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### Expansion of Bessel and g-Bessel sequences to dual frames and dual g-frames

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**Abstract.** In this paper we study the duality of Bessel and *g*-Bessel sequences in Hilbert spaces. We show that a Bessel sequence is an inner summand of a frame and the sum of any Bessel sequence with Bessel bound less than one with a Parseval frame is a frame. Next we develop this results to the g-frame situation.

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# 1. Introduction

Let  $\mathcal{H}$  denote a separable Hilbert space. A sequence  $\{f_i\}_{i \in I}$  in  $\mathcal{H}$  is called a frame if there exist constants  $0 < A \leq B < \infty$  such that

$$A\|f\|^2 \leqslant \sum_{i \in I} |\langle f, f_i \rangle|^2 \leqslant B\|f\|^2 \qquad \forall f \in \mathcal{H}.$$
 (1)

We call A and B the lower and upper frame bounds, respectively. The sequence  $\{f_i\}_{i \in I}$  is a Bessel sequence if at least the upper bound in (1) is satisfied. For any frame  $\{f_i\}_{i \in I}$  there exists at least one dual frame, i.e., a frame  $\{g_i\}_{i \in I}$  for which  $f = \sum_{i \in I} \langle f, g_i \rangle f_i$  for all  $f \in \mathcal{H}$ . If  $\{f_i\}_{i \in I}$  is a Bessel sequence with bound B < 1, how can we find two

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sequences  $\{g_i\}_{i\in I}$  and  $\{p_i\}_{i\in I}$  such that  $\{f_i + g_i\}_{i\in I}$  and  $\{p_i\}_{i\in I}$  are dual frames, i.e., such that

$$f = \sum_{i \in I} \langle f, p_i \rangle (f_i + g_i) = \sum_{i \in I} \langle f, f_i + g_i \rangle p_i.$$
<sup>(2)</sup>

In this paper we aim at the more general results of the type in (2). For any Bessel sequence  $\mathcal{F} = \{f_i\}_{i \in I}$  the synthesis operator is defined as follows:

$$T_{\mathcal{F}}: \ell^2(I) \to \mathcal{H}, \quad \text{with} \quad T_{\mathcal{F}}(\{c_i\}_{i \in I}) = \sum_{i \in I} c_i f_i.$$

The analysis operator for  $\mathcal{F}$  is  $T^*_{\mathcal{F}}$  and is given by  $T^*_{\mathcal{F}}f = \{\langle f, f_i \rangle\}_{i \in I}$ . The frame operator is the positive self-adjoint invertible operator  $S_{\mathcal{F}} = T_{\mathcal{F}}T^*_{\mathcal{F}}$  and satisfies  $S_{\mathcal{F}}f = \sum_{i \in I} \langle f, f_i \rangle f_i$ . The reconstruction formulas are as follows:

$$f = \sum_{i \in I} \langle f, f_i \rangle S_{\mathcal{F}}^{-1} f_i = \sum_{i \in I} \langle f, S_{\mathcal{F}}^{-1} f_i \rangle f_i \quad \forall f \in \mathcal{H}.$$
 (3)

Frames were first introduced in 1952 by Duffin and Schaeffer [4] in the study of nonharmonic Fourier series, reintroduced in 1986 by Daubechies, Grossman and Meyer in [3]. *G*-frames for Hilbert spaces first formally were defined by Sun in [6].

Let  $\mathcal{H}, \mathcal{K}$  be two separable Hilbert spaces and  $\{W_i\}_{i \in I}$  be a sequence of closed subspaces of  $\mathcal{K}$ , where I is a subset of  $\mathbb{Z}$ . Let  $\mathcal{L}(\mathcal{H}, W_i)$  be the collection of all bounded linear operators from  $\mathcal{H}$  into  $W_i$ . Recall that a family of operators  $\Lambda = \{\Lambda_i \in \mathcal{L}(\mathcal{H}, W_i) : i \in I\}$ is said to be a generalized frame, or simply a g-frame for  $\mathcal{H}$  with respect to  $\{W_i\}_{i \in I}$  if there exist constants  $0 < C \leq D < \infty$  such that

$$C\|f\|^2 \leqslant \sum_{i \in I} \|\Lambda_i f\|^2 \leqslant D\|f\|^2 \qquad \forall f \in \mathcal{H}.$$
(4)

The constants C and D are called g-frame bounds and  $\sup_{i \in I} \Lambda_i$  is called the multiplicity of the g-frame. We call  $\Lambda$  a tight g-frame if C = D and it is a Parseval g-frame if C = D = 1. If the right-hand side of (4) holds, then  $\Lambda$  is said a g-Bessel sequence for  $\mathcal{H}$ with respect to  $\{W_i\}_{i \in I}$ . The representation space associated with a g-Bessel sequence  $\Lambda = \{\Lambda_i\}_{i \in I}$  is defined by

$$\left(\sum_{i\in I} \oplus W_i\right)_{\ell^2} = \left\{\{g_i\}_{i\in I} | g_i \in W_i \text{ and } \sum_{i\in I} \|g_i\|^2 < \infty\right\}.$$
(5)

The synthesis operator of  $\Lambda$  given by

$$T_{\Lambda}: \left(\sum_{i \in I} \oplus W_i\right)_{\ell^2} \to \mathcal{H} \qquad T_{\Lambda}(\{g_i\}_{i \in I}) = \sum_{i \in I} \Lambda_i^* g_i.$$

The adjoint operator of  $T_{\Lambda}$ , which is called the analysis operator also obtain as follows

$$T^*_{\Lambda} : \mathcal{H} \to \left(\sum_{i \in I} \oplus W_i\right)_{\ell^2} \qquad T^*_{\Lambda}f = \{\Lambda_i f\}_{i \in I}.$$

By composing  $T_{\Lambda}$  with its adjoint  $T^*_{\Lambda}$ , we obtain the *g*-frame operator

$$S_{\Lambda}: \mathcal{H} \to \mathcal{H}$$
  $S_{\Lambda}f = T_{\Lambda}T_{\Lambda}^{*}f = \sum_{i \in I} \Lambda_{i}^{*}\Lambda_{i}f,$ 

which is a bounded, self-adjoint, positive and invertible operator and  $CI_{\mathcal{H}} \leq S_{\Lambda} \leq DI_{\mathcal{H}}$ . The canonical dual *g*-frame for  $\{\Lambda_i\}_{i \in I}$  is defined by  $\{\tilde{\Lambda}_i\}_{i \in I}$  where  $\tilde{\Lambda}_i = \Lambda_i S_{\Lambda}^{-1}$ , which is also a *g*-frame for  $\mathcal{H}$  with *g*-frame bounds  $\frac{1}{D}$  and  $\frac{1}{C}$ , respectively. The reconstruction formulas are also as follows:

$$f = \sum_{i \in I} \Lambda_i^* \widetilde{\Lambda}_i f = \sum_{i \in I} \widetilde{\Lambda}_i^* \Lambda_i f \qquad \forall f \in \mathcal{H}.$$

For more details about the theory and applications of frames we refer the readers to [1, 5], about g-frames to [5, 6].

For using of the reconstruction formulas we need to invert the g-frame operator, which can be complicated. In the following similar to frame algorithm we use a g-frame algorithm to obtain approximations of linear operator  $U \in B(\mathcal{H}, \mathcal{K})$ .

**Theorem 1.1** Let  $\Lambda = {\Lambda_i}_{i \in I}$  be a *g*-frame with *g*-frame bounds C, D. For every  $U \in B(\mathcal{H}, \mathcal{K})$ , we define the sequence  ${U_n}_{n \in \mathbb{N}}$  by

$$U_{n} = \begin{cases} 0 & n = 0\\ U_{n-1} + \frac{2}{C+D}(U - U_{n-1})S_{\Lambda} & n \ge 1 \end{cases}$$

Then we have  $U = \lim_{n \to \infty} U_n$  with the error estimate

$$||U - U_n|| \leq \left(\frac{D - C}{D + C}\right)^n ||U||.$$

**Proof.** By the *g*-frame condition for each  $f \in \mathcal{H}$  we have

$$-\frac{D-C}{D+C}\|f\|^2 \leqslant \langle (I_{\mathcal{H}} - \frac{2}{C+D}S_{\Lambda})f, f \rangle \leqslant \frac{D-C}{D+C}\|f\|^2.$$

Thus

$$\left\|Id_{\mathcal{H}} - \frac{2}{C+D}S_{\Lambda}\right\| \leq \frac{D-C}{D+C}.$$

Using the definition of  $\{U_n\}_{n\in\mathbb{N}}$  we obtain

$$U - U_n = (U - U_0) \left( I d_{\mathcal{H}} - \frac{2}{C + D} S_{\Lambda} \right)^n.$$

Hence,

$$||U - U_n|| \leq \left(\frac{D - C}{D + C}\right)^n ||U||$$

A family of operators  $\Lambda = \{\Lambda_i \in \mathcal{L}(\mathcal{H}, W_i) : i \in I\}$  is called an orthonormal *g*-basis for  $\mathcal{H}$  if it holds in the following conditions.

(1) 
$$f = \sum_{i \in I} \Lambda_i^* \Lambda_i f$$
  $\forall f \in \mathcal{H}.$   
(2)  $\langle \Lambda_i^* g, \Lambda_j^* g' \rangle = \delta_{ij} \langle g, g' \rangle$   $\forall g, g' \in W_i, \forall i, j \in I.$ 

**Lemma 1.2** Let  $\Lambda = \{\Lambda_i \in \mathcal{L}(\mathcal{H}, W_i) : i \in I\}$  be a collection of partial isometries. Then the sequence  $\{\Lambda_i\}_{i \in I}$  is a Parseval *g*-frame for  $\mathcal{H}$ , if and only if the sequence  $\{\Lambda_i^*\Lambda_i\}_{i \in I}$ be an orthonormal *g*-basis for  $\mathcal{H}$ .

**Proof.** First note that  $\Lambda_i$  is a partial isometry, if and only if  $\Lambda_i^* \Lambda_i$  is an orthogonal projection on  $\mathcal{H}$ . Now the claim follows from

$$\|f\|^2 = \sum_{i \in I} \|\Lambda_i f\|^2 = \sum_{i \in I} \langle \Lambda_i^* \Lambda_i f, f \rangle = \sum_{i \in I} \|\Lambda_i^* \Lambda_i f\|^2 \quad \forall f \in \mathcal{H}.$$

## 2. Duality of Bessel and *g*-Bessel sequences

Li and Sun in [5] expanded every Bessel sequence to a tight frame by adding some elements. In this section we show that the sum of any Bessel sequence with Bessel bound less than one with a Parseval frame is a frame for  $\mathcal{H}$  and we prove that a Bessel sequence is an inner summand of a frame.

Let  $\mathcal{F} = \{f_i\}_{i \in I}$  and  $\mathcal{G} = \{g_i\}_{i \in I}$  be two Bessel sequences for  $\mathcal{H}$  with synthesis operators  $T_{\mathcal{F}}$  and  $T_{\mathcal{G}}$  respectively. Then we say that  $\mathcal{F}$  and  $\mathcal{G}$  are dual frames for  $\mathcal{H}$  if  $T_{\mathcal{F}}T^*_{\mathcal{G}} = I_{\mathcal{H}}$  or  $T_{\mathcal{G}}T^*_{\mathcal{F}} = I_{\mathcal{H}}$ , i.e.,

$$f = \sum_{i \in I} \langle f, g_i \rangle f_i = \sum_{i \in I} \langle f, f_i \rangle g_i \quad \forall f \in \mathcal{H}.$$

Notation 2.1 For every countable (or finite) index set I, we define the space  $\ell(I, \mathcal{H})$  by

$$\ell(I,\mathcal{H}) = \Big\{ \{f_i\}_{i \in I} : f_i \in \mathcal{H}, \text{ and } \sup_{i \in I} \|f_i\| < \infty \Big\}.$$

It is easy to check that  $\ell(I, \mathcal{H})$  with the pointwise operations and norm defined by

$$\|\{f_i\}_{i\in I}\| = \sup_{i\in I} \|f_i\|,$$

is a Banach space. Let  $\mathcal{B}(I, \mathcal{H})$  be the set of all Bessel sequences,  $\mathcal{F}(I, \mathcal{H})$  be the collection of all frames and  $\mathcal{P}(I, \mathcal{H})$  denote the set of all Parseval frames indexed by I for  $\mathcal{H}$ respectively. Then  $\mathcal{B}(I, \mathcal{H})$  is a subspace of  $\ell(I, \mathcal{H})$  and  $\mathcal{P}(I, \mathcal{H}) \subset \mathcal{F}(I, \mathcal{H}) \subset \mathcal{B}(I, \mathcal{H}) \subset$  $\ell(I, \mathcal{H})$ .

The following theorem shows that the sum of any Bessel sequence with Bessel bound less than one with a Parseval frame is a frame for  $\mathcal{H}$ .

**Theorem 2.2** Let  $\mathcal{F} = \{f_i\}_{i \in I} \in \mathcal{B}(I, \mathcal{H})$  with Bessel bound B < 1. Then

$$\mathcal{F} + \mathcal{P}(I, \mathcal{H}) \subset \mathcal{F}(I, \mathcal{H}).$$

**Proof.** Let  $\mathcal{G} = \{g_i\}_{i \in I} \in \mathcal{P}(I, \mathcal{H})$ . Then for all  $f \in \mathcal{H}$  we have

$$\begin{aligned} \|T_{\mathcal{F}}T_{\mathcal{G}}^*f\|^2 &= \sup_{\|g\|=1} \left| \langle T_{\mathcal{F}}T_{\mathcal{G}}^*f,g \rangle \right|^2 = \sup_{\|g\|=1} \left| \sum_{i \in I} \langle f,g_i \rangle \langle f_i,g \rangle \right|^2 \\ &\leqslant \sup_{\|g\|=1} \sum_{i \in I} |\langle f,g_i \rangle|^2 \sum_{i \in I} |\langle g,f_i \rangle|^2 \leqslant B \|f\|^2. \end{aligned}$$

Thus  $||T_{\mathcal{F}}T^*_{\mathcal{G}}|| \leq \sqrt{B} < 1$ , and so  $I_{\mathcal{H}} + T_{\mathcal{F}}T^*_{\mathcal{G}}$  an invertible operator in  $\mathcal{L}(\mathcal{H})$ . If we set  $\Theta = (I_{\mathcal{H}} + T_{\mathcal{F}}T^*_{\mathcal{G}})^{-1}$  and  $\mathcal{U} = \mathcal{F} + \mathcal{G}$  with  $\mathcal{U} = \{u_i\}_{i \in I}$ . Then we compute

$$f = (I_{\mathcal{H}} + T_{\mathcal{F}}T_{\mathcal{G}}^*)\Theta f$$
  
=  $\sum_{i \in I} \langle \Theta f, g_i \rangle f_i + \sum_{i \in I} \langle \Theta f, g_i \rangle g_i$   
=  $\sum_{i \in I} \langle f, \Theta^* g_i \rangle u_i,$ 

for all  $f \in \mathcal{H}$ . This shows that  $\mathcal{U} \in \mathcal{F}(I, \mathcal{H})$  with frame bounds  $\|\Theta\|^{-2}$  and  $(1 + \sqrt{B})^2$ .

The next result shows that a Bessel sequence is an inner summand of a frame.

**Corollary 2.3** Let  $\mathcal{F} \in \mathcal{B}(I, \mathcal{H})$  be a Bessel sequence. Then there exists a tight frame  $\mathcal{G} \in \mathcal{F}(I, \mathcal{H})$  such that  $\mathcal{F} + \mathcal{G} \in \mathcal{F}(I, \mathcal{H})$ .

**Proof.** Let *B* be the Bessel bound for  $\mathcal{F} = \{f_i\}_{i \in I}$ , then  $\{\frac{1}{\sqrt{2B}}f_i\}_{i \in I}$  is a Bessel sequence with the Bessel bound less than one. By Theorem 2.2  $\{\frac{1}{\sqrt{2B}}f_i + e_i\}_{i \in I}$  is a frame for  $\mathcal{H}$ , where  $\{e_i\}_{i \in I}$  is an arbitrary Parseval frame. Define  $g_i = \sqrt{2B}e_i$  for all  $i \in I$ , then  $\mathcal{G} = \{g_i\}_{i \in I}$  is a tight frame and  $\mathcal{F} + \mathcal{G} = \{\sqrt{2B}(\frac{1}{\sqrt{2B}}f_i + e_i)\}_{i \in I}$  is also a frame for  $\mathcal{H}$ .

The next theorem changes every Bessel sequence to a dual frame by summing it with any Parseval frame.

**Theorem 2.4** Let  $\mathcal{F} \in \mathcal{B}(I, \mathcal{H})$  with Bessel bound B < 1 and let  $\mathcal{E} \in \mathcal{P}(I, \mathcal{H})$ . Then there exists a  $\mathcal{G} \in \mathcal{B}(I, \mathcal{H})$  such that  $\mathcal{F} + \mathcal{E}$  and  $\mathcal{G} + \mathcal{E}$  are dual frames.

**Proof.** Let  $\mathcal{F} = \{f_i\}_{i \in I}$  and  $\mathcal{E} = \{e_i\}_{i \in I}$ . Since B < 1, hence  $I_{\mathcal{H}} + T_{\mathcal{F}} T_{\mathcal{E}}^*$  is an invertible operator in  $\mathcal{L}(\mathcal{H})$ . If we define  $\Theta = -(I_{\mathcal{H}} + T_{\mathcal{F}} T_{\mathcal{E}}^*)^{-1} T_{\mathcal{F}} T_{\mathcal{E}}^*$  and  $g_i = \Theta^* e_i$  for all  $i \in I$ . Then  $\mathcal{G} = \{g_i\}_{i \in I}$  is a Bessel sequence for  $\mathcal{H}$  and for all  $f \in \mathcal{H}$  we have

$$\begin{split} f &= (I_{\mathcal{H}} + T_{\mathcal{F}} T_{\mathcal{E}}^*) \Theta f + T_{\mathcal{E}} T_{\mathcal{E}}^* f + T_{\mathcal{F}} T_{\mathcal{E}}^* f \\ &= T_{\mathcal{E}} T_{\mathcal{E}}^* \Theta f + T_{\mathcal{E}} T_{\mathcal{E}}^* f + T_{\mathcal{F}} T_{\mathcal{E}}^* \Theta f + T_{\mathcal{F}} T_{\mathcal{E}}^* f \\ &= \sum_{i \in I} \langle \Theta f, e_i \rangle e_i + \sum_{i \in I} \langle f, e_i \rangle e_i + \sum_{i \in I} \langle \Theta f, e_i \rangle f_i + \sum_{i \in I} \langle f, e_i \rangle f_i \\ &= \sum_{i \in I} \langle f, g_i + e_i \rangle (f_i + e_i), \end{split}$$

which this finishes the proof.

**Corollary 2.5** Let  $\mathcal{F} \in \mathcal{B}(I, \mathcal{H})$  with Bessel bound B < 1 and let  $\mathcal{E} \in \mathcal{P}(I, \mathcal{H})$ . Then there exists a  $\mathcal{G} \in \mathcal{B}(I, \mathcal{H})$  such that  $\mathcal{F} + \mathcal{E}$  and  $\mathcal{G}$  are dual frames.

**Proof.** Suppose that  $\mathcal{F} = \{f_i\}_{i \in I}$  and  $\mathcal{E} = \{e_i\}_{i \in I}$ . Since  $I_{\mathcal{H}} + T_{\mathcal{F}}T_{\mathcal{E}}^*$  is invertible on  $\mathcal{H}$ . Thus if we set  $\Theta = (I_{\mathcal{H}} + T_{\mathcal{F}}T_{\mathcal{E}}^*)^{-1}$  and  $g_i = \Theta^* e_i$  for all  $i \in I$ . Then for all  $f \in \mathcal{H}$  we have

$$f = (I_{\mathcal{H}} + T_{\mathcal{F}}T_{\mathcal{E}}^*)\Theta f = T_{\mathcal{E}}T_{\mathcal{E}}^*\Theta f + T_{\mathcal{F}}T_{\mathcal{E}}^*\Theta f$$
$$= \sum_{i\in I} \langle \Theta f, e_i \rangle e_i + \sum_{i\in I} \langle \Theta f, e_i \rangle f_i = \sum_{i\in I} \langle f, g_i \rangle (f_i + e_i)$$

From this completes the proof.

**Corollary 2.6** For every  $\mathcal{F} \in \mathcal{B}(I, \mathcal{H})$  there exist  $\mathcal{G} \in \mathcal{B}(I, \mathcal{H})$  and a tight frame  $\mathcal{U} \in \mathcal{F}(I, \mathcal{H})$  such that  $\mathcal{F} + \mathcal{U}$  and  $\mathcal{G}$  are dual frames for  $\mathcal{H}$ .

**Proof.** Let *B* be the Bessel bound of  $\mathcal{F} = \{f_i\}_{i \in I}$  and let  $\{e_i\}_{i \in I}$  denote any Parseval frame for  $\mathcal{H}$ . By Theorem 2.4 there exists a Bessel sequence  $\{v_i\}_{i \in I}$  for  $\mathcal{H}$  such that  $\{\frac{1}{\sqrt{2B}}f_i + e_i\}_{i \in I}$  and  $\{v_i + e_i\}_{i \in I}$  are dual frames for  $\mathcal{H}$ . Put  $\mathcal{G} = \{g_i\}_{i \in I}, \mathcal{U} = \{u_i\}_{i \in I}$  with  $g_i = \frac{1}{\sqrt{2B}}v_i + \frac{1}{\sqrt{2B}}e_i$  and  $u_i = \sqrt{2B}e_i$  for all  $i \in I$ . Then for all  $f \in \mathcal{H}$  we have

$$\sum_{i \in I} \langle f, g_i \rangle (f_i + u_i) = \sum_{i \in I} \langle f, \frac{1}{\sqrt{2B}} v_i + \frac{1}{\sqrt{2B}} e_i \rangle (f_i + \sqrt{2B} e_i)$$
$$= \sum_{i \in I} \langle f, v_i + e_i \rangle (\frac{1}{\sqrt{2B}} f_i + e_i) = f.$$

From this the claim follows immediately.

In the following theorem we show that every Bessel sequence can be expanded to a dual frame by adding it to a Parseval frame. Another form of this result can be found in [5] Corollary 3.2.

**Theorem 2.7** Let  $\mathcal{F} = \{f_i\}_{i \in I}$  be a Bessel sequence with Bessel bound B and  $\mathcal{E} = \{e_i\}_{i \in I}$  be a Parseval frame for  $\mathcal{H}$ . Then for all  $\alpha > B$ , there exists a Bessel sequence  $\{g_i\}_{i \in I}$  for  $\mathcal{H}$  such that  $\{f_i, e_i\}_{i \in I}$  and  $\{\frac{1}{\alpha}f_i, g_i\}_{i \in I}$  are dual frames for  $\mathcal{H}$ .

**Proof.** Since  $\alpha > B$ , hence  $\Theta = I_{\mathcal{H}} - \frac{1}{\alpha}T_{\mathcal{F}}T_{\mathcal{F}}^*$  is a linear bounded and positive operator on  $\mathcal{H}$ . Thus if we define  $g_i = \Theta^* e_i$  for all  $i \in I$ . Then  $\{g_i\}_{i \in I}$  is a Bessel sequence for  $\mathcal{H}$ and for all  $f \in \mathcal{H}$  we have

$$\sum_{i \in I} \langle f, \frac{1}{\alpha} f_i \rangle f_i + \sum_{i \in I} \langle f, g_i \rangle e_i = \sum_{i \in I} \langle f, \frac{1}{\alpha} f_i \rangle f_i + \sum_{i \in I} \langle \Theta f, e_i \rangle e_i$$
$$= \frac{1}{\alpha} T_{\mathcal{F}} T_{\mathcal{F}}^* f + \Theta f = f$$

which this finishes the proof.

Let  $\Lambda = {\Lambda_i}_{i \in I}$  and  $\Gamma = {\Gamma_i}_{i \in I}$  be g-Bessel sequences for  $\mathcal{H}$  with synthesis operators  $T_{\Lambda}$  and  $T_{\Lambda}$  respectively. Then we say that  $\Lambda$  and  $\Gamma$  are dual g-frames for  $\mathcal{H}$  if  $T_{\Lambda}T_{\Gamma}^* = I_{\mathcal{H}}$  or  $T_{\Gamma}T_{\Lambda}^* = I_{\mathcal{H}}$ . In the following we show that any pair of g-Bessel sequences can be extended to pair of dual g-frames. This result, generalizes a result of Christensen, Oh Kim and Young Kim [2] to the situation of g-frames.

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**Theorem 2.8** Let  $\Lambda = {\Lambda_i}_{i \in I}$  and  $\Gamma = {\Gamma_i}_{i \in I}$  be two *g*-Bessel sequences for  $\mathcal{H}$  with respect to  ${W_i}_{i \in I}$ . Then there exist *g*-Bessel sequences  ${\{\Xi_j\}}_{j \in J}$  and  ${\{\Omega_j\}}_{j \in J}$  for  $\mathcal{H}$  with respect to  ${\{V_j\}}_{j \in J}$ , such that  ${\{\Lambda_i\}}_{i \in I} \cup {\{\Xi_j\}}_{j \in J}$  and  ${\{\Gamma_i\}}_{i \in I} \cup {\{\Omega_j\}}_{j \in J}$  form a pair of dual *g*-frames for  $\mathcal{H}$  with respect to  ${\{W_i\}}_{i \in I} \cup {\{W_i\}}_{i \in I} \cup {\{V_j\}}_{j \in J}$ .

**Proof.** Assume that  $\{\Phi_j\}_{j\in J}$  and  $\{\Psi_j\}_{j\in J}$  are any pair of dual *g*-frames for  $\mathcal{H}$  respect to  $\{V_j\}_{j\in J}$  and let  $\Theta = I_{\mathcal{H}} - T_{\Gamma}T^*_{\Lambda}$ . Then for any  $f \in \mathcal{H}$  we have

$$f = \Theta f + T_{\Gamma} T_{\Lambda}^* f = \sum_{j \in J} \Psi_j^* \Phi_j \Theta f + \sum_{i \in I} \Gamma_i^* \Lambda_i f.$$

If we set  $\Xi_j = \Phi_j \Theta$  and  $\Omega_j = \Psi_j$  for all  $j \in J$ . Then  $\{\Lambda_i\}_{i \in I} \cup \{\Xi_j\}_{j \in J}$  and  $\{\Gamma_i\}_{i \in I} \cup \{\Omega_j\}_{j \in J}$  are dual g-frames for  $\mathcal{H}$  with respect to  $\{W_i\}_{i \in I} \cup \{V_j\}_{j \in J}$ .

The following corollaries are generalizations of the above results to the *g*-frames situation. We leave the proofs to interested readers.

**Corollary 2.9** Let  $\Lambda = {\Lambda_i}_{i \in I}$  be a *g*-Bessel sequence for  $\mathcal{H}$  with respect to  ${W_i}_{i \in I}$ with *g*-Bessel bound B < 1. Then there exists *g*-Bessel sequence  ${\Gamma_i}_{i \in I}$  for  $\mathcal{H}$  with respect to  ${W_i}_{i \in I}$ , such that  ${\Xi_i + \Lambda_i}_{i \in I}$  and  ${\Xi_i + \Gamma_i}_{i \in I}$  are dual *g*-frames for  $\mathcal{H}$  with respect to  ${W_i}_{i \in I}$ , where  ${\Xi_i}_{i \in I}$  is a Parseval *g*-frame for  $\mathcal{H}$  with respect to  ${W_i}_{i \in I}$ .

**Corollary 2.10** For every g-Bessel sequence  $\Lambda = {\Lambda_i}_{i \in I}$  with Bessel bound B < 1 and each Parseval g-frame  $\Xi = {\Xi_i}_{i \in I}$  for  $\mathcal{H}$  with respect to  ${W_i}_{i \in I}$ , there exists g-Bessel sequence  ${\Gamma_i}_{i \in I}$  for  $\mathcal{H}$  with respect to  ${W_i}_{i \in I}$  such that  ${\Lambda_i + \Xi_i}_{i \in I}$  and  ${\Gamma_i}_{i \in I}$  are dual g-frames for  $\mathcal{H}$  with respect to  ${W_i}_{i \in I}$ .

**Corollary 2.11** For every g-Bessel sequence  $\{\Lambda_i\}_{i \in I}$  for  $\mathcal{H}$  with respect to  $\{W_i\}_{i \in I}$  there exist g-Bessel sequence  $\{\Gamma_i\}_{i \in I}$  and a tight g-frame  $\{\Xi_i\}_{i \in I}$  for  $\mathcal{H}$  with respect to  $\{W_i\}_{i \in I}$  such that  $\{\Lambda_i + \Xi_i\}_{i \in I}$  and  $\{\Gamma_i\}_{i \in I}$  are dual g-frames for  $\mathcal{H}$  with respect to  $\{W_i\}_{i \in I}$ .

#### 3. Conclusions

In this paper, using of g-frame algorithm we obtain an approximation of a bounded linear operator  $U(\mathcal{H}, \mathcal{K})$ . We also show that the sum of any Bessel sequence with Bessel bound less than one with a Parseval frame is a frame for  $\mathcal{H}$  and we prove that a Bessel sequence is an inner summand of a frame. The important result of this paper changing every Bessel sequence to a dual frame by summing it with any Parseval frame.

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