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Amelioration of Verdegay's Approach for Fuzzy Linear Programs with Stochastic Parameters

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Abstract

This article examines a new approach which solves Linear Programming (LP) problems with stochastic parameters as a generalized model of the fuzzy mathematical model analyzed by Verdegay. An expectation model is provided for solving the problem. A multi-parametric programming is applied to access to a solution with different desired degrees as well as problem constraints. Additionally, we present a numerical example to demonstrate the state and method efficiency.

Keywords

Fuzzy decision making, Verdegay's method, stochastic linear programming, expectation method, multi-parametric programming.

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Introduction

Fuzzy Linear Programming (FLP) programs will appear when a kind of ambiguity is occurred in the parameters of the model and/or in the lack of deterministic information in the classical linear programming (LP) programs. In this case, the mentioned model will be established based on a type of ambiguity instead of the crisp data such as fuzzy inputs and so on. In the literature of fuzzy optimization, there are a lot of researches which are focused on fuzzy linear programming. An interesting approach is using parametric programming methods in the fuzzy linear programming. Afterward, various types of LP problems along with their solving methods have been presented by different authors. Also, this is important to distinguish between flexibility and uncertainty which appears in the parameters of models including the aim coefficients, the coefficients matrix, and its constraints and also in the right-hand-side data. Flexibility concept is modeled using fuzziness and reflects the concept that feasibility of the solutions for every constraint will be valid based on its satisfaction degree. Furthermore, every constraint has an accepted tolerance which is predetermined by the decision maker. Moreover, there is uncertainty that concerns an objective variability in the desired model parameters (random uncertainty) or the lack of some information of the parameter values.

Verdegay (1982) showed that an LP problem with the crisp target and some fuzzy constraints is equal to a common parametric LP model and thus, we are able to use parametric approaches to solve these FLP problems. Cadenas and Verdegay (2000) provided a multi-objective LP problem, where the objective coefficients in the objective function appeared as fuzzy numbers, and also applied fuzzy ordering approach to solve these models. Nasseri and Bavandi (2017) considered a Stochastic Interval-Valued Linear Fractional Programming (SIVLFP) problem, where in their model, the coefficients and scalars of the objective function are fractional intervals, and technological coefficients and the quantities of the constraints in the mentioned model were random variables with the specific distribution. Also, an interactive approach was proposed for Multi-Objective Fuzzy Stochastic Linear Programming (MOFSLP) program by Mohan and Nguyen (2001). After that, Iskander (2003) employed a state of fuzzy weighted objective function to solve a MOFSLP problem. Also after that, other kind of LP with fuzzy random coefficient is studied by some authors (Iskander, 2004a; Iskander, 2004b; Iskander, 2005). Furthermore, Stochastic Linear Programming (SLP) is studied by Ben Abdelaziz and Masri (2005), in which they used fuzzy and/or crisp inequality for every constraint instead of the probability distribution. In fact, they used the α−cut approach for defuzzifing the associated probability distribution. Nasseri et al. (2005) considered a FLP problem and suggested simplex method to solve these programs using linear ranking functions. In the current decade, also some serious studies are investigated in the literature. One of these works is given by Luhandjula (2006). In this work, the author presented a survey of the essential models and methods which are presented on fuzzy stochastic programming area. For Goal Programming with Fuzzy Stochastic parameters (FSGP) problem, where all parameters are considered as kind of fuzzy random variables, Hop (Hop, 2007a; Hop, 2007b; Hop, 2007c) introduced a novel approach for solving these problems. Rommelfanger (2007) studied an LP with multi-criteria crisp, fuzzy or also stochastic values. Recently, Attari and Nasseri (2014) introduced a novel concept for the Fuzzy Mathematical Programming (FMP) which concerns the feasibility of the optimal solutions of these models as an extension of the classical concept of feasibility, also it has appeared in the literature of operations research. They just consider fuzziness in the constraints of the mentioned model while in the many real situations a kind of ambiguity is occurred in the objective coefficients. So, we are going to extend their model to a generalized form, which the objective coefficients are including stochastic parameters. Based on Verdegay's method which is applied for fuzzy linear programming, we suggest an amelioration method to solve FLP programs which includes stochastic parameters in the objective function.

Different parts of this research are prepared in five sections. In

Section 2, we present the essential concepts and results on stochastic programming that will be useful in our methodology. In Section 3, we define two new concepts: $\bar{\alpha}-$feasible and $\bar{\alpha}-$efficient solutions, and then based on the theoretical discussions, we provide a computational method for obtaining a solution for the mentioned problems. In Section 4, an application of the method is described in FLP problems. We finally present some important results in the last section.

Stochastic Programming

The major part of this section regarding the fundamental and necessary concepts and definitions of probability theory is taken from Casella and Berger (2001) and Grimmett and Stirzaker (2001). Hence, to bring the preliminaries, we omit the details of the backgrounds here. In particular, the Stochastic Linear Programming (SLP) problems are concentrated. Furthermore, we introduce a new decision making model according to E-model, which is one of the most efficient models in the sense of the SLP programs.

Assume the following Stochastic Linear Programming (SLP) program:

$$\begin{aligned} &Max \quad z(x) = c^s x \\ &s.t. \quad x \in S = \left\{ x \in \mathbf{R}^n : Ax \geq 0, x \geq 0 \right\}, \end{aligned} \tag{1}$$

where $x = (x_1, \ldots, x_n)$ is a $1 \times n$ vector which is including the decision variables, matrix $A = [a_{ij}]_{m \times n}$ and vector $b^T = (b_1, b_2, \ldots, b_m)$ are respectively a matrix and a vector including the crisp real numbers. Also, $c^s = (c_1^s, c_2^s, \ldots, c_n^s)$ is a row vector of random interval data.

Since the objective function coefficients of Problem (1) are random, so this problem is not well-defined. As a result, we cannot optimize it similar to deterministic cases. In order to deal with such SLP problems, several decision approaches have been appeared in the literature. Here, we focus on the Expectation Optimization (EO) method and then present an extended version of this model which is famous as the expected value model, and using this approach replaces the objective function parameters by its mean value. Now, assume that some or all of the coefficients of $c^s = \left(c_1^s, c_2^s, ..., c_n^s\right)$ in the SLP problem are random, hence, the main target of Problem (1) in the expectation model is represented as:

$$E\left[\sum_{j=1}^{n} c_j^s x_j\right] = \sum_{J=1}^{n} E\left[c_j^s\right] x_j = m_{c^s} x$$

where $m_{c^s} = \left(m_{c_1^s}, ..., m_{c_n^s}\right)$ denotes the mean of random variables c_j^s and $E[.]$ means the expectation. Hence, Problem (1) will be transformed into the following problem:

$$\begin{aligned} & Max \quad E\left[z(x)\right] = E\left[c^s x\right] \\ & s.t. \quad x \in S \end{aligned} \qquad \mathbf{(2)}$$

where $S = \left\{x \in \mathbf{R}^n : Ax \geq b, x \geq 0\right\}$, and $E\left[c^s x\right]$ indicates the mean value of $c^s x$.

Fuzzy Mathematical Programming

In this section, consider FMP model as follows:

$$\begin{aligned} & Max \ f\left(x, c^s\right) \\ & s.t. \ g_i\left(x, a_i\right) \precsim 0, \\ & \qquad x \geq 0, \\ & \qquad i \in I, \end{aligned} \qquad (3)$$

where $I = \{1, ..., m\}$, and $x = \left(x_1, x_2, ..., x_n\right)^T$ is a real vector including the decision variables, and random vector $c^s = \left(c_1^s, c_2^s, ..., c_n^s\right)^T$ is including the objective coefficients. The row vector a_i shows the ith row of $A = \left(a_{ij}\right)$, where A is a real $m \times n -$ dimensional matrix of technical coefficients. And, functions f and g_i where $i \in I$ possess continuous property up to the second derivatives, that is f and $g_i \in C^2, i = 1,...,m$. Also "$\preceq$" denotes a fuzzy extension of "$\leq$" on $\mathbf{R}$ which is used to compare the left and the right side of fuzzy constraints (Dubois & Prade, 1980).

With regards to $g_i(x, a_i) \preceq 0, i \in I$ does not make a crisp feasible region, so, in order to produce a deterministic feasible area, the idea does not provide confidence levels α_i at which it is desirable that the corresponding i-th fuzzy constraint holds. Therefore, in order to obviate those mentioned restrictions, we introduce the following model:

$$\begin{aligned} &Max\ E\left[f\left(x, c^s\right)\right] \\ &s.t.\ \ g_i\left(x, a_i\right) \preceq 0, \\ &\qquad x \geq 0, \\ &\qquad i \in I \end{aligned} \tag{4}$$

where, $E\left[f\left(x, c^s\right)\right]$ shows the mean of $f\left(x, c^s\right)$. Let m_{c^s} denote the mean of c^s, and hence, the objective function can be clearly written as:

$$E\left[f\left(x, c^s\right)\right] = f\left(x, E\left[c^s\right]\right) = f\left(x, m_{c^s}\right),$$

So, Model (2) can be equivalently transformed to:

$$\begin{aligned} &Max\ f\left(x, m_{c^s}\right) \\ &s.t.\ \ g_i\left(x, a_i\right) \preceq 0,\ i \in I, \\ &\qquad x \geq 0, \end{aligned} \tag{5}$$

In order for a significant selection of the membership function for each fuzzy constraint, it refers to, if $g_i\left(x, a_i\right) \leq 0$, thus, this constraint is wholly satisfied, if $g_i\left(x, a_i\right) \geq p_i$, so that the parameter p_i is the predefined maximum tolerance from zero, which is determined by an expert decision maker, therefore, the i-th constraint is certainly violated. Note that, for $g_i\left(x, a_i\right) \in \left(0, p_i\right)$, the membership function is monotonically decreasing. Furthermore, when membership functions of the constraints are considered in the linear form, we have:

$$\mu_i\left(A_i x, b_i\right)=\begin{cases} 1, & g_i\left(x, a_i\right)<0, \\ 1-\dfrac{g_i\left(x, a_i\right)}{p_i}, & 0 \le g_i\left(x, a_i\right) \le p_i,\ i=1,2,3, \\ 0, & g_i\left(x, a_i\right)>p_i \end{cases} \qquad (6)$$

where $i \in I=\{1,\ldots,m\}$.

Now, let us to begin with the concept of feasible solution to the fuzzy programming problem with stochastic parameters of Model (3).

The following definition is prepared for this aim.

Definition 1. Let $\bar{\alpha}=\left(\alpha_1,\ldots,\alpha_m\right)\in(0,1]^m$ be a vector, and

$X_{\bar{\alpha}}=\left\{x \in \mathbf{R}^n \mid x \ge 0, \mu_i\left\{g_i\left(x, a_i\right) \preceq 0\right\} \ge \alpha_i, i \in I=\{1,\ldots,m\}\right\}.$

Then, the vector $x \in X_{\bar{\alpha}}$ is named an $\bar{\alpha}-$feasible solution of Model (3).

Following proposition enables us to define feasible set of Model (3) as an intersection of all $\alpha-$cuts corresponding to fuzzy constraints.

Proposition 1. Let $\bar{\alpha}=\left(\alpha_1,\ldots,\alpha_m\right)\in(0,1]^m$, then $X_{\bar{\alpha}}=\bigcap_{i=1}^{m} X_{\alpha_i}^{i}$, where

$X_{\alpha_i}^{i}=\left\{x \in \mathbf{R}^n \mid x \ge 0, \mu_i\left\{g_i\left(x, a_i\right) \preceq 0\right\} \ge \alpha_i\right\},$

For $i \in I=\{1,\ldots,m\}$ (Namely, X_{α}^{i} is the α-cut of the i-th constraint).

Proof. For $\bar{\alpha}=\left(\alpha_1,\ldots,\alpha_m\right)\in(0,1]^m$, let $x \in X_{\bar{\alpha}}$. Therefore,

$$\mu_i\left\{g_i\left(x, a_i\right) \preceq 0\right\} \ge \alpha$$

and from $X_{\alpha_i}^{i}=\left\{x \in \mathbf{R}^n \mid x \ge 0, \mu_i\left\{g_i\left(x, a_i\right) \preceq 0\right\} \ge \alpha_i\right\}$, we have $x \in X_{\alpha_i}^{i}, i \in I$. Therefore, $x \in \bigcap_{i=1}^{m} X_{\alpha_i}^{i}$. Moreover, if $x \in \bigcap_{i=1}^{m} X_{\alpha_i}^{i}$, we have $x \in X_{\alpha_i}^{i}$, $i \in I$, so $\mu_i\left\{g_i\left(x, a_i\right) \preceq 0\right\} \ge \alpha_i$ and hence, $x \in X_{\bar{\alpha}}$. Therefore, the proof is completed.□

Proposition 2. Let $\bar{\alpha}'=(\alpha_1',...,\alpha_m')$ and $\bar{\alpha}''=(\alpha_1'',...,\alpha_m'')$, where $\alpha_i' \le \alpha_i''$ for all i. Then $\bar{\alpha}''-$feasibility of x implies the $\bar{\alpha}'-$feasibility of it.

Proof. By the use of definition of $\alpha-$cuts and also $\alpha-$feasibility of the solution the proof is straightforward.

For a given $\alpha \in (0,1]$, let a solution $x \in \mathbf{R}^n$ be usual $\alpha-$feasible to Problem (4) (a solution in which has the same satisfaction degree in the mentioned constraints). It means that $\mu_i\{g_i(x,a_i)\preceq 0\} \ge \alpha$, or $x \in X_\alpha^i$, for all $i \in I$ If $\bar{\alpha}=(\alpha,...,\alpha)\in(0,1]^m$, then $x \in X_{\bar{\alpha}}$, which implies that the $\alpha-$feasibility of Problem (3) can be understood as a special case of the $\bar{\alpha}-$feasibility. Therefore, we immediately have the next result.

Remark 1. If Problem (3) is feasible, we clearly conclude that $X_{\bar{\alpha}}$ is not empty.

Definition 2. Let $\preceq$ be a fuzzy extension of common relation $\le$ and also a solution $x=(x_1,...,x_n)^T \in \mathbf{R}^n$ be $\bar{\alpha}-$feasible to Problem (3), where $\bar{\alpha}=(\alpha_1,...,\alpha_m)\in(0,1]^m$ and let $f(x,c^s)$ be a stochastic objective in the form of maximization. Therefore, $x=(x_1,...,x_n)$, where $x_j \in \mathbf{R}^n$ is an $\bar{\alpha}-$efficient solution to Problem (3), if there is no $x' \in X_{\bar{\alpha}}$ so that $E[f(x,c^s)] < E[f(x',c^s)]$.

Similarly, an $\bar{\alpha}-$efficient solution for the form of minimization can be defined.

It is clear that any $\bar{\alpha}-$efficient solution to the mentioned FMP is indeed a $\bar{\alpha}-$feasible solution to the FMP with some additional properties.

Now, we give the following theorem which is concerned to both important conditions (that is, necessary and also sufficient) for an $\bar{\alpha}-$efficient solution to Problem (3).

We will see that this theorem has an important role in the given theoretical discussion in our study.

Theorem 1. Let $\bar{\alpha}=(\alpha_1,...,\alpha_m)\in(0,1]^m$ and also $x^*=(x_1^*,...,x_n^*)$, where $x_j^*\geq 0, j\in J=\{1,2,...,n\}$ be a $\bar{\alpha}-$feasible solution to Problem (3). Then, $x^*\in\mathbf{R}^n$ is an $\bar{\alpha}-$efficient optimal solution to Problem (3), where the objective function is assumed in the type of maximization, if and only if the decision making vector x^* is an optimal solution to the following program:

$$\begin{aligned} &Max\ f(x, m_{c^s}) \\ &s.t.\quad g_i(x,a_i)\leq(1-\alpha_i)\mathrm{p}_i,\ i\in I=\{1,...,m\}, \\ &\qquad x_j\geq 0, \mathrm{j}\in J, \end{aligned} \tag{7}$$

where p_i is the predefined maximum tolerance.

Proof. Let $\bar{\alpha}=(\alpha_1,...,\alpha_m)\in(0,1]^m$ and let $x^*=(x_j^*)_{1\times n}$, such that $x_j^*\geq 0, \mathrm{j}\in\mathrm{J},$ be an $\bar{\alpha}-$efficient solution to Problem (3). Attari and Nasseri (2014) by using Definition 1 and Equation (6) concluded that x^* is feasible to Model (7), because $\mu_i\{g_i(x^*,a_i)\preceq 0\}\geq\alpha_i$ or equivalently $g_i(x^*,a_i)\leq(1-\alpha_i)p_i$ for $i\in I$. Also, according to Definition 2, there is no $x'\in X_{\bar{\alpha}}$ such that $E[f(x,c^s)]<E[f(x',c^s)]$, it means that x^* is optimal to Model (7). Conversely, if x^* is an optimal solution to Model (7), obviously, x^* is an $\bar{\alpha}-$feasible solution to Model (3) and hence, the optimality of x^* implies that the $\bar{\alpha}-$efficiency of x^*.

In Theorem 1, we have discussed a method to fuzzy mathematical programming problems to obtain an $\bar{\alpha}-$efficient solution. If the resulting Problem (7) has only one optimal solution, then we have proved that this solution is an $\bar{\alpha}-$efficient solution to the given problem. In the case of which Problem (7) has some multiple optimal solutions, in order to achieve a maximum efficient solution, that is an $\bar{\alpha}'-$efficient solution with $\alpha'\geq\alpha, i=1,...,m$, we perform the following two-phase approach. Note that the current suggested approach is different from the classical two-phase method which is common for solving linear programming. In fact, in the proposed two-phase approach, Equation (7) can be solved in the first phase, while in the second phase; a solution is obtained which has higher satisfaction degrees than the previous solution. Therefore, we obtain a more comfortable assignment of the available resources by using this approach. Moreover, the achieved solution by this method is also an $\bar{\alpha}-$efficient solution for the mentioned problem.

Let us call Problem (7) as phase 1 problem. Let $\bar{\alpha}^0=\left(\alpha_1^0,...,\alpha_m^0\right)$ and $\left(x^*,E\left(f\left(x^*,c\right)\right)\right)$ be the optimal solution of phase 1 with $\bar{\alpha}^0$ degree of efficiency.

Set $\alpha_i^*=\mu_i\left\{g_i\left(x^*,a_i\right)\preceq 0\right\}\geq\alpha_i^0, i\in I$.

In Phase 2, we solve the following problem

$$
\begin{aligned}
& Max\ \sum_{i=1}^{m}\alpha_i \\
& \qquad f\left(x,m_{c^s}\right)\geq f\left(x^*,m_{c^s}\right), \\
& s.t.\quad g_i\left(x,a_i\right)\leq\left(1-\alpha_i\right)p_i, \\
& \qquad \alpha_i^*\leq\alpha_i\leq 1, \\
& \qquad x_j\geq 0,
\end{aligned}
\tag{8}
$$

where $i\in I$ and $j\in J$.

Let $\left(x^{**},\alpha_1^{**},...,\alpha_m^{**}\right)$ be an optimal solution to Problem (8) (Phase 2). Then, the next important result is at hand. Clearly, this result can help us to understand the valuable relation between Problems (3) and (8).

Theorem 2. The optimal solution x^{**} to Problem (8) is a maximum $\bar{\alpha}-$efficient solution to Problem (3).

Proof. Using Problem (8), Proposition 2 and due to $\alpha_i^*\geq\alpha_i^0$, it results that x^{**} is an α^0-feasible solution to Problem (3) and this shows that it is feasible in Model (5). With optimality of x^* in Model

(5) and moreover, $f\left(x^{**},m_{c^S}\right)\geq f\left(x^{*},m_{c^S}\right)$, we will have optimality of x^{**} in Problem (5) and $f\left(x^{**},m_{c^S}\right)=f\left(x^{*},m_{c^S}\right)$. Thus, x^{**} is an $\bar{\alpha}^0-$efficient solution for Problem (3). Also, due to the optimality $\left(x^{**},\alpha_1^{**},...,\alpha_m^{**}\right)$ and the positivity of the objective function coefficient in Problem (8), we have $\alpha_i^{**}=\mu_i\left\{g_i\left(x^{*},a_i\right)\preceq 0\right\},i\in I$. Now, assume that $\bar{x}^{**}$ is not a maximum $\bar{\alpha}^0-$efficient solution for Problem (3). Thus, there is an α^0-efficient solution $\bar{x}^{**}$ for Problem (3), so that

$\beta_i\geq\alpha_i^{**},i\in I$

and for some k,

$\beta_k\geq\alpha_k^{**}$

where, $\beta_i=\mu_i\left\{g_i\left(x^{*},a_i\right)\preceq 0\right\},i\in I$ and also $f\left(x',m_{c^S}\right)\geq f\left(x^{*},m_{c^S}\right)$.

Thus, $\left(x',\beta_1,...,\beta_m\right)$ feasible to Problem (8) and

$$\sum_{i=1}^{m}\alpha_i^{**}=\sum_{i=1,i\neq k}^{m}\alpha_i^{**}+\alpha_k^{**}<\sum_{i=1,i\neq k}^{m}\beta_i+\beta_k=\sum_{i=1}^{m}\beta_i$$

The proof is completed now.☐

Next section is prepared to describe our suggested approach.

Numerical Discussions

Now, we are at the place that we would like to explore the solving process for the extended model by some illustrative examples.

Example 1: Assume that the following model is given to solve:

$$\begin{aligned}
&Max\ c_j^s\left(\omega_i\right)x_j\\
&s.t.\ \ 5x_1+x_2+3x_3+x_4\preceq 16,\\
&\qquad 7x_1+4x_2+3x_3+x_4\preceq 70,\\
&\qquad 2x_1+4x_2+9x_3+12x_4\preceq 90,\\
&\qquad\quad x_j\geq 0,j=1,...,4,
\end{aligned}\qquad (9)$$

where $c_j^s\left(\omega_i\right),j=1,...,4,\ i=1,2,3,$ is random variable which is defined on some probability spaces $\left(\Omega,F,P\right)$. Table 1 describes the value of random variable coefficients in objective functions.

Table 1. Value of the Random Variable Coefficients

	ω_1	ω_2	ω_3
$c_1^s(\omega)$	3	2	5
$c_2^s(\omega)$	3	6	5
$c_3^s(\omega)$	7	8	9
$c_4^s(\omega)$	9	12	15
P	0.25	0.5	0.25

The expectation model with fuzzy constraints is formulated as:

$$\begin{aligned} &Max\ \ m_{c_1^s(\omega_i)} + m_{c_2^s(\omega_i)} + m_{c_3^s(\omega_i)} + m_{c_4^s(\omega_i)} \\ &s.t.\ \ 5x_1 + x_2 + 3x_3 + x_4 \preceq 16, \\ &\quad\ 7x_1 + 4x_2 + 3x_3 + x_4 \preceq 70, \\ &\quad\ 2x_1 + 4x_2 + 9x_3 + 12x_4 \preceq 90, \\ &\quad\quad x_j \geq 0, j = 1,...,4. \end{aligned} \tag{10}$$

Based on the concept of the expectation, from the given data from Table 1, we have:

$$m_{c_1^s(\omega_i)} = 3,$$

$$m_{c_2^s(\omega_i)} = 5,$$

$$m_{c_3^s(\omega_i)} = 8,$$

$$m_{c_4^s(\omega_i)} = 12,$$

so, the current program can be re-written as:

$$\begin{aligned} &Max\ \ 3x_1 + 5x_2 + 8x_3 + 12x_4 \\ &s.t.\ \ 5x_1 + x_2 + 3x_3 + x_4 \preceq 16, \\ &\quad\ 7x_1 + 4x_2 + 3x_3 + x_4 \preceq 70, \\ &\quad\ 2x_1 + 4x_2 + 9x_3 + 12x_4 \preceq 90, \\ &\quad\quad x_j \geq 0, j = 1,...,4. \end{aligned} \tag{11}$$

with the membership functions also defined in Model (6) as following:

$$\mu_i\left(A_i x, b_i\right)=\begin{cases} 1, & A_i x \in\left(-\infty, b_i\right], \\ \frac{p_i+b_i-A_i x}{p_i} & A_i x \in\left[b_i, b_i+p_i\right],\ i=1,2,3, \\ 0, & A_i x \in\left[b_i+p_i,+\infty\right) \end{cases}$$

such that the elements of $P=\left(p_1, p_2, p_3\right)=(5,40,30)$ is respectively the predefined maximum tolerance from $b_i, i=1,2,3$ (Fiacco, 1983). By Theorem 1, we can rewrite Model (11) as follows:

$$\begin{aligned} & Max\ 3x_1+5x_2+8x_3+12x_4 \\ & s.t.\ \ 5x_1+x_2+3x_3+x_4 \leq 16+5\left(1-\alpha_1\right), \\ & \quad 7x_1+4x_2+3x_3+x_4 \leq 70+40\left(1-\alpha_2\right), \\ & \quad 2x_1+4x_2+9x_3+12x_4 \leq 90+30\left(1-\alpha_3\right), \\ & \qquad 0<\alpha_i \leq 1, i=1,2,3, \\ & \qquad x_1, \ldots, x_4 \geq 0. \end{aligned} \tag{12}$$

Some $\bar{\alpha}-$efficient solutions with satisfaction degrees which decision maker's desire can be found in Table 2.

Table 2. Some Optimal Solutions to Model (11) with Different Satisfactions

a	b	c	d	e	f
$\bar{a}$	(0.5,0.5,0.2)	(0.5,0.5,0.8)	(0.5,0.1,0.5)	(0.5,0.9,0.5)	(0.5,0.5,0.5)
$c^T x$	127.500	111.7500	119.6250	119.6250	119.6250
x_1	0.0	0.0	0.0	0.0	0.0
x_2	13.5000	15.7500	14.6250	14.6250	14.6250
x_3	0.0	0.0	0.0	0.0	0.0
x_4	5.00000	2.75000	3.87500	3.87500	3.87500

If all of the satisfaction degrees are equal, then the $\bar{\alpha}-$feasibility and $\bar{\alpha}-$efficiency reduce to classic $\alpha-$feasibility and $\alpha-$optimality (see Table 1, column f).

Let x^* be an $(0.5,0.9,0.5)-$efficient solution with $c^T x^*=119.6250$ as an optimal objective value (see Table 1, column e). In Phase 2, it is necessary to solve a multi-parametric LP program which is shown below:

$$
\begin{aligned}
&Max \;\; \alpha_1+\alpha_2+\alpha_3\\
&\quad 3x_1+5x_2+8x_3+12x_4 \ge 119.6250\\
&s.t. \;\; 5x_1+x_2+3x_3+x_4 \le 16+5(1-\alpha_1),\\
&\quad 7x_1+4x_2+3x_3+x_4 \le 70+40(1-\alpha_2),\\
&\quad 2x_1+4x_2+9x_3+12x_4 \le 90+30(1-\alpha_3),\\
&\qquad 0.5<\alpha_1\le 1,\\
&\qquad 0.9<\alpha_2\le 1,\\
&\qquad 0.5<\alpha_3\le 1,\\
&\quad x_1,\ldots,x_4 \ge 0.
\end{aligned}
\tag{13}
$$

By solving the above multi-parametric programs, an optimal solution will be achieved as:

$x^{**}=(0.0,14.6250,0.0,3.87500)$

Also, $c^T x^{**}=c^T x^{*}=119.6250$. We have

$\mu_1(A_1x^{**},b_1)=\mu_3(A_3x^{**},b_3)=0.5$

and

$\mu_2(A_2x^{**},b_2)=1$

Thus, solving the model by the mentioned approach, we can achieve a new optimal solution for Problem (9), which not only it has the optimal value for the given objective, but also it gives us a higher membership value in μ_2.

Example 2: A manufacturing company produces three products in three processes. The machining time of each product in each process is given in Table 3. The maximum available time of the processes per week for Process I is approximately 600 minutes, for Process II approximately 400 minutes, and for Process III approximately 200 minutes. The profit of each unit of product A, B, and C is respectively c_1^s, c_2^s and c_3^s, which are random variables with the expectation 30, 12, and 11.

Table 3. The Machining Time

Process	Product A	Product B	Product C
I	9	3	5
II	5	4	7
III	3	2	4

To maximize profits, the number of products produced per week is obtained. Suppose that the amounts of products A, B. and C will be respectively produced by x_1, x_2 and x_3 variables. Then, by using the above values, the mentioned program is formulated as:

$$\begin{aligned} Max\ & 30x_1+12x_2+11x_3 \\ s.t.\ & 9x_1+3x_2+5x_3 \preceq 600, \\ & 5x_1+4x_2+7x_3 \preceq 400, \\ & 3x_1+2x_2+4x_3 \preceq 200, \\ & x_1, x_2, x_3 \geq 0. \end{aligned} \tag{14}$$

Suppose that predefined maximum tolerance from $b_i, i=1,2,3$ are determined by manager of the company as $p_1=60, p_2=40$, and $p_3=20$, respectively. Now, by using the membership function which is defined in Model (6), we can rewrite Problem (14) as follows:

$$\begin{aligned} Max\ & 30x_1+12x_2+11x_3 \\ s.t.\ & 9x_1+3x_2+5x_3 \leq 600+60(1-\alpha_1), \\ & 5x_1+4x_2+7x_3 \leq 400+40(1-\alpha_2), \\ & 3x_1+2x_2+4x_3 \leq 200+20(1-\alpha_3), \\ & 0 \leq \alpha_i \leq 1,\ i=1,2,3, \\ & x_1, x_2, x_3 \geq 0. \end{aligned} \tag{15}$$

So, for $\alpha_i=0.5,\ i=1,2,3$, this problem can be solved by using software of LINGO 14.0 and an optimal solution is obtained as follows:

$x^*=(70,0,0)$

with the optimal objective value

$c^T x^* = 2100$

Now, in order to obtain a maximum $\bar{\alpha}-$efficient solution for the above mentioned problem, it is necessary to solve the following multi-parametric LP problem,

$$
\begin{aligned}
Max\ \ & \alpha_1+\alpha_2+\alpha_3 \\
& 30x_1+12x_2+11x_3 \geq 1200, \\
s.t.\ \ & 9x_1+3x_2+5x_3 \leq 600+60(1-\alpha_1), \\
& 5x_1+4x_2+7x_3 \leq 400+40(1-\alpha_2), \qquad (16) \\
& 3x_1+2x_2+4x_3 \leq 200+20(1-\alpha_3), \\
& \quad 0.5 \leq \alpha_1 \leq 1, \\
& \quad 0.5 \leq \alpha_2 \leq 1, \\
& \quad 0.5 \leq \alpha_3 \leq 1, \\
& \quad x_j \geq 0,\ j=1,2,3.
\end{aligned}
$$

By solving the above parametric linear model, we achieve the optimal solution below:

$x^{**} = (70, 0, 0)$

Also, $c^T x^* = c^T x^{**} = 2100$. While we have:

$\mu_1\left(A_1 x^{**}, b_1\right) = 0.5, \quad \mu_2\left(A_2 x^{**}, b_2\right) = 1, \quad \mu_3\left(A_3 x^{**}, b_3\right) = 0.5.$

The above results conclude that the best value for the satisfaction degree of the obtained optimal solution is achieved. We see that in the second resource, we may need to reduce the tolerance to 400.

Conclusions

In this research, we first extended the common FLP to the stochastic environment. In particular, we introduced the novel approach to solve linear programming models with stochastic parameters and flexible constraints. Also, we presented the concepts of $\bar{\alpha}-$feasibility and $\bar{\alpha}-$efficiency, where $\bar{\alpha}$ is a vector of satisfaction degrees which are determined by the decision maker. These concepts help us to obtain more flexible solutions to FMP problems. In addition, in the case of linear problems, we have proved that the desired solution can be achieved by solving a corresponding multi-parametric LP.

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