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RESEARCH PAPER

Solving Generalized DEA/AR Model With Fuzzy Data and Its Application to Evaluate the Performance of Manufacturing Enterprises

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Abstract
The use of conventional data envelopment analysis (DEA) models in real-world problems are limited because of some restrictions that must be considered in the model such as imprecise or vague data in inputs and outputs as well as additional information or assumptions. One way to handle this problem is by using fuzzy DEA with assurance regions (FDEA/AR) models. There is a common approach in almost all the suggested methods for solving FDEA/AR models. However, in this paper, we show that in some DEA/AR models, applying this approach can be led to inappropriate results. Four theorems are given to provide some sufficient conditions for a DMU to be the DEA/AR efficient. These theorems can be used to check the accuracy of the presented methods for solving FDEA/AR models, too. Moreover, a new method for solving a generalized FDEA/AR model that includes established DEA models such as CCR model (Charnes et al., 1978), BCC model (Banker et al., 1984), FG model (Färe & Grosskopf, 1985), and ST model (Seiford & Thrall, 1990) is proposed. These models are constant, variable, non-decreasing, and non-increasing returns to scale models, respectively. The proposed method is applied to evaluate the performance of manufacturing enterprises.

Keywords: Data envelopment analysis, Fuzzy DEA, Assurance regions, Fuzzy numbers, Efficiency.

Introduction

Data envelopment analysis (DEA) is a famous and effective method for evaluating the relative efficiency of homogeneous decision-making units (DMUs) that use multiple inputs to produce multiple outputs. DEA was first introduced by Charnes et al. in 1978. Since 1978, DEA has had continuous growth and extensive applications. The most popular DEA models are the CCR model, the BCC model, the FG model, and the ST model. These models were introduced by Charnes et al. (1978), Banker et al. (1984), Färe and Grosskopf (1985), and Seiford and Thrall (1990), and they are constant, variable, non-decreasing, and non-increasing returns to scale models, respectively. Yu et al. (1996a, 1996b) proposed a generalized model that included the popular DEA models mentioned above. However, in the presence of additional information or assumptions in a DEA problem, we have to restrict some weights. Therefore, we face a DEA problem with assurance regions.

A DEA problem with assurance regions (DEA/AR) was first introduced by Thompson et al. (1986) and made DEA more applicable. In conventional DEA/AR problems, inputs and outputs must be measured by exact values, but in the face of imprecise or vague data,

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traditional DEA/AR models are not applicable. For example, to provide more flexibility in DEA/AR models, Cooper et al. (2001) used imprecise data to a DEA/AR model and then, they applied it to evaluate the efficiencies of the Korean mMobile Telecommunication Company's branch offices. An appropriate method to model imprecise or vague data is to use fuzzy numbers that convert conventional DEA/AR models to fuzzy DEA/AR (FDEA/AR) models.

Since the time the first study was done by Sengupta (1992), many papers have been published about FDEA. For example, Saati et al. (2002) proposed an advantageous method to evaluate the efficiency of DMUs in the fuzzy CCR (FCCR) model based on α-cuts. Lertworasirikul et al. (2003) developed the possibility and credibility approaches for solving fuzzy BCC (FBCC) models. To apply qualitative factors in the input and output data, Liao et al. (2007) developed a new FDEA model using cloud theory. In their model, for different amounts of α, fuzzy data was converted to interval data and then, two conventional DEA models were employed to evaluate the upper and lower bounds of DMUs' efficiencies. Using fuzzy arithmetic, Wang et al. (2009) introduced different models to evaluate the lower bounds, middle values, and upper bounds of the DMU's efficiencies in an FDEA model. Then, by a preference degree approach, they combined the obtained efficiency measures and proposed a complete ranking of DMUs. In addition, the proposed method was applied to evaluate the performance of eight manufacturing enterprises in China. Later, Amirteimoori et al. (2020) applied Wang et al. (2009) approach for two-stage FDEA models and obtained the optimistic and pessimistic measures of both the overall and stage-based efficiencies of DMUs. Wen et al. (2011) proposed some sensitivity and stability analysis on the FDEA models. Emrouznejad et al. (2011) proposed two methods for computing the Malmquist productivity index using DEA models whose data and price vectors are fuzzy or vary in intervals. Moreover, Emrouznejad et al. (2014) presented a taxonomy and review of the published papers in this field from 1999 to 2013. Using a transformation function, Foroughi and Shureshjani (2017) converted a generalized FDEA model into a generalized parametric DEA model that was dependent on the α-cuts. By applying the lower and upper bounds of fuzzy input and output data, Hatami-Marbini et al. (2017a) proposed a stepwise FDEA method to characterize weakly efficient frontiers. Their method guaranteed a feasible solution for the FDEA problem. In another work, Hatami-Marbini et al. (2017b) characterized both data and variables by trapezoidal fuzzy numbers and suggested a lexicographic multi-objective linear programming method to solve the proposed fully fuzzy DEA (FFDEA) models in constant and variable returns to scale conditions. Furthermore, a super efficiency FFDEA model was introduced to compare the fuzzy efficient DMUs. Similar to Hatami-Marbini et al. (2017b), Namakin et al. (2018) considered fuzziness in all inputs, outputs, and parameters of a DEA model. They proposed a new method to solve a fully fuzzy DEA model with Z-numbers. Inspired by the TOPSIS (Technique for Order of Preference by Similarity to Ideal Solution) technique, Hu et al. (2017) proposed a method for evaluating the efficiency of DMUs in an FDEA model with common weights. In this method, first two appropriate models were used to obtain the positive and negative ideal solutions. Then, using two distance functions, the initial FDEA model was converted to a fuzzy bi-objective nonlinear programming problem. In addition, supposing the linearity of membership functions, their models became linear and simple to use. In a case study, Wanke et al. (2018) compared the obtained efficiency results of Angolan banks from different stochastic DEA and fuzzy DEA models. They concluded that although the obtained efficiency scores from different models are roughly similar, the ranking of DMUs may be substantially different. Hatami-Marbini (2019) considered trapezoidal fuzzy numbers as inputs, intermediate products, and outputs in a two-stage DEA model. Then, by standard arithmetic operators, Hatami-Marbini (2019) proposed four different DEA models to evaluate the efficiency of DMUs. These four obtained efficiency results were then used to obtain the fuzzy measure of the DMUs' scale efficiencies. Besides, the proposed method was applied to compare and rank 39 Spanish airports. By considering the optimistic, pessimistic, or compromise tendency of the decision-maker, Peykani et al. (2019) first proposed a general fuzzy measure on the possibility of space. Then, they applied the proposed measure to an FDEA model with trapezoidal fuzzy input and output data. In this case, the FDEA model was converted to a parametric DEA model with optimistic-pessimistic parameters. As an example, they applied the proposed model to evaluate the efficiency of 38 hospitals in the East Virginia Department of Health and Human Services in the USA.

Despite many works in the FDEA field, few methods are offered for solving FDEA/AR models. First, Liu (2008) entered fuzzy numbers into DEA/AR models and based on Zadeh's extension principle (Zadeh, 1978), he transformed an FDEA/AR model into a family of crisp DEA/AR models to offer the lowest and the highest bounds for the efficiency of DMUs at different α-levels. Then by applying Chen and Klein's (1997) index, Liu (2008) obtained a crisp number and ranked DMUs. In a comment, Jahanshahloo et al. (2009) made some corrections to the theorem-proof presented by Liu (2008) as well as their proposed model. Again, Zhou et al. (2010) provided some corrections and proposed a different proof for Jahanshahloo et al.'s (2009) theorem. Liu and Chuang (2009) used the FDEA/AR model proposed by Liu (2008) to evaluate the performance of 24 university libraries in Taiwan. Later, in a note, Zhou et al. (2012a) corrected the models and proof of a proposition in Liu and Chuang (2009) and proposed a generalized FDEA/AR model. In addition, Zhou et al. (2012b) proposed a generalized FDEA/AR model and developed the Liu (2008) method to solve well-known DEA/AR models such as CCR, BCC, ST, and FG models with fuzzy inputs and outputs and assurance regions. Zhou et al. (2012b) applied Chen and Wang's (2009) index to rank DMUs. The proposed method was applied to evaluate the efficiency of ten manufacturing enterprises. As a real-world application of the FDEA/AR models, Hongmei et al. (2015) applied an FDEA/AR model to assess the seismic behavior of reservoir dams. Similar to Liu's (2008) method, they first calculated the lowest and highest bounds of the damaged reservoir dams' efficiencies by two appropriate models for different α-levels and then, using a ranking index, they compared 19 damaged reservoir dams in Luojiang County, Sichuan Province.

In this paper, four sufficient conditions for BCC/AR, FG/AR, or ST/AR efficiency of a DMU are provided. Then, a new method for solving a generalized FDEA/AR model is proposed. The proposed method converts an FDEA/AR model to a parametric DEA/AR model that is dependent on α-levels. In this method, the decision maker's opinion contributes to the decision process by the selection of α. Furthermore, the proposed method is illustrated in an example and the obtained results are compared with two other methods in this field.

The reminder of the paper is organized as follows. In Section 2, generalized DEA/AR and generalized FDEA/AR models are briefly described. Liu's (2008) method is the most famous approach to solving FDEA/AR models. In Section 3, two proposed models by Liu (2008) for evaluating lower and upper bounds of DMUs' efficiencies in an FDEA model are introduced. In Section 4, four theorems are proved to propose sufficient conditions for the BCC/AR, ST/AR, or FG/AR efficiency of a DMU. Moreover, a new approach for solving the generalized FDEA/AR model is proposed in Section 5. In Section 6, our method is applied to evaluate the performance of manufacturing enterprises. Furthermore, the obtained results from our method are compared with the obtained results from Liu's (2008) and Zhou's (2012b) methods. Finally, some conclusions are drawn in Section 7.

Generalized DEA/AR and FDEA/AR Models

Let x_{ij} $(i=1,...,m)$ and y_{rj} $(r=1,...,s)$ be the positive inputs and outputs of DMU_j $(j=1,...,n)$, then the generalized DEA model (Yu et al., 1996a; Yu et al., 1996b) is formulated as follows.

$$\begin{aligned} Z &= \max \sum_{r=1}^{s} u_r y_{ro} - \delta_1 u_0 \\ s.t.\ & \sum_{i=1}^{m} v_i x_{io} = 1 \\ & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} - \delta_1 u_0 \le 0, \quad j=1,...,n \\ & \delta_1 \delta_2 (-1)^{\delta_3} u_0 \ge 0, \quad v_i, u_r \ge 0, \quad i=1,...,m \ , r=1,...,s \end{aligned} \tag{1}$$

where, v_i and u_r are the weights given to input *i* and output *r*, respectively. In addition, $\delta_1, \delta_2, \delta_3$ are binary parameters, and we have:

(i) If δ_1=0, then the generalized DEA model is reduced to the CCR model.

(ii) If δ_1= 1 and δ_2= 0, then the generalized DEA model is reduced to the BCC model.

(iii) If $\delta_1 = \delta_2 = 1$ and δ_3 = 0, then the generalized DEA model is reduced to the FG model.

(iv) If $\delta_1 = \delta_2 = \delta_3 = 1$, then the generalized DEA model is reduced to the ST model.

However, due to additional information or assumptions that should be considered in real-world problems, some of the weights in a DEA problem might need to be restricted. Hence, Zhou et al. (2012b) developed Model (1) into Model (2), considering assurance regions as follows.

$$\begin{aligned} Z &= \max \sum_{r=1}^{s} u_r y_{ro} - \delta_1 u_0 \\ s.t.\ & \sum_{i=1}^{m} v_i x_{io} = 1 \\ & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} - \delta_1 u_0 \le 0, \quad j=1,...,n \\ & C_{pq}^{L} v_p \le v_q \le C_{pq}^{U} v_p, \quad 1 \le p < q = 2,...,m \\ & D_{pq}^{L} u_p \le u_q \le D_{pq}^{U} u_p, \quad 1 \le p < q = 2,...,s \\ & \delta_1 \delta_2 (-1)^{\delta_3} u_0 \ge 0, \quad v_i, u_r \ge 0, \quad i=1,...,m \ , r=1,...,s \end{aligned} \tag{2}$$

Here, we assume that $0 \le C_{pq}^{L} < C_{pq}^{U}$ and $0 \le D_{pq}^{L} < D_{pq}^{U}$.

In the model (2), we should provide many pairwise comparisons to determine the lower and upper bounds of the ratio between two arbitrary input weights or the ratio between two arbitrary output weights. However, pairwise comparisons between all the input weights or output weights are a very significant challenge and can lead to inconsistency. One of the appropriate ways to handle this problem is to determine one input as an “input numeraire” and one output as an “output numeraire” and then, compare the weights of all inputs and outputs with the weight of their numeraires. In this case, it is easier for the decision-maker to express his/her judgment; fewer comparisons are needed and so, less inconsistency will occur. This technique is common to reduce comparisons and to provide more accurate and consistent models. For example, Rezaei (2015) applied this approach and proposed the best-worst method (BWM) that is an MCDM method for evaluating multiple alternatives with multiple decision criteria. He proved that in comparison with the analytic hierarchy process (AHP) method, the BWM method is more consistent and needs fewer comparisons.

Here, we select one of the inputs, say x_1, as an "input numeraire" and one of the outputs, say y_1, as an "output numeraire" (Cooper et al., 2007; Thompson et al., 1986; Thompson et al., 1990). In this case, the assurance regions are simplified as:

$$C_{1q}^{L}v_1 \le v_q \le C_{1q}^{U}v_1,\quad q=2,\dots,m \quad \text{and} \qquad D_{1q}^{L}u_1 \le u_q \le D_{1q}^{U}u_1,\quad q=2,\dots,s \tag{3}$$

So, model (2) will be converted to the following DEA/AR model:

$$\begin{aligned}
Z &= \max \sum_{r=1}^{s} u_r y_{ro} - \delta_1 u_0 \\
s.t.\quad & \sum_{i=1}^{m} v_i x_{io} = 1 \\
& \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} - \delta_1 u_0 \le 0,\quad j=1,\dots,n \\
& C_{1q}^{L}v_1 \le v_q \le C_{1q}^{U}v_1,\quad q=2,\dots,m \\
& D_{1q}^{L}u_1 \le u_q \le D_{1q}^{U}u_1,\quad q=2,\dots,s \\
& \delta_1\delta_2(-1)^{\delta_3}u_0 \ge 0,\ v_i,u_r \ge 0,\ i=1,\dots,m,\ r=1,\dots,s
\end{aligned} \tag{4}$$

And by considering fuzzy input ($\tilde{x}_{ij}$) and fuzzy output ($\tilde{y}_{rj}$) data, the generalized DEA/AR model (model 4) can be converted to the following generalized FDEA/AR model. The difference between the following model and the generalized FDEA/AR model proposed by Zhou et al. (2012b) is that in the following model, the first inputs and outputs are considered as "numeraires."

$$\begin{aligned}
\tilde{Z} &= \max \sum_{r=1}^{s} u_r \tilde{y}_{ro} - \delta_1 u_0 \\
s.t.\quad & \sum_{i=1}^{m} v_i \tilde{x}_{io} = 1 \\
& \sum_{r=1}^{s} u_r \tilde{y}_{rj} - \sum_{i=1}^{m} v_i \tilde{x}_{ij} - \delta_1 u_0 \le 0,\quad j=1,\dots,n \\
& C_{1q}^{L}v_1 \le v_q \le C_{1q}^{U}v_1,\quad q=2,\dots,m \\
& D_{1q}^{L}u_1 \le u_q \le D_{1q}^{U}u_1,\quad q=2,\dots,s \\
& \delta_1\delta_2(-1)^{\delta_3}u_0 \ge 0,\quad v_i,u_r \ge 0,\quad i=1,\dots,m\ ,r=1,\dots,s
\end{aligned} \tag{5}$$

Lower and Upper Bounds of DMUs' Efficiencies in an FDEA/AR Model

Calculating the lower and upper bounds of DMUs' efficiencies is one of the most famous approaches for solving FDEA models in the literature. In addition, as mentioned in the introduction, most of the proposed methods for solving FDEA/AR models are inspired by the work of Liu (2008). Using two appropriate models, Liu (2008) calculated the lower and upper bounds of a DMU's fuzzy efficiency for different amounts of α-cuts. Here we briefly introduce the Liu (2008) models.

Let $\tilde{x}_{ij}$ and $\tilde{y}_{rj}$ be the fuzzy amounts of *i*th input and *r*th output of the *j*th DMU, respectively. Also, suppose $S(\tilde{x}_{ij})$ and $S(\tilde{y}_{rj})$ indicate the support of $\tilde{x}_{ij}$ and $\tilde{y}_{rj}$. The α-cuts of $\tilde{x}_{ij}$ and $\tilde{y}_{rj}$ are defined as:

$$(x_{ij})_\alpha = \{x_{ij} \in S(\tilde{x}_{ij}) | \mu_{\tilde{x}_{ij}}(x_{ij}) \geq \alpha\} = \left[\min\{x_{ij} \in S(\tilde{x}_{ij}) | \mu_{\tilde{x}_{ij}}(x_{ij}) \geq \alpha\}, \max\{x_{ij} \in S(\tilde{x}_{ij}) | \mu_{\tilde{x}_{ij}}(x_{ij}) \geq \alpha\}\right], \ \forall i, j \quad (6)$$

$$(y_{rj})_\alpha = \{y_{rj} \in S(\tilde{y}_{rj}) | \mu_{\tilde{y}_{rj}}(y_{ik}) \geq \alpha\} = \left[\min\{y_{rj} \in S(\tilde{y}_{rj}) | \mu_{\tilde{y}_{rj}}(y_{rj}) \geq \alpha\}, \max\{y_{rj} \in S(\tilde{y}_{rj}) | \mu_{\tilde{y}_{rj}}(y_{rj}) \geq \alpha\}\right], \ \forall r, j \quad (7)$$

, that can be represented as:

$$(x_{ij})_\alpha = [(x_{ij})^L_\alpha, (x_{ij})^U_\alpha], \quad \text{and} \quad (y_{rj})_\alpha = [(y_{rj})^L_\alpha, (y_{rj})^U_\alpha] \quad (8)$$

For an arbitrary amount of α-cut, the lower and upper bounds of efficiencies for each DMU under evaluation are calculated as follows, respectively (Liu, 2008).

$$\begin{aligned} (Z_o)^L_\alpha = \max & \sum_{r=1}^{s} u_r (y_{ro})^L_\alpha \\ \text{s.t.} & \sum_{i=1}^{m} v_i (x_{io})^U_\alpha = 1, \\ & \sum_{r=1}^{s} u_r (y_{rj})^U_\alpha - \sum_{i=1}^{m} v_i (x_{ij})^L_\alpha \leq 0, \ j = 1, \ldots, n \\ & C^L_{pq} v_p \leq v_q \leq C^U_{pq} v_p, \ 1 \leq p < q = 2, \ldots, m \\ & D^L_{pq} u_p \leq u_q \leq D^U_{pq} u_p, \ 1 \leq p < q = 2, \ldots, s \\ & v_i, u_r \geq \varepsilon > 0, \ i = 1, \ldots, m, \ r = 1, \ldots, s \end{aligned} \quad (9)$$

, and

$$\begin{aligned} (Z_o)^U_\alpha = \max & \sum_{r=1}^{s} u_r (y_{ro})^U_\alpha \\ \text{s.t.} & \sum_{i=1}^{m} v_i (x_{io})^L_\alpha = 1, \\ & \sum_{r=1}^{s} u_r (y_{rj})^L_\alpha - \sum_{i=1}^{m} v_i (x_{ij})^U_\alpha \leq 0, \ j = 1, \ldots, n \\ & C^L_{pq} v_p \leq v_q \leq C^U_{pq} v_p, \ 1 \leq p < q = 2, \ldots, m \\ & D^L_{pq} u_p \leq u_q \leq D^U_{pq} u_p, \ 1 \leq p < q = 2, \ldots, s \\ & v_i, u_r \geq \varepsilon > 0, \ i = 1, \ldots, m, \ r = 1, \ldots, s \end{aligned} \quad (10)$$

From the two above-mentioned models, it can be seen that to obtain the lower bound of the efficiency of the DMU under evaluation, the lower bounds of its outputs and the upper bounds of its inputs are considered, whereas for the other DMUs, the upper bounds of their outputs and the lower bounds of their inputs are applied. Moreover, to calculate the efficiency of the upper bound of the DMU under evaluation, the upper bounds of its outputs and the lower bounds of its inputs are considered, whereas for the other DMUs, the lower bounds of outputs and the upper bounds of inputs are used. Therefore, it is clear that for each amount of α, we obtain a wide interval as the efficiency of the DMU under evaluation, and as a result, it will be difficult to compare and rank DMUs. To solve this problem, Liu (2008) applied the following index, which was first proposed by Chen and Klein (1997) for ranking fuzzy numbers, to combine the obtained efficiency intervals and rank DMUs.

$$I(Z_o) = \sum_{i=0}^{n} \left((Z_o)^U_{\alpha_i} - c\right) \Bigg/ \left[\sum_{i=0}^{n} \left((Z_o)^U_{\alpha_i} - c\right) - \sum_{i=0}^{n} \left((Z_o)^L_{\alpha_i} - d\right)\right], \quad n \to \infty \quad (11)$$

, where $c = \min_{i,o} \{(Z_o)^L_{\alpha_i}\}$ and $d = \max_{i,o} \{(Z_o)^U_{\alpha_i}\}$

Although most of the proposed methods for solving FDEA/AR models are inspired by Liu's (2008) method, in the following sections, it will be shown that using this method can lead to inappropriate results in some FDEA/AR models.

Four Sufficient Conditions for BCC/AR, FG/AR, or ST/AR Efficiency of a DMU

Cooper et al. (2007) proved that a DMU is BCC efficient if it has a minimum input value for any input item or a maximum output value for any output item. This theorem may not be applied to BCC models with assurance regions (BCC/AR models). However, considering model 4 as a DEA/AR model, we can prove the following theorems. These theorems provide some sufficient conditions for BCC/AR, FG/AR, and ST/AR efficiency of a DMU, respectively.

Theorem 1 A DMU_o with $y_o \geq y_j, y_o \neq y_j,\ j=1,...,n,\ j \neq o$, is BCC/AR efficient.

Proof. We can write the BCC/AR model (model 4) corresponding to DMU_o as follows:

$$\begin{aligned} &\max uy_o - u_0 \\ &s.t.\ vx_o = 1 \\ &\quad -vX + uY - u_0 \leq 0 \\ &\quad vP \leq 0,\ uQ \leq 0 \\ &\quad u, v \geq 0,\ u_0 \textit{ is free} \end{aligned}$$

Where,

$$P = \begin{pmatrix} C_{12}^L & -C_{12}^U & C_{13}^L & -C_{13}^U & ... & C_{1m}^L & -C_{1m}^U \\ -1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 1 & \cdots & 0 & 0 \\ & \vdots & & & \ddots & \vdots & \\ 0 & 0 & 0 & 0 & \cdots & -1 & 1 \end{pmatrix},$$

and

$$Q = \begin{pmatrix} D_{12}^L & -D_{12}^U & D_{13}^L & -D_{13}^U & ... & D_{1s}^L & -D_{1s}^U \\ -1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 1 & \cdots & 0 & 0 \\ & \vdots & & & \ddots & \vdots & \\ 0 & 0 & 0 & 0 & \cdots & -1 & 1 \end{pmatrix},$$

In addition, as it was assumed, we have $0 \leq C_{1i}^L < C_{1i}^U, i=2,...,m$ and $0 \leq D_{1j}^L < D_{1j}^U, j=2,...,s$.

The dual of this model can be formulated as:

$$\begin{aligned} &\min \theta \\ &s.t.\ \theta x_o - X\lambda + P\pi \geq 0 \\ &\quad Y\lambda + Q\tau \geq y_o \quad (*) \\ &\quad e\lambda = 1 \\ &\quad \lambda, \pi, \tau \geq 0,\ \theta \textit{ is free} \end{aligned}$$

Let $(\lambda, \pi, \tau, \theta)$ be an optimal solution for this problem. From $y_o \geq y_j, y_o \neq y_j, j=1,...,n, j \neq o, \lambda \geq 0$ and $e\lambda = 1$, we have $y_o \geq Y\lambda$. Hence, $(*)$ implies:

$$Q\tau=\begin{pmatrix} D_{12}^{L} & -D_{12}^{U} & D_{13}^{L} & -D_{13}^{U} & \dots & D_{1s}^{L} & -D_{1s}^{U} \\ -1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 1 & \cdots & 0 & 0 \\ & \vdots & & & \ddots & \vdots & \\ 0 & 0 & 0 & 0 & \cdots & -1 & 1 \end{pmatrix}\begin{pmatrix} \tau_{21} \\ \tau_{22} \\ \vdots \\ \tau_{s1} \\ \tau_{s2} \end{pmatrix}\geq 0,$$

thus:

$$(**)\begin{cases} D_{12}^{L}\tau_{21}-D_{12}^{U}\tau_{22}+\cdots+D_{1s}^{L}\tau_{s1}-D_{1s}^{U}\tau_{s2}\geq 0 \\ -\tau_{21}+\tau_{22}\geq 0\rightarrow\tau_{21}\leq\tau_{22}\rightarrow D_{12}^{L}\tau_{21}\leq D_{12}^{U}\tau_{22}\rightarrow D_{12}^{L}\tau_{21}-D_{12}^{U}\tau_{22}\leq 0 \\ \vdots \\ -\tau_{s1}+\tau_{s2}\geq 0\rightarrow\tau_{s1}\leq\tau_{s2}\rightarrow D_{1s}^{L}\tau_{s1}\leq D_{1s}^{U}\tau_{s2}\rightarrow D_{1s}^{L}\tau_{s1}-D_{1s}^{U}\tau_{s2}\leq 0 \end{cases}$$

From (**), we have $D_{12}^{L}\tau_{21}=D_{12}^{U}\tau_{22},\dots,D_{1s}^{L}\tau_{s1}=D_{1s}^{U}\tau_{s2}$, and by considering $0\leq D_{1j}^{L}<D_{1j}^{U},\ j=2,\dots,s$ and $\tau_{21}\leq\tau_{22},\cdots,\tau_{s1}\leq\tau_{s2}$, we conclude $\tau=0$.

By setting $\tau=0$ in (*) we have $Y\lambda\geq y_o$, so from $y_o\geq y_j, y_o\neq y_j, j\neq o$ we should have $\lambda_o=1,\lambda_j=0, j\neq o$. Furthermore, $\theta x_o-X\lambda+P\pi\geq 0$ and by using the found λ we have $\theta x_o-x_o+P\pi\geq 0$, therefore $P\pi\geq x_o-\theta x_o\geq 0$ (note that θ is the optimal objective value of the problem, so obviously $\theta\leq 1$).

Similarly, from $P\pi\geq 0$ we conclude $\pi=0$, therefore $\theta=1$ and DMU_o is BCC/AR efficient. □

Theorem 2 A DMU_o with $x_o\leq x_j, x_o\neq x_j, j=1,\dots,n, j\neq o$, is BCC/AR efficient.

Proof. The proof is similar to the proof of theorem 4.1 and is thus omitted.

Theorem 3 A DMU_o with $y_o\geq y_j, y_o\neq y_j,\ j=1,\dots,n,\ j\neq o$, is FG/AR efficient.

Proof. To prove this theorem, you just need to replace $e\lambda=1$ with $e\lambda\leq 1$ at the proof of theorem 1.

Theorem 4 A DMU_o with $x_o\leq x_j, x_o\neq x_j, j=1,\dots,n, j\neq o$, is ST/AR efficient.

Proof. The proof of this theorem will be similar to the proof of theorem 1 by considering $e\lambda\geq 1$.

These theorems provide some sufficient conditions for the DEA/AR efficiency of a DMU and can be used to check the accuracy of the proposed methods for solving FDEA/AR problems. The reason is that most of the proposed methods for solving FDEA/AR problems are inspired by Liu's (2008) method, and as can be seen in Section 3, after the selection of alpha parameter, we should solve the models in the same way as models 9 and 10 – which are traditional DEA/AR models – are solved.

The Proposed Method

In this section, we present a new method for solving the generalized fuzzy data envelopment analysis model with assurance regions (generalized FDEA/AR model). First, some definitions are given about fuzzy numbers and then, the function Q_α that is proposed by Abbasi Shureshjani and Darehmiraki (2013) is introduced. They applied this function to compare and rank fuzzy numbers. In their method, first the fuzzy numbers are replaced with assigned Q_α

functions. Then, by an appropriate selection of α, the fuzzy numbers are compared and ranked based on the obtained amounts of Q_α .

Definition 1 According to Abbasi Shureshjani and Darehmiraki (2013), a fuzzy number $\tilde{A}$ in parametric form is an ordered pair $(\underline{A}(r), \overline{A}(r))$ of functions $\underline{A}(r)$ and $\overline{A}(r)$, $0 \le r \le \omega$, which satisfy the following requirements:

1. $\underline{A}(r)$ is a bounded monotonic increasing left continuous function over $[0,\omega]$,
2. $\overline{A}(r)$ is a bounded monotonic decreasing left continuous function over $[0,\omega]$,
3. $\underline{A}(r) \le \overline{A}(r)$, $0 \le r \le \omega$.

ω is an arbitrary constant such that $0 < \omega \le 1$.

In the above-mentioned definition, if $\omega = 1$ then $\tilde{A}$ is called a normal fuzzy number.

Definition 2 According to Abbasi Shureshjani and Darehmiraki (2013), the α-cut of an arbitrary fuzzy number is defined as $[\tilde{A}]_\alpha = [\underline{A}(\alpha), \overline{A}(\alpha)]$, $0 \le \alpha \le \omega$.

Definition 3 According to Abbasi Shureshjani and Darehmiraki (2013), a normal trapezoidal fuzzy number $\tilde{A}$ can be characterized by a trapezoidal membership function parametrized as (a,b,c,d) where a,b,c and d are real values. If $b = c$, then we have a normal triangular fuzzy number, which for simplicity purposes will be represented by (a,b,d).

In Figure 1, two arbitrary normal trapezoidal and triangular fuzzy numbers are shown.

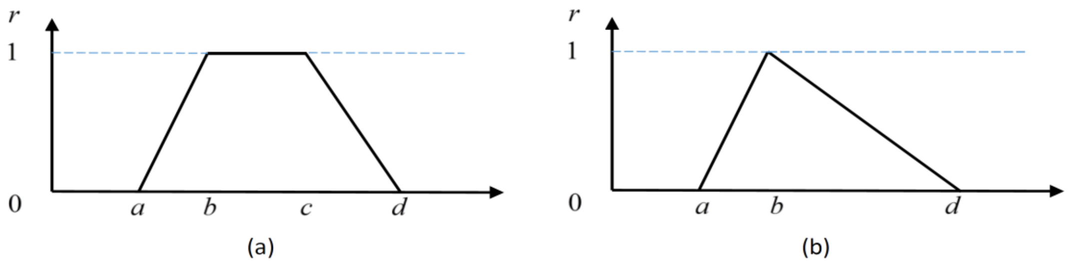

Figure 1. (a) A Normal Trapezoidal Fuzzy Number, (b) A Normal Triangular Fuzzy Number

Let $\tilde{A} = (\underline{A}(r), \overline{A}(r))$, $0 \le r \le 1$ be a normal fuzzy number (Figure 2). The function $Q_\alpha(\tilde{A})$ is defined as follows (Abbasi Shureshjani & Darehmiraki, 2013):

$$Q_\alpha(\tilde{A}) = \int_\alpha^1 (\underline{A}(r) + \overline{A}(r))dr, \quad 0 \le \alpha < 1 \tag{12}$$

The amount of function $Q_\alpha(\tilde{A})$ is graphically shown in Figure 2. This value is the summation of the dotted area and the crosshatched area. Using this function, Abbasi Shureshjani and Darehmiraki (2013) proposed the following definitions for ranking normal fuzzy numbers.

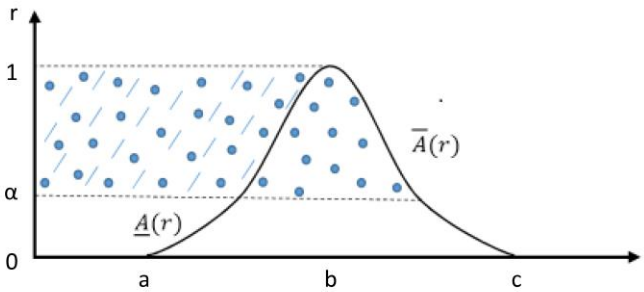

Figure 2. $\tilde{A} = (\underline{A}(r), \overline{A}(r))$ (Abbasi Shureshjani & Darehmiraki, 2013)

Let $\tilde{A}=(a,b,c,d)$ be a normal trapezoidal fuzzy number and $\tilde{B}=(a,b,d)$ be a normal triangular fuzzy number. Then we have:

$$Q_\alpha(\tilde{A})=(b+c).(1-\alpha)+\frac{(a-b-c+d)}{2}.(1-\alpha)^2, \tag{13}$$

$$Q_\alpha(\tilde{B})=(2b).(1-\alpha)+\frac{(a-2b+d)}{2}.(1-\alpha)^2 \tag{14}$$

Definition 4 If $\tilde{A}$ and $\tilde{B}$ are two arbitrary normal fuzzy numbers, then we have:

$\tilde{A}\le\tilde{B}$ if and only if $\forall\alpha\in[0,1],\ Q_\alpha(\tilde{A})\le Q_\alpha(\tilde{B})$

$\tilde{A}=\tilde{B}$ if and only if $\forall\alpha\in[0,1],\ Q_\alpha(\tilde{A})=Q_\alpha(\tilde{B})$

$\tilde{A}\ge\tilde{B}$ if and only if $\forall\alpha\in[0,1],\ Q_\alpha(\tilde{A})\ge Q_\alpha(\tilde{B})$

Definition 5 If we compare two arbitrary normal fuzzy numbers including $\tilde{A}$ and $\tilde{B}$ at a decision level higher than "α" and $\alpha\in[0,1]$, then we have:

$\tilde{A}\le_\alpha\tilde{B}$ if and only if $Q_\alpha(\tilde{A})\le Q_\alpha(\tilde{B})$

$\tilde{A}=_\alpha\tilde{B}$ if and only if $Q_\alpha(\tilde{A})=Q_\alpha(\tilde{B})$

$\tilde{A}\ge_\alpha\tilde{B}$ if and only if $Q_\alpha(\tilde{A})\ge Q_\alpha(\tilde{B})$

where, $\tilde{A}\le_\alpha\tilde{B}$ means that, at decision level higher than α, $\tilde{B}$ is greater than or equal to $\tilde{A}$.

From the definition of the Q_α function, we can see that unlike the two models that are proposed by Liu (2008) (models 9 and 10), this function considers both the left and right parts of a fuzzy number. Therefore, it is sensitive to any changes in the left and right parts of a fuzzy number simultaneously and can be an appropriate representative of fuzzy inputs and outputs in the fuzzy DEA/AR model.

By replacing fuzzy inputs and outputs with their assigned Q_α functions, the generalized fuzzy DEA/AR model is transformed into a parametric linear programming problem that is dependent on α-levels. We can write this transformed generalized FDEA/AR model as follows:

$$\begin{aligned}
Z_\alpha &= \max \sum_{r=1}^{s} u_r Q_\alpha(\tilde{y}_{ro})-\delta_1 u_0\\
s.t.\ & \sum_{i=1}^{m} v_i Q_\alpha(\tilde{x}_{io})=1\\
& \sum_{r=1}^{s} u_r Q_\alpha(\tilde{y}_{rj})-\sum_{i=1}^{m} v_i Q_\alpha(\tilde{x}_{ij})-\delta_1 u_0\le 0,\quad j=1,\dots,n\\
& C_{1q}^{L}v_1\le v_q\le C_{1q}^{U}v_1,\quad q=2,\dots,m\\
& D_{1q}^{L}u_1\le u_q\le D_{1q}^{U}u_1,\quad q=2,\dots,s\\
& \delta_1\delta_2(-1)^{\delta_3}u_0\ge 0,\quad v_i,u_r\ge 0,\quad i=1,\dots,m\ ,r=1,\dots,s
\end{aligned} \tag{15}$$

Then, by choosing an appropriate α (that is a confidence level and depends on the decision maker's ideas), model (15) will be converted to a conventional DEA/AR model and the efficiency of DMUs can be easily computed. As can be seen from Figure 2, if α is defined close to 1, just the elements with high membership quantities of fuzzy numbers (fuzzy inputs and outputs) are considered (a low-risk decision). Alternatively, if α is defined close to zero, it means that the elements with low membership quantities of fuzzy numbers are considered in addition to the high membership quantities (a high-risk decision). Thus, the model (15) is a flexible model based on the decision maker's idea. For example, a risk-averse decision-maker can evaluate the efficiency of decision-making units by the model (15) with the alpha parameter near to zero, whereas a risk-prone decision-maker prefers to solve model (15) with the alpha parameter close to 1 for evaluating the efficiency of the decision-making units. In addition, neutrality in risk can be chosen in the model (15) by selecting the alpha parameter close to 0.5.

We can provide the following efficiency definitions for a DMU in the transformed FDEA/AR model (model (15)).

Definition 6 DMU_o is FBCC/AR efficient if and only if $\forall\alpha\in[0,1)$, DMU_o is efficient in the transformed FBCC/AR model.

Definition 7 DMU_o is FBCC/AR α-efficient if and only if for the selected α, DMU_o is efficient in the transformed FBCC/AR model.

Similar definitions for FCCR/AR, fuzzy FG/AR (FFG/AR) and fuzzy ST/AR (FST/AR) models can be obtained. Now we can rewrite the previous theorems for DMUs with fuzzy inputs and outputs as follows.

Corollary 1 For a selected α, if DMU_o has $\tilde{x}_o \leq_\alpha \tilde{x}_j, \tilde{x}_o \neq_\alpha \tilde{x}_j, j=1,...,\text{n}, j\neq o$, or $\tilde{y}_o \geq_\alpha \tilde{y}_j, \tilde{y}_o \neq_\alpha \tilde{y}_j, j=1,...,n, j\neq o$, then DMU_o is FBCC/AR α-efficient.

Proof. According to definition 5.5, for a selected α, $\tilde{x}_o \leq_\alpha \tilde{x}_j, \tilde{x}_o \neq_\alpha \tilde{x}_j, j=1,...,\text{n}, j\neq o$, if and only if $Q_\alpha(\tilde{x}_o)\leq Q_\alpha(\tilde{x}_j), Q_\alpha(\tilde{x}_o)\neq Q_\alpha(\tilde{x}_j), j=1,...,\text{n}, j\neq o$. In this case, according to theorem 4.2, DMU_o is efficient in the transformed FBCC/AR model and according to definition 5.7, DMU_o is FBCC/AR α-efficient. Proof for the case $\tilde{y}_o \geq_\alpha \tilde{y}_j, \tilde{y}_o \neq_\alpha \tilde{y}_j, j=1,...,n, j\neq o$ is the same. □

Corollary2 If for all α, $\alpha\in[0,1)$, DMU_o has the assumptions of corollary 5.1, DMU_o is FBCC/AR efficient.

Proof. According to definition 5.4, for all α, $\alpha\in[0,1)$, $\tilde{x}_o \leq_\alpha \tilde{x}_j, \tilde{x}_o \neq_\alpha \tilde{x}_j, j=1,...,\text{n}, j\neq o$, if and only if $Q_\alpha(\tilde{x}_o)\leq Q_\alpha(\tilde{x}_j), Q_\alpha(\tilde{x}_o)\neq Q_\alpha(\tilde{x}_j), j=1,...,\text{n}, j\neq o$. In this case, according to theorem 4.2, for all α, DMU_o is efficient in the transformed FBCC/AR model and according to definition 5.6, DMU_o is FBCC/AR efficient. Proof for the case that for all α, $\alpha\in[0,1)$, $\tilde{y}_o \geq_\alpha \tilde{y}_j, \tilde{y}_o \neq_\alpha \tilde{y}_j, j=1,...,n, j\neq o$ is the same. □

Similar to the above-mentioned proofs, we can also prove the following corollaries:

Corollary 3 For a selected α, if DMU_o has $\tilde{y}_o \geq_\alpha \tilde{y}_j, \tilde{y}_o \neq_\alpha \tilde{y}_j, j=1,...,n, j\neq o$, then DMU_o is FFG/AR α-efficient.

Corollary 4 If for all α, $\alpha\in[0,1)$, DMU_o has the assumptions of corollary 5.3, DMU_o is FFG/AR efficient.

Corollary 5 For a selected α, if DMU_o has $\tilde{x}_o \leq_\alpha \tilde{x}_j, \tilde{x}_o \neq_\alpha \tilde{x}_j, j=1,...,\text{n}, j\neq o$, then DMU_o is FST/AR α-efficient.

Corollary 6 If for all α, $\alpha\in[0,1)$, DMU_o has the assumptions of corollary 5.5, DMU_o is FST/AR efficient.

Evaluating the Performance of Manufacturing Enterprises

Here, we consider the same data that was used by Zhou et al. (2012b) (Table 1) to illustrate the proposed approach and to compare the obtained efficiency results from our method with the two other methods in this field. In Table 1, there are ten manufacturing enterprises with three inputs and two outputs. In the input data, the manufacturing cost data (MC) are normal trapezoidal fuzzy numbers, but the number of employee data (NOE) and the floor space data (FS) are crisp. In the output data, the gross output value data (GOV) are represented by normal trapezoidal fuzzy numbers, and the product quality data (PQ) are presented as normal triangular fuzzy numbers.

Table 1. Input and Output Data for the Ten Manufacturing Enterprises (Zhou et al., 2012b)

DMU	Input			Output	
	MC	NOE	FS	GOV	PQ
A	(21.00,21.30,21.70,22.10)	1780	17.30	(147.50,147.90,148.00,148.70)	(3,4,5)
B	(14.10,14.50,14.60,15.00)	1430	16.40	(125.80,126.20,127.20,128.10)	(1,2,3)
C	(25.00,25.50,25.70,26.10)	2630	11.20	(179.00,180.00,182.60,184.50)	(3,4,5)
D	(22.00,22.50,23.50,24.00)	2000	10.50	(149.70,152.70,154.00,155.00)	(3,4,5)
E	(14.80,15.00,15.20,15.60)	1570	9.50	(138.90,142.60,143.30,145.40)	(1,2,3)
F	(19.60,20.00,20.30,21.00)	1670	4.80	(140.50,143.10,144.60,145.70)	(2,3,4)
G	(22.00,22.40,22.60,23.20)	1890	6.20	(164.50,168.70,170.80,175.40)	(2,3,4)
H	(24.00,24.60,25.20,25.50)	2350	11.10	(176.70,179.60,181.20,185.30)	(3,4,5)
I	(15.80,16.30,16.80,17.60)	1750	9.80	(139.80,146.20,148.30,150.00)	(1,2,3)
J	(14.90,15.30,15.80,16.00)	1690	8.50	(140.00,142.80,143.50,144.50)	(2,3,4)

We consider x_1 as an "input numeraire" and y_1 as an "output numeraire" and the assurance regions (AR) on input and output weights are provided as:

$$\frac{0.17}{0.83} \le \frac{v_2}{v_1} \le \frac{0.25}{0.75}, \quad \frac{0.32}{0.83} \le \frac{v_3}{v_1} \le \frac{0.40}{0.75}, \quad \frac{0.28}{0.50} \le \frac{u_2}{u_1} \le \frac{0.35}{0.41} \tag{16}$$

The efficiencies of DMUs obtained from the proposed method are shown in Tables 2 and 3. The results show that for different amounts of α, different efficiency measures are obtained. According to the presented definitions, DMU E is FCCR/AR efficient; DMUs B, C, E, G, and H are FBCC/AR efficient; DMUs C, E, G, and H are FFG/AR efficient; and DMUs B, and E are FST/AR efficient.

Table 2. FCCR/AR and FBCC/AR Efficiency Scores

DMU	FCCR/AR efficiency score				FBCC/AR efficiency score			
	α=0.0	α=0.3	α=0.6	α=0.9	α=0.0	α=0.3	α=0.6	α=0.9
A	0.915320	0.914423	0.913500	0.912592	0.918098	0.917422	0.916717	0.916015
B	0.970259	0.969157	0.968029	0.966916	1.000000	1.000000	1.000000	1.000000
C	0.768134	0.767187	0.766219	0.765264	1.000000	1.000000	1.000000	1.000000
D	0.846859	0.846956	0.847028	0.849557	0.850807	0.851337	0.851859	0.854937
E	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
F	0.949608	0.949574	0.949513	0.949465	0.950287	0.950325	0.950336	0.950361
G	0.990507	0.989522	0.988509	0.987512	1.000000	1.000000	1.000000	1.000000
H	0.852165	0.851006	0.849824	0.848657	1.000000	1.000000	1.000000	1.000000
I	0.920709	0.922172	0.923604	0.925048	0.922431	0.924165	0.925904	0.927674
J	0.939652	0.939677	0.939675	0.939686	0.940114	0.940191	0.940240	0.940305

Table 3. FFG/AR and FST/AR Efficiency Scores

DMU	FFG/AR efficiency score				FST/AR efficiency score			
	α=0.0	α=0.3	α=0.6	α=0.9	α=0.0	α=0.3	α=0.6	α=0.9
A	0.918098	0.917422	0.916717	0.916015	0.915320	0.914423	0.913500	0.912592
B	0.970259	0.969157	0.968029	0.966916	1.000000	1.000000	1.000000	1.000000
C	1.000000	1.000000	1.000000	1.000000	0.768134	0.767187	0.766219	0.765264
D	0.850807	0.851337	0.851859	0.854937	0.846859	0.846956	0.847028	0.849557
E	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
F	0.950287	0.950325	0.950336	0.950361	0.949608	0.949574	0.949513	0.949465
G	1.000000	1.000000	1.000000	1.000000	0.990507	0.989522	0.988509	0.987512
H	1.000000	1.000000	1.000000	1.000000	0.852165	0.851006	0.849824	0.848657
I	0.922431	0.924165	0.925904	0.927674	0.920709	0.922172	0.923604	0.925048
J	0.940114	0.940191	0.940240	0.940305	0.939652	0.939677	0.939675	0.939686

However, as mentioned before, in Liu (2008) and Zhou et al. (2012b) methods, just a boundary for efficiency measure of a DMU is suggested for each amount of the alpha parameter, and these boundaries include a wide range of efficiency measures. Especially for ranking DMUs, combining these obtained boundaries can lead to inaccurate results.

As an example, in fuzzy number ranking methods used in some papers (e.g., Abbasi Shureshjani & Darehmiraki, 2013; Chen & Klein, 1997; Chen & Wang, 2009; Liu, 2008; Zhou et al., 2012b), we can show that DMU C has the biggest outputs among all DMUs. From Table 1, we see that output 1 of DMU C is clearly the biggest among output 1 of all DMUs except for output 1 of DMU H.

We set $\tilde{y}_1^C = (179.00,180.00,182.60,184.50)$ and $\tilde{y}_1^H = (176.70,179.60,181.20,185.30)$ that are the output 1 of DMUs C and H, respectively. $\tilde{y}_1^C$ and $\tilde{y}_1^H$ are two intersected fuzzy numbers, and by applying the ranking fuzzy numbers proposed by Abbasi Shureshjani and Darehmiraki (2013), we have:

$$Q_\alpha(\tilde{y}_1^C) = 362.6(1-\alpha) + 0.45(1-\alpha)^2, \quad Q_\alpha(\tilde{y}_1^H) = 360.8(1-\alpha) + 0.6(1-\alpha)^2$$

We can see that $\forall \alpha \in [0,1],\ Q_\alpha(\tilde{y}_1^C) \geq Q_\alpha(\tilde{y}_1^H)$, then according to definition 5.4, we have: $\tilde{y}_1^C \geq \tilde{y}_1^H$. In addition, by calculating *I* index proposed by Chen and Klein (1997) (used in Liu (2008) method) and *RI* index proposed by Chen and Wang (2009) (used in Zhou et al. (2012b) method) for $\tilde{y}_1^C$ and $\tilde{y}_1^H$, we have $\tilde{y}_1^C \geq \tilde{y}_1^H$, too, as shown in Table 4. It means that in all these three ranking methods, the output 1 of DMU C is bigger than output 1 of DMU H, too.

Similarly, output 2 of DMU C is biggest among output 2 of all the DMUs.

Therefore, if we replace fuzzy inputs and outputs of the DMUs with their assigned values from Abbasi Shureshjani and Darehmiraki (2013) index, Chen and Klein index (1997) (used in Liu (2008) method), or Chen and Wang (2009) index (used in Zhou et al. (2012b) method), we can see that DMU C has the biggest outputs among all the DMUs, and based on Theorem 4.1, it is natural to expect that DMU C be an efficient DMU. This result is obtained by our proposed approach (Table 2). But, if we solve this example with the method of Liu (2008) or Zhou et al. (2012b), DMU C will be inefficient and will obtain rank 9 among these 10 DMUs, which is an inappropriate result (Tables 5 and 6).

Moreover, if we replace fuzzy inputs and outputs of the DMUs with their assigned values from Abbasi Shureshjani and Darehmiraki (2013) index, Chen and Klein index (1997) (used in Liu (2008) method), or Chen and Wang (2009) index (used in Zhou et al. (2012b) method), we can see that DMU B has the smallest inputs among all the DMUs, and based on theorem 4.2, it is natural to expect that DMU B is an efficient DMU. As can be seen from Tables 2, 5, and 6, this result is confirmed by all the above-mentioned approaches.

Table 4. I and RI Indexes for Output 1 of DMUs C and H

DMU		α												
		0.0	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	**1.0**	***I***	***RI***
$\tilde{y}_1^C$	L	179	179.1	179.2	179.3	179.4	179.5	179.6	179.7	179.8	179.9	180	0.9087	0.8687
	U	184.5	184.31	184.12	183.93	183.74	183.55	183.36	183.17	182.98	182.79	182.60		
$\tilde{y}_1^H$	L	176.7	176.99	177.28	177.57	177.86	178.15	178.44	178.73	179.02	179.31	179.6	0.8893	0.8510
	U	185.3	184.89	184.48	184.07	183.66	183.25	182.84	182.43	182.02	181.61	181.2		

Table 5. Efficiency Scores for FBCC/AR Model from Liu (2008) Method

DMU	**A**	**B**	**C**	**D**	**E**	**F**	**G**	**H**	**I**	**J**
Index	0.6227	1.0000	0.5667	0.3817	0.9765	0.7388	1.0000	0.7798	0.6387	0.7122
Rank	8	1	9	10	3	5	1	4	7	6

Table 6. The Lower and Upper Bounds of Efficiency Scores for FBCC/AR Model From Zhou et al. (2012b) Method

DMU		α												
		0.0	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	**1.0**	**RI**	**Rank**
A	L	0.883	0.886	0.889	0.891	0.894	0.897	0.900	0.903	0.906	0.909	0.912	0.1156	8
	U	0.965	0.960	0.956	0.951	0.947	0.942	0.938	0.934	0.929	0.925	0.921		
B	L	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.1850	1
	U	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000		
C	L	0.773	0.784	0.795	0.806	0.816	0.827	0.837	0.848	0.858	0.868	0.879	0.1080	9
	U	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000		
D	L	0.803	0.806	0.810	0.814	0.818	0.823	0.827	0.831	0.835	0.840	0.844	0.0625	10
	U	0.904	0.900	0.895	0.891	0.886	0.882	0.878	0.874	0.870	0.865	0.861		
E	L	0.980	0.984	0.988	0.992	0.996	1.000	1.000	1.000	1.000	1.000	1.000	0.1842	3
	U	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000		
F	L	0.910	0.915	0.918	0.920	0.923	0.925	0.928	0.931	0.934	0.938	0.942	0.1416	5
	U	1.000	1.000	0.999	0.994	0.989	0.984	0.979	0.974	0.969	0.964	0.959		
G	L	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.1850	1
	U	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000		
H	L	0.829	0.855	0.880	0.903	0.926	0.947	0.968	0.987	1.000	1.000	1.000	0.1675	4
	U	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000		
I	L	0.869	0.873	0.877	0.881	0.885	0.889	0.893	0.897	0.904	0.911	0.917	0.1204	7
	U	0.994	0.988	0.983	0.977	0.972	0.966	0.961	0.955	0.950	0.945	0.940		
J	L	0.907	0.909	0.912	0.914	0.917	0.919	0.922	0.925	0.928	0.931	0.935	0.1349	6
	U	0.996	0.991	0.986	0.981	0.976	0.971	0.966	0.961	0.956	0.952	0.947		

Conclusions

Fuzzy data envelopment analysis with assurance regions (FDEA/AR) is an appropriate method for evaluating the efficiency of DMUs when we face fuzzy data in inputs and outputs, or when we come to assumptions or additional information that must be considered in a DEA model.

In this paper, some sufficient conditions for DEA/AR and FDEA/AR efficiency of a DMU are provided. These theorems can be used to check the accuracy of the proposed methods in this field. Then, a new approach for solving a generalized FDEA/AR model included in four popular DEA models, i.e., CCR, BCC, FG, and ST models, is proposed. The proposed method converts a generalized FDEA/AR model to a generalized parametric DEA/AR model dependent on α-level sets. Finally, the proposed method is applied to evaluate the performance of manufacturing enterprises. In this example, we show that although calculating the lower and upper bounds of DMUs' efficiencies is a common method for solving FDEA/AR problems, , this method in practice can lead to inappropriate results.

In real-world problems, such as supply chains, some DMUs have important interior structures; considering the system as a black box ignores these interior structures when evaluating the efficiency of DMUs. Network DEA models study the internal structure of DMUs (Guo et al., 2017). Because of the structure of Liu's (2008) approach, applying it to network FDEA models will be more complicated. As future works, we want to apply our proposed method to different kinds of network FDEA models such as two-stage FDEA, multi-stage FDEA, dynamic network FDEA, etc.

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