-a_{23}^2 + LN\left(\phi_w\phi_a\right)^2\right) + a_{13}a_{23}\phi_w W + a_{12}a_{23}M\phi_a + a_{12}NW\left(\phi_a\right)^2\phi_w + a_{33}L\left(\phi_w\right)^2\zeta \\ & + a_{22}\left(\phi_a\right)^2 N\zeta - a_{21}a_{23}\xi - a_{31}L\left(\phi_w\right)^2\xi + a_{22}M\phi_a\xi - a_{23}a_{31}\zeta - a_{21}N\left(\phi_a\right)^2\varsigma + a_{33}W\phi_w\varsigma \end{aligned} \tag{A.29}$$

$$A_4 = a_{23}^2\zeta - a_{23}\left(W\phi_w\xi + M\phi_a\varsigma\right) - \phi_w\phi_a\left(L\phi_w\left(N\phi_a\zeta + M\xi\right) + NW\phi_a\varsigma\right) \tag{A.30}$$

$$a_{11} = \left(\tau_w - 1\right)\phi_w\rho_w\omega^2 + \left(\tau_a - 1\right)\phi_a\rho_a\omega^2 - ib_w\omega - ib_a\omega + \rho_s\left(1-\phi\right)\omega^2 \tag{A.31}$$

$$a_{12} = a_{21} = -\left(\tau_w - 1\right)\phi_w\rho_w\omega^2 + ib_w\omega \tag{A.32}$$

$$a_{13} = a_{31} = -\left(\tau_a - 1\right)\phi_a\rho_a\omega^2 + ib_a\omega \tag{A.33}$$

$$a_{22} = \left(\tau_w - 1\right)\phi_w\rho_w\omega^2 - ib_w\omega + \phi_w\rho_w\omega^2 \tag{A.34}$$

$$a_{23} = a_{32} = -\phi_w\phi_a C \tag{A.35}$$

$$a_{33} = \left(\tau_a - 1\right)\phi_a\rho_a\omega^2 - ib_a\omega + \phi_a\rho_a\omega^2 \tag{A.36}$$

The parameters of dispersion equation of shear waves are as below:

$$b_{11} = \left(\tau_w - 1\right)\phi_w \rho_w \omega^2 + \left(\tau_a - 1\right)\phi_a \rho_a \omega^2 - ib_w \omega - ib_a \omega + \rho_s (1-\phi)\omega^2 \tag{A.37}$$

$$b_{12} = b_{21} = -\left(\tau_w - 1\right)\phi_w \rho_w \omega^2 + ib_w \omega \tag{A.38}$$

$$b_{13} = b_{31} = -\left(\tau_a - 1\right)\phi_a \rho_a \omega^2 + ib_a \omega \tag{A.39}$$

$$b_{22} = \left(\tau_w - 1\right)\phi_w \rho_w \omega^2 - ib_w \omega S_w + \omega^2 \phi_w \rho_w \tag{A.40}$$

$$b_{33} = \left(\tau_a - 1\right)\phi_a \rho_a \omega^2 - ib_a \omega + \omega^2 \phi_a \rho_a \tag{A.41}$$

Appendix B

Coefficients of system of equation (35) :

$$H_{1i} = ((\phi_w^u \left(C^u - L^u\right)\Gamma_i^u - \Pi_i^u \phi_a^u (C^u - N))\chi^u - C^u \Gamma_i^u \phi_w^u - N\Pi_i^u \phi_a^u - 2G^u - H^u) d_{pi}^{u2} + k^2((\phi_w^u (C^u - L^u)\Gamma_i^u - \Pi_i^u \phi_a^u (C^u - N))\chi^u - C^u \Gamma_i^u \phi_w^u - N\Pi_i^u \phi_a^u - H^u); \quad (i = 1,2,3) \tag{B.1}$$

$$H_{14} = -2G^u d_s^u k \tag{B.2}$$

$$H_{1(i+4)} = ((-\phi_w (C-L)\Gamma_i + \Pi_i \phi_a (C-N))\chi + C\Gamma_i \phi_w + N\Pi_i \phi_a + 2G + H) d_{pi}^2 - k^2((\phi_w (C-L)\Gamma_i - \Pi_i \phi_a (C-N))\chi - C\Gamma_i \phi_w - N\Pi_i \phi_a - H); \quad (i = 1,2,3) \tag{B.3}$$

$$H_{18} = -2Gd_s k \tag{B.4}$$

$$q_1 = ((-\phi_w^u \left(C^u - L^u\right)\Gamma_1^u + \Pi_1^u \phi_a^u (C^u - N))\chi^u + C^u \Gamma_1^u \phi_w^u + N\Pi_1^u \phi_a^u + 2G^u + H^u) d_{p0}^{u2} - k^2((\phi_w^u (C^u - L^u)\Gamma_1^u - \Pi_1^u \phi_a^u (C^u - N))\chi^u - C^u \Gamma_1^u \phi_w^u - N\Pi_1^u \phi_a^u - H^u); \tag{B.5}$$

$$H_{2i} = -2G^u d_{pi}^u k; \ (i = 1,2,3) \tag{B.6}$$

$$H_{24} = G^u \left(d_s^u - k\right)\left(d_s^u + k\right) \tag{B.7}$$

$$H_{2(i+4)} = -2Gd_{pi} k; \ (i = 1,2,3) \tag{B.8}$$

$$H_{28} = G\left(-d_s + k\right)\left(d_s + k\right) \tag{B.9}$$

$$q_2 = -2G^u d_{p0}^u k \tag{B.10}$$

$$H_{3i} = \omega d_{pi}^u ; (i = 1,2,3) \tag{B.11}$$

$$H_{34} = k\omega \tag{B.12}$$

$$H_{3(i+4)} = -\omega d_{pi}; \ (i = 1,2,3) \tag{B.13}$$

$$H_{38} = -k\omega \tag{B.14}$$

$$q_3 = \omega d_{p0}^u \tag{B.15}$$

$$H_{4i} = \left(1/(\Psi-1)\right)(k((-2G^u d^u_{pi} + T\omega)\Psi + 2G^u d^u_{pi}); \ (i=1,2,3) \quad \text{(B.16)}$$

$$H_{44} = \left(1/(\Psi-1)\right)((k^2 - d^{u2}_s)(\Psi-1)G^u - \Psi T d^u_s \omega) \quad \text{(B.17)}$$

$$H_{4(i+4)} = \left(1/(1-\Psi)\right)(\Psi T k\omega)(i=1,2,3) \quad \text{(B.18)}$$

$$H_{48} = \left(1/(1-\Psi)\right)(\Psi T d_s \omega) \quad \text{(B.19)}$$

$$q_4 = \left(1/(1-\Psi)\right)(2G^u(\Psi-1)d^u_{p0} + \Psi T\omega)k \quad \text{(B.20)}$$

$$H_{5i} = \left(1/(Z-1)\right)(((C^u\Pi^u_i\phi^u_a + \Gamma^u_i L^u\phi^u_w - W^u)(d^{u2}_{pi} + k^2) + \omega(\Gamma^u_i - 1)d^u_{p1})Z$$
$$-(k^2 + d^{u2}_{pi})(C^u\Pi^u_i\phi^u_a + \Gamma^u_i L^u\phi^u_w - W^u)); \ (i=1,2,3) \quad \text{(B.21)}$$

$$H_{54} = \left(1/(Z-1)\right)(Zk\omega(J^u_1 - 1)) \quad \text{(B.22)}$$

$$H_{5(i+4)} = -(d^2_{pi} + k^2)(C\Pi_i\phi_a + \Gamma_i L\phi_w - W); \ (i=1,2,3) \quad \text{(B.23)}$$

$$H_{58} = 0 \quad \text{(B.24)}$$

$$q_5 = \left(1/(Z-1)\right)(((-C^u\Pi^u_1\phi^u_a - \Gamma^u_1 L^u\phi^u_w + W^u)(k^2 + d^u_{p0}) + \omega(\Gamma^u_1 - 1)d^u_{p0})Z + (k^2 + d^{u2}_{p0})(C^u\Pi^u_1\phi^u_a + \Gamma^u_1 L^u\phi^u_w - W^u)) \quad \text{(B.25)}$$

$$H_{6i} = \left(1/(Z-1)\right)(((C^u\Gamma^u_i\phi^u_w + N\Pi^u_i - M)(d^{u2}_{pi} + k^2) + \omega(\Pi^u_i - 1)d^u_{ai})Z$$
$$-(k^2 + d^{u2}_{pi})(C^u\Gamma^u_1\phi^u_w + N\Pi^u_1\phi^u_a - M)); \ (i=1,2,3) \quad \text{(B.26)}$$

$$H_{64} = \left(1/(Z-1)\right)(Zk\omega(J^u_2 - 1)) \quad \text{(B.27)}$$

$$H_{6(i+4)} = -(d^2_{pi} + k^2)(C\Gamma_i\phi_w + N\Pi_i\phi_a - M); \ (i=1,2,3) \quad \text{(B.28)}$$

$$H_{68} = 0 \quad \text{(B.29)}$$

$$q_6 = \left(1/(Z-1)\right)(((-C^u\Gamma^u_1\phi^u_w - N\Pi^u_1\phi^u_a + M)d^{u2}_{p0} + \omega(\Pi^u_1 - 1)d^u_{p0} - k^2(C^u\Gamma^u_1\phi^u_w + N\Pi^u_1\phi^u_a - M))Z$$
$$+(k^2 + d^{u2}_{p0})(C^u\Gamma^u_1 + N\Pi^u_1\phi^u_a - M)) \quad \text{(B.30)}$$

$$H_{7i} = -\omega d^u_{pi}(\Pi^u_i - 1); \ (i=1,2,3) \quad \text{(B.31)}$$

$$H_{74} = -k\omega(J^u_2 - 1) \quad \text{(B.32)}$$

$$H_{7(i+4)} = \omega d_{pi}\left(\Gamma_i - 1\right); \ (i=1,2,3) \quad \text{(B.33)}$$

$$H_{78} = k\omega\left(J_2 - 1\right) \quad \text{(B.34)}$$

$$q_7 = -d^u_{p0}\omega(\Pi^u_1 - 1) \quad \text{(B.35)}$$

$$H_{8i} = -\omega d^u_{pi}(\Gamma^u_i - 1); \ (i=1,2,3) \quad \text{(B.36)}$$

$$H_{84} = -k\omega(J_1^u - 1) \tag{B.37}$$

$$H_{8(i+4)} = \omega d_{pi}(\Gamma_i - 1);\ (i = 1,2,3) \tag{B.38}$$

$$H_{88} = k\omega(J_1 - 1) \tag{B.39}$$

$$q_8 = -d_{p0}^u \omega(\Gamma_1^u - 1) \tag{B.40}$$

Appendix C

Energy flux relations of reflected and refracted waves:

$$\begin{aligned} P_{00}^u &= (k^2 + d_{p0}^{u2})(\Gamma_1^{u2} L^u \phi_w^u + (C^u(\phi_a^u + \phi_w^u)\Pi_1^u + (-\chi^u C^u + L^u(\chi^u - 1))\phi_w^u - W^u)\Gamma_1^u + N\Pi_1^{u2}\phi_a^u \\ &+ (((\chi^u - 1)C^u - \chi^u N)\phi_a^u - M)\Pi_1^u + 2G^u + H^u + M + W^u)\omega d_{p0}^u \end{aligned} \tag{C.1}$$

$$\begin{aligned} P_{01}^u &= -(((\Gamma_1^{u2} L^u \phi_w^u + (C^u(\phi_a^u + \phi_w^u)\Pi_1^u + (-\chi^u C^u + L^u(\chi^u - 1))\phi_w^u - W^u)\Gamma_1^u + N\Pi_1^{u2}\phi_a^u + (((\chi^u - 1)C^u - \chi^u N)\phi_a^u - M)\pi_1^u \\ &+ 2G^u + H^u + M + W^u)d_{p0}^{u2} + k^2\left(\Gamma_1^{u2} L^u \phi_w^u + (C^u(\phi_a^u + \phi_w^u\right)\Pi_1^u + (-\chi C^u + L^u(\chi^u - 1))\phi_w^u - W^u)\Gamma_1^u + N\Pi_1^{u2}\phi_a^u \\ &+ (((\chi^u - 1)C^u - \chi^u N)\phi_a^u - M)\Pi_1^u + H^u + M + W^u))d_{p1}^u - 2G^u k^2 d_{p0}^u)\omega R_1 \end{aligned} \tag{C.2}$$

$$\begin{aligned} P_{0i}^u &= -(((\phi_w^u((\Pi_i^u - \chi^u)C^u + L^u(\chi^u + \Gamma_i^u - 1))\Gamma_1^u + \Pi_1^u((\chi^u + \Gamma_i^u - 1)C^u + N(\Pi_i^u - \chi^u))\phi_a^u - M\Pi_i^u - \Gamma_i^u W^u + 2G^u + H^u + M + W^u)d_{p0}^{u2} \\ &+ (\phi_w^u((\Pi_i^u - \chi^u)C^u + L^u(\chi^u + \Gamma_i^u - 1))\Gamma_1^u + \Pi_1^u((\chi^u + \Gamma_i^u - 1)C^u + N(\Pi_i^u - \chi^u))\phi_a^u - M\Pi_i^u - \Gamma_i^u W^u + H^u + M + W^u)k^2 d_{pi}^u \\ &- 2G^u k^2 d_{p0}^u)\omega R_i;\ (i = 2,3) \end{aligned} \tag{C.3}$$

$$\begin{aligned} P_{04}^u &= -((\phi_w^u((J_2^u - \chi^u)C^u + L^u(J_1^u + \chi^u - 1))\Gamma_1^u + \Pi_1^u((J_1^u + \chi^u - 1)C^u + N(J_2^u - \chi^u))\phi_a^u - J_1^u W^u - J_2^u M + 2G^u + H^u + M + W^u)d_{p0}^{u2} \\ &+ 2G^u d_s^u d_{p0}^u + k^2(\phi_w^u((J_2^u - \chi^u)C^u + L^u(J_1^u + \chi^u - 1))\Gamma_1^u + \Pi_1^u((J_1^u + \chi^u - 1)C^u \\ &+ N(J_2^u - \chi^u))\phi_a^u - J_1^u W^u - J_2^u M + H^u + M + W^u)) \end{aligned} \tag{C.4}$$

$$\begin{aligned} P_{10}^u &= (((\Gamma_1^{u2} L^u \phi_w^u + (C^u(\phi_a^u + \phi_w^u)\Pi_1^u + (-\chi^u C^u + L^u(\chi^u - 1))\phi_w^u - W^u)\Gamma_1^u + N\Pi_1^{u2}\phi_a^u + (((\chi^u - 1)C^u - \chi^u N)\phi_a^u - M)\pi_1^u \\ &+ 2G^u + H^u + M + W^u)d_{p1}^{u2} + k^2\left(\Gamma_1^{u2} L^u \phi_w^u + (C^u(\phi_a^u + \phi_w^u\right)\Pi_1^u + (-\chi C^u + L^u(\chi^u - 1))\phi_w^u - W^u)\Gamma_1^u + N\Pi_1^{u2}\phi_a^u \\ &+ (((\chi^u - 1)C^u - \chi^u N)\phi_a^u - M)\Pi_1^u + H^u + M + W^u))d_{p0}^u - 2G^u k^2 d_{p1}^u)\omega R_1 \end{aligned} \tag{C.5}$$

$$\begin{aligned} P_{11}^u &= -(k^2 + d_{p1}^{u2})(\Gamma_1^{u2} L^u \phi_w^u + (C^u(\phi_a^u + \phi_w^u)\Pi_1^u + (-\chi^u C^u + L^u(\chi^u - 1))\phi_w^u - W^u)\Gamma_1^u + N\Pi_1^{u2}\phi_a^u \\ &+ (((\chi^u - 1)C^u - \chi^u N)\phi_a^u - M)\Pi_1^u + 2G^u + H^u + M + W^u)\omega d_{p1}^u R_1^2 \end{aligned} \tag{C.6}$$

$$\begin{aligned} P_{1i}^u &= (((-((\Pi_i^u - \chi^u)C^u + L^u(\chi^u + \Gamma_i^u - 1))\phi_w^u \Gamma_1^u - ((\chi^u + \Gamma_i^u - 1)C^u + N(\Pi_i^u - \chi^u))\Pi_1^u \phi_a^u \\ &+ M\Pi_i^u + \Gamma_i^u W^u - 2G^u - H^u - M - W^u)d_{p1}^{u2} + k^2(((\Pi_i^u - \chi^u)C^u + L^u(\chi^u + \Gamma_i^u - 1))\phi_w^u \Gamma_1^u \\ &- \Pi_1^u((\chi^u + \Gamma_i^u - 1)C^u + N(\Pi_i^u - \chi^u))\phi_a^u + M\Pi_2^u + \Gamma_2^u W^u - H^u - M - W^u)k^2 d_{pi}^u - 2G^u k^2 d_{p1}^u)\omega R_1 R_i;\ (i = 2,3) \end{aligned} \tag{C.7}$$

$$\begin{aligned} P_{14}^u &= ((-\phi_w^u((J_2^u - \chi^u)C^u + L^u(J_1^u + \chi^u - 1))\Gamma_1^u + \Pi_1^u((J_1^u + \chi^u - 1)C^u \\ &+ N(J_2^u - \chi^u))\phi_a^u + J_1^u W^u + J_2^u M - 2G^u - H^u - M - W^u)d_{\alpha 1}^{u2} + 2G^u d_s^u d_{p1}^u \\ &+ (-\phi_w^u((J_2^u - \chi^u)C^u + L^u(J_1^u + \chi^u - 1))\Gamma_1^u - ((J_1^u + \chi^u - 1)C^u + N(J_2^u - \chi^u))\Pi_1^u \phi_a^u + J_1^u W^u + J_2^u M - H^u - M - W^u))R_1 \mathrm{R}_4 k\omega \end{aligned} \tag{C.8}$$

$$\begin{aligned} P_{20}^{u} &= (((\phi_{w}^{u}((\Pi_{1}^{u}-\chi^{u})C^{u}+L^{u}(\chi^{u}+\Gamma_{1}^{u}-1))\Gamma_{2}^{u} \\ &+\Pi_{2}^{u}((\chi^{u}+\Gamma_{1}^{u}-1)C^{u}+N(\Pi_{1}^{u}-\chi^{u}))\phi_{a}^{u}-M\Pi_{1}^{u}-\Gamma_{1}^{u}W^{u}+2G^{u}+H^{u}+M+W^{u})d_{p2}^{u2} \\ &+\Gamma_{2}^{u}\,\mathrm{k}((\Pi_{1}^{u}-\chi^{u})C^{u}+L^{u}(\chi^{u}+\Gamma_{1}^{u}-1))\phi_{w}^{u}d_{a2}^{u}+k^{2}(((\chi^{u}+\Gamma_{1}^{u}-1)C^{u}+N(\Pi_{1}^{u}-\chi^{u}))\Pi_{2}^{u}\phi_{a}^{u} \\ &-M\Pi_{1}^{u}-\Gamma_{1}^{u}W^{u}+H^{u}+M+W^{u}))d_{p0}^{u}-2G^{u}k^{2}d_{p2}^{u})\omega R_{2} \end{aligned} \tag{C.9}$$

$$\begin{aligned} P_{2i}^{u} &= (((-((\Pi_{i}^{u}-\chi^{u})C^{u}+L^{u}(\chi^{u}+\Gamma_{i}^{u}-1))\phi_{w}^{u}\Gamma_{2}^{u}+\Pi_{2}^{u}((\chi^{u}+\Gamma_{i}^{u}-1)C^{u}+N(\Pi_{i}^{u}-\chi^{u}))\phi_{a}^{u} \\ &+M\Pi_{i}^{u}+\Gamma_{i}^{u}W^{u}-2G^{u}-H^{u}-M-W^{u})d_{p2}^{u2}-\Gamma_{2}^{u}\,\mathrm{k}((\Pi_{i}^{u}-\chi^{u})C^{u}+L^{u}(\chi^{u}+\Gamma_{i}^{u}-1))\phi_{w}^{u}d_{a2}^{u} \\ &+((-(\chi^{u}+\Gamma_{i}^{u}-1)C^{u}+N(\Pi_{i}^{u}-\chi^{u}))\Pi_{i}^{u}\phi_{a}^{u}+M\Pi_{i}^{u}+\Gamma_{i}^{u}W^{u}-H^{u}-M-W^{u})k^{2})d_{pi}^{u} \\ &-2G^{u}k^{2}d_{p2}^{u})\omega R_{2}R_{i};\ \left(i=1,3\right) \end{aligned} \tag{C.10}$$

$$\begin{aligned} P_{22}^{u} &= -d_{a2}^{u}(\Gamma_{2}^{u2}L^{u}\phi_{w}^{u}+(C^{u}(\phi_{a}^{u}+\phi_{w}^{u})\Pi_{2}^{u}+(-\chi^{u}C^{u}+L^{u}(\chi^{u}-1))\phi_{w}^{u}-W^{u})\Gamma_{2}^{u}+N\Pi_{2}^{u2}\phi_{a}^{u} \\ &+(((\chi^{u}-1)C^{u}-\chi^{u}N)\phi_{a}^{u}-M)\Pi_{2}^{u}+2G^{u}+H^{u}+M+W^{u})d_{p2}^{u2} \\ &+\phi_{w}^{u}(\Gamma_{2}^{u}L^{u}+C^{u}\pi_{2}^{u}-C^{u}\chi^{u}+L^{u}(\chi^{u}-1))k\Gamma_{2}^{u}d_{p2}^{u}+((C^{u}\pi_{2}^{u}\phi_{a}^{u}-W^{u})\Gamma_{2}^{u}+N\Pi_{2}^{u2}\phi_{a}^{u} \\ &+(((\chi^{u}-1)C^{u}-N\chi^{u})\phi_{a}^{u}-M)\pi_{2}^{u}+2G^{u}+H^{u}+M+W^{u})k^{2})\omega R_{2}^{2} \end{aligned} \tag{C.11}$$

$$\begin{aligned} P_{24}^{u} &= ((-\phi_{w}^{u}((J_{2}^{u}-\chi^{u})C^{u}+L^{u}(J_{1}^{u}+\chi^{u}-1))\Gamma_{2}^{u}+\Pi_{2}^{u}((J_{1}^{u}+\chi^{u}-1)C^{u}+N(J_{2}^{u}-\chi^{u}))\phi_{a}^{u}+J_{1}^{u}W^{u}+J_{2}^{u}M-2G^{u}-H^{u}-M-W^{u})d_{p2}^{u2} \\ &+(-\phi_{w}^{u}((J_{2}^{u}-\chi^{u})C^{u}+L^{u}(J_{1}^{u}+\chi^{u}-1))\Gamma_{2}^{u}k+2d_{s}^{u}G^{u})d_{a2}^{u} \\ &+(-((J_{1}^{u}+\chi^{u}-1)C^{u}+N(J_{2}^{u}-\chi^{u}))\Pi_{1}^{u}\phi_{a}^{u}+J_{1}^{u}W^{u}+J_{2}^{u}M-H^{u}-M-W^{u})\mathrm{k}^{2})R_{2}R_{4}k\omega \end{aligned} \tag{C.12}$$

$$\begin{aligned} P_{30}^{u} &= (((\phi_{w}^{u}((\Pi_{1}^{u}-\chi^{u})C^{u}+L^{u}(\chi^{u}+\Gamma_{1}^{u}-1))\Gamma_{3}^{u}+\Pi_{3}^{u}((\chi^{u}+\Gamma_{1}^{u}-1)C^{u} \\ &+N(\Pi_{1}^{u}-\chi^{u}))\phi_{a}^{u}-M\Pi_{1}^{u}-\Gamma_{1}^{u}W^{u}+2G^{u}+H^{u}+M+W^{u})d_{p3}^{u2}+(((\Pi_{1}^{u}-\chi^{u})C^{u}+L^{u}(\chi^{u}+\Gamma_{1}^{u}-1))\phi_{w}^{u}\Gamma_{3}^{u} \\ &+((\chi^{u}+\Gamma_{1}^{u}-1)C^{u}+N(\Pi_{1}^{u}-\chi^{u}))\Pi_{3}^{u}\phi_{a}^{u}-M\Pi_{1}^{u}-\Gamma_{1}^{u}W^{u}+H^{u}+M+W^{u})\mathrm{k}^{2})d_{p0}^{u}-2G^{u}k^{2}d_{p3}^{u})\omega R_{3} \end{aligned} \tag{C.13}$$

$$\begin{aligned} P_{3i}^{u} &= (((-((\Pi_{i}^{u}-\chi^{u})C^{u}+L^{u}(\chi^{u}+\Gamma_{i}^{u}-1))\phi_{w}^{u}\Gamma_{3}^{u}-((\chi^{u}+\Gamma_{i}^{u}-1)C^{u} \\ &+N(\Pi_{i}^{u}-\chi^{u}))\Pi_{3}^{u}\phi_{a}^{u}+M\Pi_{i}^{u}+\Gamma_{i}^{u}W^{u}-2G^{u}-H^{u}-M-W^{u})d_{p3}^{u2}+(-((\Pi_{i}^{u}-\chi^{u})C^{u}+L^{u}(\chi^{u}+\Gamma_{i}^{u}-1))\phi_{w}^{u}d_{a3}^{u}-((\chi^{u}+\Gamma_{i}^{u}-1)C^{u}+N(\Pi_{i}^{u}-\chi^{u}))\Pi_{3}^{u}\phi_{a}^{u} \\ &+M\Pi_{i}^{u}+\Gamma_{i}^{u}W^{u}-H^{u}-M-W^{u})k^{2})d_{pi}^{u}-2G^{u}k^{2}d_{p3}^{u})\omega R_{i}R_{3};\ \left(i=1,2\right) \end{aligned} \tag{C.14}$$

$$\begin{aligned} P_{33}^{u} &= -(k^{2}+d_{p3}^{u2})(\Gamma_{3}^{u2}L^{u}\phi_{w}^{u}+(C^{u}(\phi_{a}^{u}+\phi_{w}^{u})\Pi_{3}^{u} \\ &+(-\chi^{u}C^{u}+L^{u}(\chi^{u}-1))\phi_{w}^{u}-W^{u})\Gamma_{3}^{u}+N\Pi_{3}^{u2}\phi_{a}^{u} \\ &+(((\chi^{u}-1)C^{u}-\chi^{u}N)\phi_{a}^{u}-M)\Pi_{3}^{u}+2G^{u}+H^{u}+M+W^{u})\omega d_{p3}^{u}R_{3}^{2}a \end{aligned} \tag{C.15}$$

$$\begin{aligned} P_{34}^{u} &= ((-\phi_{w}^{u}((J_{2}^{u}-\chi^{u})C^{u}+L^{u}(J_{1}^{u}+\chi^{u}-1))\Gamma_{3}^{u} \\ &+\Pi_{3}^{u}((J_{1}^{u}+\chi^{u}-1)C^{u}+N(J_{2}^{u}-\chi^{u}))\phi_{a}^{u}+J_{1}^{u}W^{u}+J_{2}^{u}M-2G^{u}-H^{u}-M-W^{u})d_{p3}^{u2} \\ &+2G^{u}\,\mathrm{d}_{s}^{u}\mathrm{d}_{p3}^{u}+(-\phi_{w}^{u}((J_{2}^{u}-\chi^{u})C^{u}+L^{u}(J_{1}^{u}+\chi^{u}-1))\Gamma_{3}^{u}-((J_{1}^{u}+\chi^{u}-1)C^{u}+N(J_{2}^{u}-\chi^{u}))\Pi_{3}^{u}\phi_{a}^{u}+J_{1}^{u}W^{u}+J_{2}^{u}M-H^{u}-M-W^{u})\mathrm{k}^{2})R_{3}R_{4}k\omega \end{aligned} \tag{C.16}$$

$$P_{40}^{u} = G^{u}k\omega\left(d_{s}^{u2}+2d_{s}^{u}d_{p0}^{u}-k^{2}\right)R_{4} \tag{C.17}$$

$$P_{4i}^{u}=G^{u}k\omega\left(d_{s}^{u2}-2d_{s}^{u}d_{pi}^{u}-k^{2}\right)R_{i}R_{4};\left(i=1,2,3\right) \tag{C.18}$$

$$P_{44}^{u}=-G^{u}d_{s}^{u}\omega\left(d_{s}^{u2}+k^{2}\right)R_{4}^{2} \tag{C.19}$$

$$\begin{aligned}P_{11}&=(k^{2}+d_{p1})(\Gamma_{1}L\phi_{w}+(C(\phi_{a}+\phi_{w})\Pi_{1}+(-\chi C+L(\chi-1))\phi_{w}-W)\Gamma_{1}+N\Pi_{1}\phi_{a}\\&+(((\chi-1)C-\chi N)\phi_{a}-M)\Pi_{1}+2G+H+M+W)\omega d_{p1}R_{5}^{2}\end{aligned} \tag{C.20}$$

$$\begin{aligned}P_{1i}&=(((((\Pi_{i}-\chi)C+L(\chi+\Gamma_{i}-1))\phi_{w}\Gamma_{i}+((\chi+\Gamma_{i}-1)C+N(\Pi_{i}-\chi))\Pi_{1}\phi_{a}-M\Pi_{i}-\Gamma_{i}W+2G+H+M+W)d_{p1}^{2}\\&+(((\Pi_{i}-\chi)C+L(\chi+\Gamma_{i}-1))\phi_{w}\Gamma_{1}+((\chi+\Gamma_{i}-1)C+N(\Pi_{i}-\chi))\Pi_{1}\phi_{a}-M\Pi_{i}-\Gamma_{i}W+H+M+W)k^{2}d_{pi}+2Gk^{2}d_{p1})\omega R_{5}R_{(i+4)};\ (i=2,3)\end{aligned} \tag{C.21}$$

$$\begin{aligned}P_{14}&=-((\phi_{w}((J_{2}-\chi)C+L(\chi+J_{1}-1))\Gamma_{1}+\Pi_{1}((\chi+J_{1}-1)C\\&+N(J_{2}-\chi))\phi_{a}-J_{2}M-J_{1}W+2G+H+M+W)d_{p1}^{2}-2Gd_{p1}d_{s}\\&+k^{2}(\phi_{w}((J_{w}-\chi)C+L(\chi+J_{1}-1))\Gamma_{1}+\Pi_{1}((\chi+J_{1}-1)C\\&+N(J_{2}-\chi))\phi_{a}-J_{2}M-J_{1}W+H+M+W))\omega\,\mathrm{k}\,R_{5}R_{8}\end{aligned} \tag{C.22}$$

$$\begin{aligned}P_{2i}&=(((((\Pi_{i}-\chi)C+L(\chi+\Gamma_{I}-1))\phi_{w}\Gamma_{2}\\&+((\chi+\Gamma_{i}-1)C+N(\Pi_{i}-\chi))\Pi_{2}\phi_{a}-M\Pi_{i}-\Gamma_{i}W+2G+H+M+W)d_{p2}^{2}\\&+(((\Pi_{i}-\chi)C+L(\chi+\Gamma_{i}-1))\phi_{w}\Gamma_{2}\\&+((\chi+\Gamma_{i}-1)C+N(\Pi_{i}-\chi))\Pi_{2}\phi_{a}-M\Pi_{i}-\Gamma_{i}W+H+M+W)k^{2}d_{pi}+2Gk^{2}d_{p2})\omega R_{6}R_{(i+4)};\ (i=1,3)\end{aligned} \tag{C.23}$$

$$\begin{aligned}P_{22}&=(k^{2}+d_{p2}^{2})(\Gamma_{2}^{2}L\phi_{w}+(C(\phi_{a}+\phi_{w})\Pi_{2}+(-\chi C+L(\chi-1))\phi_{w}-W)\Gamma_{2}+N\Pi_{2}^{2}\phi_{a}\\&+(((\chi-1)C-\chi N)\phi_{a}-M)\Pi_{2}+2G+H+M+W)\omega d_{p2}R_{6}^{2}\end{aligned} \tag{C.24}$$

$$\begin{aligned}P_{24}&=-((\phi_{w}((J_{2}-\chi)C+L(\chi+J_{1}-1))\Gamma_{2}+\Pi_{2}((\chi+J_{1}-1)C+N(J_{2}-\chi))\phi_{a}-J_{2}M-J_{1}W+2G+H+M+W)d_{p2}^{2}-2Gd_{p2}d_{s}\\&+k^{2}(\phi_{w}((J_{2}-\chi)C+L(\chi+J_{1}-1))\Gamma_{2}+\Pi_{2}((\chi+J_{1}-1)C+N(J_{2}-\chi))\phi_{a}-J_{2}M-J_{1}W+H+M+W))\omega\,\mathrm{k}\,R_{6}R_{8}\end{aligned} \tag{C.25}$$

$$\begin{aligned}P_{3i}&=(((((\Pi_{i}-\chi)C+L(\chi+\Gamma_{i}-1))\phi_{w}\Gamma_{3}+((\chi+\Gamma_{i}-1)C\\&+N(\Pi_{i}-\chi))\Pi_{3}\phi_{a}-M\Pi_{i}-\Gamma_{i}W+2G+H+M+W)d_{p3}^{2}+(((\Pi_{i}-\chi)C\\&+L(\chi+\Gamma_{i}-1))\phi_{w}\Gamma_{3}+((\chi+\Gamma_{i}-1)C+N(\Pi_{i}-\chi))\Pi_{3}\phi_{a}-M\Pi_{i}-\Gamma_{i}W+H+M+W)k^{2}d_{p1}\\&+2Gk^{2}d_{p3})\omega R_{(i+4)}R_{7};\ (i=1,2)\end{aligned} \tag{C.26}$$

$$\begin{aligned}P_{33}&=(k^{2}+d_{p3}^{2})(\Gamma_{3}^{2}L\phi_{w}+(C(\phi_{a}+\phi_{w})\Pi_{3}+(-\chi C+L(\chi-1))\phi_{w}-W)\Gamma_{3}+N\Pi_{3}^{2}\phi_{a}\\&+(((\chi-1)C-\chi N)\phi_{a}-M)\Pi_{3}+2G+H+M+W)\omega d_{p3}R_{7}^{2}\end{aligned} \tag{C.27}$$

$$\begin{aligned}P_{34}&=-((\phi_{w}((J_{2}-\chi)C+L(\chi+J_{1}-1))\Gamma_{3}+\Pi_{3}((\chi+J_{1}-1)C+N(J_{2}-\chi))\phi_{a}-J_{2}M-J_{1}W+2G+H+M+W)d_{p3}^{2}-2Gd_{p3}d_{s}\\&+k^{2}(\phi_{w}((J_{2}-\chi)C+L(\chi+J_{1}-1))\Gamma_{3}+\Pi_{3}((\chi+J_{1}-1)C+N(J_{2}-\chi))\phi_{a}-J_{2}M-J_{1}W+H+M+W))\omega\,\mathrm{k}\,R_{7}R_{8}\end{aligned} \tag{C.28}$$

$$P_{4i}=-Gk\omega\left(2d_{pi}d_{s}-d_{s}^{2}+k^{2}\right)R_{(i+4)}R_{8};\ \left(i=1,2,3\right) \tag{C.29}$$

$$P_{44}=Gd_{s}\omega\left(d_{s}^{2}+k^{2}\right)R_{8}^{2} \tag{C.30}$$