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Computation of the q-th roots of circulant matrices

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Abstract. In this paper, we investigate the reduced form of circulant matrices and we show that the problem of computing the q-th roots of a nonsingular circulant matrix A can be reduced to that of computing the q-th roots of two half size matrices $B - C$ and $B + C$.

Keywords: Circulant matrix, Matrix q-th root, Principle q-th root of circulant matrix, Nonsingular matrix, Reduced form.

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1. Introduction

Circulant matrices is a kind of important patterned matrices which arise in many areas of physics, electromagnetics, signal processing, molecular vibration and applied mathematics [1, 2]. For recent years the properties and applications of circulant matrices have been extensively investigated [3].

X is a q-th root of $A \in \mathbb{C}^{n\times n}$, if $X^q = A$. The q-th root of a matrix, for $q > 2$, arises less frequently than the square root, but nevertheless is of interest both in theory and in practice. One application is in the computation of the matrix logarithm through the relation $\log A = q \log A^{1/q}$ [5], where q is chosen so that $A^{1/q}$ can be well approximated by a polynomial or rational function. It can be arising most frequently in the context of symmetric positive definite matrices. The key roles that the q-th root plays in, for example, the matrix sign function. The rich variety of methods for computing the q-th root of a matrix, and principal q-th root of A are the Newtons method given in [5, 6],

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Inverse Newton iteration [4] and Schur method given in [7]. The matrix q-th root is an interesting object of study because algorithms and results for the case $q = 2$ do not always generalize easily, or in the manner that they might be expected to.

This paper is organized as follows. In Section 2, we review some basic definitions and notations. In Section 3, we investigate the properties of circulant matrices, then discuss the form of q-th root of circulant matrices.

2. preliminaries

Throughout this paper we denote the set of all $n \times n$ complex matrices by $\mathbb{C}^{n\times n}$ and the set of all $n \times n$ real matrices by $\mathbb{R}^{n\times n}$.

We first review the structure and reducibility of circulant matrices. All the formulas become slightly more complicated when n is odd; similar to the most of literatures we restrict our attention to the case of even $n = 2m$.

Fix a positive integer $n \geqslant 2$, and let $a = (a_0, \cdots, a_{n-1})$ be a row vector in $\mathbb{R}^n$. Define the operator $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ by $T(a_0, \cdots, a_{n-1}) = (a_{n-1}, a_0, \cdots, a_{n-2})$, the circulant matrix associated to a is the $n \times n$ matrix whose rows are given by iterations of the operator acting on a, that is to say, the matrix whose k-th row is given by $T^{k-1}a$, $k = 1, 2, \cdots, n$. Such a matrix will be denoted by

$$Cir(a) = \begin{pmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ a_{n-1} & a_0 & \cdots & a_{n-2} \\ \vdots & \ddots & \ddots & \\ a_1 & \cdots & a_{n-1} & a_0 \end{pmatrix}.$$

Another equivalent definition of a circulant matrix is as follows:

$A \in \mathbb{R}^{n\times n}$ is a circulant matrix, if and only if $A = G^T AG$, where $G = Cir([0, 1, 0, 0, \cdots, 0])$.

Definition 2.1 Let $< A >= (a_{ij})_{n\times n}$. If

$$< A >= (a_{ij}) = \begin{cases} \mid a_{ij} \mid, \; i = j, & i, j = 1, 2, \cdots n, \\ - \mid a_{ij} \mid, \; i \neq j, & i, j = 1, 2, \cdots n, \end{cases}$$

then $< A >$ is called a comparison matrix of A.

For simplicity, using the partition, the $n \times n$ circulant matrix A can be described as

$$A = \begin{bmatrix} B & A \\ C & B \end{bmatrix}, \tag{1}$$

where B and C are $m \times m$ matrices.

Define

$$P = \frac{\sqrt{2}}{2} \begin{bmatrix} I_m & I_m \\ -I_m & I_m \end{bmatrix}, \tag{2}$$

where I_m is m-th unit matrix.

By applying (1) and (2) we obtain the following:

Lemma 2.2 Let $A \in \mathbb{R}^{n\times n}$ is a circulant matrix, then

$$P^T AP = diag(M, N), \tag{3}$$

where $M = B - C$ and $N = B + C$. We call the matrix of the right side of (3) the reduced form of the circulant matrix A.

Lemma 2.3 Let A be a circulant matrix, then

(1) $< A >$ is a circulant matrix.
(2) If A is nonsingular, then A^{-1} is a circulant matrix.
(3) A^T is a circulant matrix.
(4) Let $B \in \mathbb{R}^{n\times n}$ be a circulant matrix, then $A \pm B$ and AB are also circulant matrices.

Proof. By the assumption that A is a circulant matrix, comparison matrix $< A >$ is as follows:

$$< A >= \begin{bmatrix} < B > & - \mid C \mid \\ - \mid C \mid & < B > \end{bmatrix}. \tag{4}$$

By applying (1)-(3) we obtain

$$P^T < A > P = \frac{\sqrt{2}}{2} \begin{bmatrix} I_m & -I_m \\ I_m & I_m \end{bmatrix} \begin{bmatrix} < B > & - \mid C \mid \\ - \mid C \mid & < B > \end{bmatrix} \frac{\sqrt{2}}{2} \begin{bmatrix} I_m & I_m \\ -I_m & I_m \end{bmatrix} \tag{5}$$

$$P^T < A > P = \begin{bmatrix} < B > + \mid C \mid & 0 \\ 0 & < B > - \mid C \mid \end{bmatrix}, \tag{6}$$

where $M =< B > + \mid C \mid$ and $N =< B > - \mid C \mid$. And similar to the part (1), the other parts can be proved by applying (1)-(3)

$$P^T AP = diag(M, N), \tag{7}$$

$$A = P^{-T} diag(M, N)P^{-1}, \tag{8}$$

where matrix A is nonsingular and $P^T = P^{-1}$

$$A^{-1} = P(diag(M, N))^{-1}P^T, \tag{9}$$

$$A^{-1} = P^{-T}(diag(M, N))^{-1}P^{-1}. \tag{10}$$

So A^{-1} is a circulant matrix.
The other parts obtain as follow:

$$A^T = P^{-T}(diag(M, N))^T P^{-1}. \tag{11}$$

$$A \pm B = P^{-T}(diag(M \pm \acute{M}, N \pm \acute{N}))P^{-1}. \tag{12}$$

■

Definition 2.4 (Matrix function via Hermite interpolation[5]). Let f be defined on the spectrum of $A \in \mathbb{C}^{n\times n}$ and let ψ be the minimal polynomial of A. Then $f(A) := p(A)$, where p is the polynomial of degree less than $\sum_{i=1}^{s} n_i = deg\psi$ that satisfies the interpolation conditions

$$p^{(j)}(\lambda_i) = f^{(j)}(\lambda_i), \quad j = 0 : n_i - 1, i = 1 : s \tag{13}$$

There is a unique such p and it is known as the Hermite interpolating polynomial.

Lemma 2.5 Let the nonsingular matrix $A \in \mathbb{C}^{n\times n}$ have the Jordan canonical form

$$Z^{-1}AZ = diag(J_1, \cdots, J_m), \tag{14}$$

and let $s \leqslant m$ be the number of distinct eigenvalues of A. Then A has precisely q^s principal q-th roots that are primary functions of A given by

$$X_j = Zdiag(L_1^{j_1}, \cdots, L_m^{j_m})Z^{-1}, \quad j = 1 : q^s \tag{15}$$

Corresponding to all possible choices of $j_1, j_2, \cdots, j_m$ subject to the constraint that $j_i = j_k$ whenever $\lambda_i = \lambda_k$.

Here for the $m_k \times m_k$ Jordan block $J_k = J_k(\lambda_k)$, $k = 1, 2, \cdots, m$, we have

$$L_k^{j_k} = J_k^{1/q} = \begin{pmatrix} f(\lambda_k) & f'(\lambda_k) & \cdots & \frac{f^{(m_k-1)}(\lambda_k)}{(m_k-1)!} \\ 0 & f(\lambda_k) & f'(\lambda_k) & \vdots \\ \vdots & \ddots & \ddots & f'(\lambda_k) \\ 0 & \cdots & 0 & f(\lambda_k) \end{pmatrix},$$

where $f(x) = x^{1/q}$ is the principal q-th roots of complex number x.

If $s < m$, A has non primary q-th roots. They form parameterized families

$$X_j(U) = ZUdiag(L_1^{j_1}, \cdots, L_m^{j_m})U^{-1}Z^{-1}, \quad j = q^s + 1 : q^m, \tag{16}$$

where $j_k \in \{1, 2, \cdots, q\}$, U is an arbitrary nonsingular matrix that commutes with J, and for each j there exist i and k, depending on j such that $\lambda_i = \lambda_k$ while $j_i \neq j_k$ [5].

Proof. Let $q \geqslant 2$ be an integer. Suppose that $A \in \mathbb{C}^{n\times n}$ has no negative real eigenvalues and all zero eigenvalues of A are semi simple. Let the Jordan canonical form of A be $Z^{-1}AZ = diag(J_1, \cdots, J_m)$, then the principal q-th root of A is

$$A^{1/q} = Zdiag(J_1^{1/q}, J_2^{1/q}, \cdots, J_m^{1/q})Z^{-1}. \tag{17}$$

So for which $j_i = j_k$ whenever, $\lambda_i = \lambda_k$,

$$Udiag(J_1^{1/q}, J_2^{1/q}, \cdots, J_m^{1/q})U^{-1} = diag(J_1^{1/q}, J_2^{1/q}, \cdots, J_m^{1/q}), \tag{18}$$

that is, U commutes with the block diagonal matrix in the middle. This commutativity follows form the explicit form for U. ■

3. The q-th roots of circulant matrices

In this section we will discuss the q-th roots of circulant matrices.

Theorem 3.1 Let $A \in \mathbb{R}^{n\times n}$ be a nonsingular circulant matrix and X as the principal q-th root of A and write $X^q = A$, where X is a primary function of A. Then all q-th roots such as X are circulant matrices.

Proof. By assumption, $X^q = A$ and $X = f(A)$, here $f(x) = x^{1/q}$, and A is a nonsingular matrix, the q-th root function is defined on spectrum of A and which $f(x) = x^{1/q}$ is the principal q-th roots of complex number x. It follows that the eigenvalues of $A^{1/q}$ lie in the segment $\{x : \frac{-\pi}{q} < \arg(x) < \frac{\pi}{q}\}$. Using the fact that the sum and product of n circulant matrices are also circulant, and from Definition 2.4, we can construct a polynomial p such that $p(A) = f(A)$, the polynomial $X = p(A)$ is a circulant matrix since A is a circulant matrix. ■

Remark 1 *Assume that $\tilde{X}$ is q-th root of A which is function of A. Then, by Lemma 2.2 and Theorem 3.1, we have that*

$$P^T\tilde{X}P = X = diag(X_1, X_2), \tag{19}$$

and

$$P^TAP = X = diag(M, N), \tag{20}$$

which means that $X_1 = f(M)$ and $X_2 = f(N)$. Thus the problem of computing q-th root of a circulant matrix A, which are function of A, can be reduced to that of computing q-th root of two half size matrix M and N, which are functions of M and N, respectively.

Theorem 3.2 Let $A \in \mathbb{R}^{n\times n}$ be a nonsingular circulant matrix. If A has a circulant q-th root, then each $M = B - C$ and $N = B + C$ in Lemma 2.2, has q-th root, respectively.

Proof. We know that A has reduced form (3). From hypothesis, denote a circulant q-th root of A by $\tilde{X}$. By Lemma 2.2, there holds

$$P^T\tilde{X}P = diag(X_1, X_2), \tag{21}$$

where P is defined by the Definition 2.1. In fact $\tilde{X}^q = A$ implies that $X_1^q = B - C$, $X_2^q = B + C$ hold simultaneously, that is, X_1 and X_2 are q-th root of $B - C$ and$B + C$, respectively. ■

Theorem 3.3 Let $A \in \mathbb{R}^{n\times n}$ be a nonsingular circulant matrix with reduced form given by (3). Assume that M has s distinct eigenvalues and N has t distinct eigenvalues. Let $M = Z_M J_M Z_M^{-1}$ with

$$J_M = diag(J_1, J_2 \cdots, J_l), \tag{22}$$

and $N = Z_N J_N Z_N^{-1}$ with

$$J_N = diag(\tilde{J}_1, \tilde{J}_2 \cdots, \tilde{J}_r), \tag{23}$$

be the Jordan decompositions of M and N, respectively.

Let $\mid (\delta(M) \cap \delta(N) \mid= \alpha$, where $\delta(M)$ denotes the spectrum of matrix M. If M and N have a common eigenvalues, then A has $q^{s+t-\alpha}$ circulant q-th roots which are functions of A and take the form

$$\hat{X} = PZ\hat{L}Z^{-1}P^T, \tag{24}$$

with

$$Z = diag(Z_M, Z_N), \tag{25}$$

and

$$\hat{L} = diag(\hat{L}_M, \hat{L}_N), \tag{26}$$

where $\hat{L}_M$ denoted a q-th root of J_M, which is a function of J_M and $\hat{L}_N$ denoted a q-th root of J_N, which is a function of J_N. Furthermore, if $s+t < l+r$, then A has q-th roots which are not function of A; they form $q^{l+r} - q^{s+t-\alpha}$ parameterized families and taking the following form:

$$\tilde{X}(U) = PZU\hat{L}U^{-1}Z^{-1}P^T, \tag{27}$$

where $Z, \hat{L}$ are defined in (25) and (26) and U is an arbitrary nonsingular matrix which commutes with $J = diag(J_M, J_N)$.

Proof. If M and N have a common eigenvalues, then A has $s+t-\alpha$ distinct eigenvalues and $l+r$ Jordan blocks. Then by Lemma 2.5 we can get that A has $q^{s+t-\alpha}$, q-th roots which are functions of A and take the form (24). By Theorem 3.1, those q-th roots are circulant. Using Lemma 2.5 again, we get the form (27).

■

4. Conclusions

In this paper, we get that any nonsingular circulant matrix has a circulant q-th roots. And by Theorem 3.2 and Definition 2.4 we developed the problem of computing the q-th roots of a circulant matrices $A \in \mathbb{R}^{n\times n}$ can be reduced to that of computing the q-th roots of two half size matrices M and N. And we developed the q-th roots of a nonsingular circulant matrix fall into two classes. The first class comprises finitely many primary q-th roots. The second class, which may be empty, comprises a finite number of parametrized families of matrices, and the q-th roots in this class may be circulant matrices or not.

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