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## G-Frames, g-orthonormal bases and g-Riesz bases

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**Abstract.** *G*-Frames in Hilbert spaces are a redundant set of operators which yield a representation for each vector in the space. In this paper we investigate the connection between *g*-frames, *g*-orthonormal bases and *g*-Riesz bases. We show that a family of bounded operators is a *g*-Bessel sequences if and only if the Gram matrix associated to its defines a bounded operator.

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# 1. Introduction

Let  $\mathcal{H}, \mathcal{K}$  be separable Hilbert spaces, let I, J and every  $J_i$  denote the countable (or finite) index sets. Let  $\{W_j\}_{j \in J}$  is a sequence of closed subspaces of  $\mathcal{K}$  and let  $B(\mathcal{H}, W_j)$  denote the algebra of all bounded linear operators from  $\mathcal{H}$  to  $W_j$ . Recall that a family  $\mathcal{F} = \{f_j\}_{j \in J}$  is called a frame for  $\mathcal{H}$  if there exist constants  $0 < A \leq B < \infty$  such that,

$$A||f||^2 \leqslant \sum_{j \in J} |\langle f, f_j \rangle|^2 \leqslant B||f||^2 \quad \forall f \in \mathcal{H}.$$
 (1)

Frames have many nice properties which make them very useful in the characterization of function spaces, signal processing and many other fields. Gabor [5], in 1946 introduced a technique for signal processing which eventually led to wavelet theory. Later in 1952, Duffin and Schaeffer [3] in the context of nonharmonic Fourier series introduced

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frame theory for Hilbert spaces. In 1986, Daubechies, Grassman and Meyer [2] showed that Duffin and Schaeffers definition was an abstraction of Gabors concept. Sun in [6] introduced g-frames and g-Riesz bases in a complex Hilbert space and discussed some properties of them. We refer to [1, 4, 7] for an introduction to the frames and g-frames and its applications.

**Definition 1.1** A family  $\Lambda = \{\Lambda_j \in B(\mathcal{H}, W_j) | j \in J\}$  is called a generalized frame, or simply a g-frame for  $\mathcal{H}$  with respect to  $\{W_i\}_{i \in J}$  if there are two positive constant C and D such that

$$C\|f\|^2 \leqslant \sum_{j \in J} \|\Lambda_j f\|^2 \leqslant D\|f\|^2 \quad \forall f \in \mathcal{H}.$$
 (2)

The real numbers  $0 < C \leq D < \infty$  are called the lower and upper g-frame bounds, respectively. We call this family a g-frame for  $\mathcal{H}$  with respect to  $\mathcal{K}$  whenever  $\mathcal{K} = W_j$ for all  $j \in J$ . The family  $\Lambda$  is called a C-tight g-frame if C = D and if C = D = 1, it's called a Parseval g-frame, the sup{ $rank(\Lambda_j) : j \in J$ } is called the multiplicity of the g-frame. If  $\|\Lambda_i\| = \|\Lambda_i\| = \lambda$  for all  $i, j \in J$ , then the g-frame is called  $\lambda$ -uniform. If we only have the upper bound, we call  $\Lambda$  a g-Bessel sequence for  $\mathcal{H}$  with respect to  $\{W_j\}_{j\in J}$ with g-Bessel bound D. The family  $\Lambda$  is called

- (i) A g-complete set for  $\mathcal{H}$  with respect to  $\{W_j\}_{j\in J}$  if  $\mathcal{H} = \overline{\operatorname{span}}\{\Lambda_j^*(W_j)\}_{j\in J}$ . (ii) A g-orthonormal system for  $\mathcal{H}$  with respect to  $\{W_j\}_{j\in J}$ , if:

$$<\Lambda_i^*g, \Lambda_j^*g'>=\delta_{ij}< g, g'> \quad \forall i, j \in J, g \in W_i, g' \in W_j.$$

(*iii*) A g-orthonormal basis for  $\mathcal{H}$  with respect to  $\{W_j\}_{j\in J}$ , if it is a g-orthonormal system for  $\mathcal{H}$  with respect to  $\{W_j\}_{j\in J}$  and  $\{\Lambda_j^*(e_{ij})\}_{j\in J, i\in J_j}$  is a basis for  $\mathcal{H}$ , where  $\{e_{ij}\}_{i\in J_j}$  is an orthonormal basis for  $W_j$  for all  $j\in J$ .

**Notation 1.2** Let  $\Lambda = {\Lambda_j}_{j \in J}$  be a g-frame for  $\mathcal{H}$  with respect to  ${W_j}_{j \in J}$ . The representation space associated to  $\Lambda$  denotes by

$$\left(\sum_{j\in J} \oplus W_j\right)_{\ell^2} = \left\{\{g_j\}_{j\in J} | g_j \in W_j \text{ and } \sum_{j\in J} \|g_j\|^2 < \infty\right\}.$$
(3)

which is a Hilbert space with inner product as follows:

$$<\{g_j\}_{j\in J}, \{g'_k\}_{k\in J}>=\sum_{j\in J}< g_j, g'_j> \quad \forall \{g_j\}_{j\in J}, \{g'_j\}_{j\in J}\in \big(\sum_{j\in J}\oplus W_j\big)_{\ell^2}$$

Moreover, if  $\{e_{ij}\}_{i \in J_j}$  is an orthonormal basis for  $W_j$  for all  $j \in J$ , then  $\{u_{ij}\}_{j \in J, i \in J_j}$  is called the standard orthonormal basis of  $\left(\sum_{j\in J} \oplus W_j\right)_{\ell^2}$  where  $u_{ij} = \{\delta_{kj}e_{ij}\}_{k\in J}$  and  $\delta_{kj}$ is the Kronecker delta.

**Definition 1.3** The synthesis operator of a *g*-frame  $\Lambda = {\Lambda_j}_{j \in J}$  is defined by

$$\Theta_{\Lambda} : \left(\sum_{j \in J} \oplus W_j\right)_{\ell^2} \longrightarrow \mathcal{H} \quad \text{with} \quad \Theta_{\Lambda}(\{g_j\}_{j \in J}) = \sum_{j \in J} \Lambda_j^* g_j \tag{4}$$

The associated adjoint operator given by

$$\Theta^*_{\Lambda} : \mathcal{H} \longrightarrow \left( \sum_{j \in J} \oplus W_j \right)_{\ell^2} \quad \text{with} \quad \Theta^*_{\Lambda}(f) = \{\Lambda_j f\}_{j \in J}.$$
(5)

is called the analysis operator. By composing  $\Theta_{\Lambda}$  and  $\Theta^*_{\Lambda}$  we obtain the g-frame operator

$$S_{\Lambda} : \mathcal{H} \longrightarrow \mathcal{H} \quad \text{with} \quad S_{\Lambda}(f) = \Theta_{\Lambda} \Theta^*_{\Lambda}(f) = \sum_{j \in J} \Lambda^*_j \Lambda_j(f)$$
 (6)

which is a bounded, invertible, and positive operator. This provides the reconstruction formula

$$f = \sum_{j \in J} \Lambda_j^* \tilde{\Lambda}_j f = \sum_{j \in J} \tilde{\Lambda}_j^* \Lambda_j f \tag{7}$$

where  $\tilde{\Lambda}_j = \Lambda_j S_{\Lambda}^{-1}$ . The family  $\tilde{\Lambda} = {\{\tilde{\Lambda}_j\}_{j \in J}}$  is also a *g*-frame for  $\mathcal{H}$  with respect to  $\{W_j\}_{j \in J}$  with *g*-frame bounds  $\frac{1}{D}$  and  $\frac{1}{C}$  respectively.

The well-known relations between a frame and associated analysis and synthesis operator also holds in *g*-frames situation.

**Theorem 1.4** Let  $\Lambda_j \in B(\mathcal{H}, W_j)$  for all  $j \in J$ . Then the following are equivalent:

- (i)  $\Lambda = {\Lambda_j}_{j \in J}$  is a g-frame for  $\mathcal{H}$  with respect to  ${W_j}_{j \in J}$ .
- (ii) The synthesis operator  $\Theta_{\Lambda}$  is bounded, linear and onto.
- (*iii*) The analysis operator  $\Theta^*_{\Lambda}$  is injective with closed range.

**Proof.** This claim holds in an analogous way as in frame theory.

A family  $\Lambda = {\Lambda_j}_{j \in J}$  is called a *g*-frame sequence for  $\mathcal{H}$  with respect to  ${W_j}_{j \in J}$  if  $\Lambda$  is a *g*-frame for  $\overline{\text{span}}{\Lambda_j^*(W_j)}_{j \in J}$  with respect to  ${W_j}_{j \in J}$ . The definition shows that if  $\Lambda$  is a *g*-frame for  $\mathcal{H}$  with respect to  ${W_j}_{j \in J}$  then  $\Lambda$  is *g*-complete set for  $\mathcal{H}$  with respect to  ${W_j}_{j \in J}$ . Theorem 1.4 leads to a statement about *g*-frame sequence.

**Corollary 1.5** A sequence  $\Lambda = {\Lambda_j}_{j \in J}$  is a *g*-frame sequence for  $\mathcal{H}$  with respect to  ${W_j}_{j \in J}$  if and only if

$$\Theta_{\Lambda}: \big(\sum_{j\in J} \oplus W_j\big)_{\ell^2} \longrightarrow \mathcal{H} \quad , \quad \Theta_{\Lambda}(\{g_j\}_{j\in J}) = \sum_{j\in J} \Lambda_j^* g_j,$$

is a well-defined bounded operator with closed range.

**Proof.** This follows immediately from Theorem 1.4.

### 2. G-Frames, g-orthonormal bases and g-Riesz bases

If  $\Lambda = {\Lambda_j}_{j \in J}$  is a g-Bessel sequence for  $\mathcal{H}$  with respect to  ${W_j}_{j \in J}$ , then the Gram matrix associated to  $\Lambda$  is defined by

$$\Theta^*_{\Lambda}\Theta_{\Lambda} = \{ <\Lambda^*_j e_{ij}, \Lambda^*_n e_{mn} > \}_{j,n \in J, i \in J_j, m \in J_n}.$$
(8)

**Theorem 2.1** Let  $\Lambda_j \in B(\mathcal{H}, W_j)$  for all  $j \in J$ , then the following are equivalent:

- (i)  $\Lambda = {\Lambda_j}_{j \in J}$  is g-Bessel sequence with bound B for  $\mathcal{H}$  with respect to  ${W_j}_{j \in J}$ .
- (*ii*) The Gram matrix associated to  $\Lambda$  defines a bounded operator on  $\left(\sum_{j\in J} \oplus W_j\right)_{\ell^2}$ , with norm at most B.

**Proof.**  $(i) \Rightarrow (ii)$  Let  $\{g_j\}_{j \in J} \in \left(\sum_{j \in J} \oplus W_j\right)_{\ell^2}$ , we show that  $\Theta_{\Lambda}(\{g_j\}_{j \in J})$  is well-defined. Fix  $I \subseteq J$  with  $|I| < \infty$ , we have

$$\begin{split} \left\| \sum_{i \in I} \Lambda_{i}^{*} g_{i} \right\|^{2} &= \sup_{\|f\|=1} \left| < f, \sum_{i \in I} \Lambda_{i}^{*} g_{i} > \right|^{2} \\ &= \sup_{\|f\|=1} \left| \sum_{i \in I} < \Lambda_{i} f, g_{i} > \right|^{2} \\ &\leqslant \sup_{\|f\|=1} \left( \sum_{i \in I} \|\Lambda_{i} f\|^{2} \right) \left( \sum_{i \in I} \|g_{i}\|^{2} \right) \leqslant B \sum_{i \in I} \|g_{i}\|^{2} \end{split}$$

It follows that  $\sum_{j\in J} \Lambda_j^* g_j$  is weakly unconditionally Cauchy and hence unconditionally convergent in  $\mathcal{H}$ . Thus  $\Theta_{\Lambda}(\{g_j\}_{j\in J})$  is well-defined. Clearly  $\Theta_{\Lambda}^*\Theta_{\Lambda}$  is bounded and  $\|\Theta_{\Lambda}^*\Theta_{\Lambda}\| \leq B$ .

$$\sum_{j \in J} \|\sum_{k \in J} \Lambda_j \Lambda_k^* g_k\|^2 \leqslant B^2 \sum_{j \in J} \|g_j\|^2.$$

Given arbitrary  $I \subseteq J$  with  $|I| < \infty$ , we have

$$\begin{split} \left\| \sum_{i \in I} \Lambda_i^* g_i \right\|^4 &= \left| < \sum_{i \in I} \Lambda_i^* g_i, \sum_{k \in I} \Lambda_k^* g_k > \right|^2 \\ &= \left| \sum_{i \in I} < g_i, \sum_{k \in I} \Lambda_i \Lambda_k^* g_k > \right|^2 \\ &\leqslant \left( \sum_{i \in I} \|g_i\|^2 \right) \left( \sum_{i \in I} \|\sum_{k \in I} \Lambda_i \Lambda_k^* g_k \|^2 \right) \\ &\leqslant B^2 \left( \sum_{i \in I} \|g_i\|^2 \right)^2. \end{split}$$

It follows that  $\Theta_{\Lambda}(\{g_j\}_{j\in J}) = \sum_{j\in J} \Lambda_j^* g_j$  is convergent and  $\|\Theta_{\Lambda}\| \leq \sqrt{B}$ . Hence for all  $f \in \mathcal{H}$  we obtain

$$\sum_{j\in J} \|\Lambda_j f\|^2 = \|\Theta^*_{\Lambda} f\|^2 \leqslant B \|f\|^2.$$

**Theorem 2.2** Let  $\{\Lambda_j\}_{j\in J}$  and  $\{\Gamma_j\}_{j\in J}$  be g-Bessel sequences for  $\mathcal{H}$  with respect to  $\{W_j\}_{j\in J}, \{V_j\}_{j\in J}$  then the series  $\sum_{j\in J} \Gamma_j^* \Lambda_j f$  converges unconditionally for all  $f \in \mathcal{H}$ .

**Proof.** Since  $\{\Lambda_j\}_{j\in J}$  is a g-Bessel sequence for  $\mathcal{H}$  with respect to  $\{W_j\}_{j\in J}$ , hence  $\{\Lambda_j f\}_{j\in J} \in \ell^2(\mathcal{K}, J)$  for all  $f \in \mathcal{H}$ . Fix  $I \subset J$  with  $|I| < \infty$ , and let B be the g-Bessel

bound of  $\{\Gamma_i\}_{i \in J}$ . Then for all  $f \in \mathcal{H}$  we have

$$\begin{split} \left|\sum_{i\in I}\Gamma_i^*\Lambda_i f\right\|^2 &= \sup_{\|g\|=1} \left|<\sum_{i\in I}\Gamma_i^*\Lambda_i f, g>\right|^2\\ &= \sup_{\|g\|=1} \left|\sum_{i\in I}<\Lambda_i f, \Gamma_i g>\right|^2\\ &\leqslant \sup_{\|g\|=1} \left(\sum_{i\in I}\|\Lambda_i f\|^2\right) \left(\sum_{i\in I}\|\Gamma_i g\|^2\right)\\ &\leqslant B\sum_{i\in I}\|\Lambda_i f\|^2. \end{split}$$

It follows that  $\sum_{j \in J} \Gamma_j^* \Lambda_j f$  is weakly unconditionally Cauchy and hence unconditionally convergent in  $\mathcal{H}$ .

**Proposition 2.3** Let  $\{\Xi_j\}_{j\in J}, \{\Xi'_j\}_{j\in J}$  be *g*-orthonormal bases for  $\mathcal{H}, \mathcal{U}$  with respect to  $\{W_j\}_{j\in J}$  respectively, and let  $T : \mathcal{H} \to \mathcal{U}$  be a bounded linear operator such that  $\Xi'_j T = \Xi_j$  for all  $j \in J$ . Then T is a unitary operator.

**Proof.** For each  $f \in \mathcal{H}$  we have

$$||Tf||^{2} = \sum_{j \in J} ||\Xi'_{j}Tf||^{2} = \sum_{j \in J} ||\Xi_{j}f||^{2} = ||f||^{2}$$

which implies that T is an isometry operator. Further for every  $g \in \mathcal{U}$  we compute

$$T^*g = \sum_{j \in J} T^* \Xi'_j \Xi'_j g = \sum_{j \in J} \Xi^*_j \Xi'_j g$$

This yields

$$\|T^*g\|^2 = <\sum_{j \in J} \Xi_j^* \Xi_j'g, \sum_{k \in J} \Xi_k^* \Xi_k'g >$$
  
$$= \sum_{j \in J} \sum_{k \in J} \delta_{jk} < \Xi_j'g, \Xi_k'g >$$
  
$$= \sum_{j \in J} \|\Xi_j'g\|^2 = \|g\|^2.$$

Thus T is a co-isometry, which finishes the proof.

**Theorem 2.4** Let  $\Xi = \{\Xi_j\}_{j \in J}$  be a g-orthonormal system for  $\mathcal{H}$  with respect to  $\{W_i\}_{i \in J}$ . Then the following conditions are equivalent:

- (i)  $\Xi$  is a g-orthonormal basis for  $\mathcal{H}$  with respect to  $\{W_j\}_{j\in J}$ .

- (i)  $f = \sum_{j \in J} \Xi_j^* \Xi_j f$   $\forall f \in \mathcal{H}.$ (ii)  $\|f\|^2 = \sum_{j \in J} \|\Xi_j^* \Xi_j f\|^2$   $\forall f \in \mathcal{H}.$ (iv)  $\|f\|^2 = \sum_{j \in J} \|\Xi_j f\|^2$   $\forall f \in \mathcal{H}.$ (v)  $\langle f, g \rangle = \sum_{j \in J} \langle \Xi_j f, \Xi_j g \rangle$   $\forall f, g \in \mathcal{H}.$ (vi) If  $\Xi_j f = 0$  for all  $j \in J$  then f = 0.

(vii)  $\mathcal{H} = \overline{\operatorname{span}} \{ \Xi_j^* \Xi_j(\mathcal{H}) \}_{j \in J}.$ (viii)  $\mathcal{H} = \overline{\operatorname{span}} \{ \Xi_j^*(W_j) \}_{j \in J}.$ 

**Proof.**  $(i) \Rightarrow (ii)$  By the assumptions  $\{\Xi_j^* e_{ij}\}_{j \in J, i \in J_j}$  is an orthonormal basis for  $\mathcal{H}$ . Therefore for all  $f \in \mathcal{H}$  we have

$$<\sum_{j\in J} \Xi_j^* \Xi_j f, f> = \sum_{j\in J} \|\Xi_j f\|^2 = \sum_{j\in J} \sum_{i\in J_j} | |^2 = \|f\|^2 = .$$

From this (ii) follows.

 $(ii) \Rightarrow (iii)$  Since  $\Xi$  is a g-orthonormal system for  $\mathcal{H}$  with respect to  $\{W_j\}_{j \in J}$ , hence for all  $f \in \mathcal{H}$  and  $j \in J$  we have  $(\Xi_j^* \Xi_j)^2 f = \Xi_j^* \Xi_j f$ . This yields

$$||f||^2 = <\sum_{j\in J} \Xi_j^* \Xi_j f, f > = \sum_{j\in J} ||\Xi_j^* \Xi_j f||^2,$$

which implies (*iii*). The implications (*iii*)  $\Rightarrow$  (*iv*)  $\Rightarrow$  (*v*)  $\Rightarrow$  (*vi*) are clear. To prove (*vi*)  $\Rightarrow$  (*vii*) assume that  $f \perp \overline{\text{span}} \{\Xi_j^* \Xi_j(\mathcal{H})\}_{j \in J}$ , hence  $\|\Xi_j f\|^2 = \langle f, \Xi_j^* \Xi_j f \rangle = 0$  and so  $\Xi_j f = 0$  for all  $j \in J$ , it shows that f = 0 and thus (*vii*) follows. Also the implication (*vii*)  $\Rightarrow$  (*viii*) is obvious. To prove (*viii*)  $\Rightarrow$  (*i*) suppose that

$$\mathcal{A} = \Big\{ f \in \mathcal{H} : \sum_{j \in J} \Xi_j^* \Xi_j f = f \Big\}.$$

It is obvious that  $\mathcal{A}$  is a closed subspace of  $\mathcal{H}$ . It follows by assumption that for every  $i, j \in J$  and  $f \in \mathcal{H}$  we have  $\langle \Xi_j^* \Xi_j f, \Xi_i^* \Xi_i f \rangle = \delta_{ij} \langle \Xi_j f, \Xi_i f \rangle$ , which implies that  $\Xi_j^* \Xi_j \Xi_i^* \Xi_i = \delta_{ij} \Xi_j^* \Xi_i$ . Thus  $\Xi_j^* \Xi_j f \in \mathcal{A}$ . Now suppose  $f \in \mathcal{A}^{\perp}$ , then for all  $j \in J$  we compute  $\|\Xi_j f\|^2 = \langle \Xi_j^* \Xi_j f, f \rangle = 0$ , hence  $\Xi_j f = 0$ . Let  $j \in J$  and  $g \in W_j$  then  $\langle f, \Xi_j^* g \rangle = \langle \Xi_j f, g \rangle = 0$ . It follows that  $f \perp \overline{\text{span}} \{\Xi_j^* (W_j)\}_{j \in J} = \mathcal{H}$  and so f = 0, therefore  $\mathcal{H} = \mathcal{A}$ . For every  $f \in \mathcal{H}$  we further have

$$f = \sum_{j \in J} \Xi_j^* \Xi_j f = \sum_{j \in J} \sum_{i \in J_j} \langle f, \Xi_j^* e_{ij} \rangle \Xi_j^* e_{ij}$$

From this the result follows.

Let  $\{Z_j\}_{j\in J}$  be a family of closed subspaces in  $\mathcal{H}$ , then  $\{Z_j\}_{j\in J}$  is called an orthonormal fusion basis for  $\mathcal{H}$  if  $\mathcal{H} = \bigoplus_{j\in J} Z_j$ .

**Corollary 2.5** Let  $\Xi = \{\Xi_j\}_{j \in J}$  be a *g*-orthonormal basis for  $\mathcal{H}$  with respect to  $\{W_j\}_{j \in J}$ , and let  $V_j = \Xi_j^* \Xi_j(\mathcal{H})$  for all  $j \in J$ . Then  $\{V_j\}_{j \in J}$  is an orthonormal fusion basis for  $\mathcal{H}$ .

**Proof.** This claim follows immediately from the fact that for each  $i, j \in J$  we have

$$\Xi_j^* \Xi_j \Xi_i^* \Xi_i = \delta_{ij} \Xi_j^* \Xi_i$$

**Corollary 2.6** Let  $\Xi = \{\Xi_j\}_{j \in J}$  be a Parseval *g*-frame of co-isometries for  $\mathcal{H}$  with respect to  $\{W_j\}_{j \in J}$ . Then  $\Xi$  is a *g*-orthonormal basis for  $\mathcal{H}$  with respect to  $\{W_j\}_{j \in J}$ .

**Proof.** Fix  $i \in J$ , since  $\Xi$  is a Parseval g-frame for  $\mathcal{H}$  with respect to  $\{W_j\}_{j\in J}$ . Thus for every  $g \in W_i$ , we have

$$\|\Xi_i^*g\|^2 = \sum_{j \in J} \|\Xi_j \Xi_i^*g\|^2 = \|\Xi_i^*g\|^2 + \sum_{j \in J \atop j \neq i} \|\Xi_j \Xi_i^*g\|^2.$$

Hence  $\sum_{\substack{j \in J \\ j \neq i}} \|\Xi_j \Xi_i^* g\|^2 = 0$ . So  $\Xi_j \Xi_i^* g = 0$  for all  $j \neq i$ . This shows that  $\Xi = \{\Xi_j\}_{j \in J}$  is a *g*-orthonormal system for  $\mathcal{H}$  with respect to  $\{W_j\}_{j \in J}$ . Now the result follows from the Theorem 2.4.

**Definition 2.7** Let  $\{\Lambda_j\}_{j\in J}$  and  $\{\Gamma_j\}_{j\in J}$  be sequences for  $\mathcal{H}$  with respect to  $\{W_j\}_{j\in J}$ and  $\{V_j\}_{j\in J}$  respectively. Then

(i)  $\{\Lambda_j\}_{j\in J}$  and  $\{\Gamma_j\}_{j\in J}$  are said to be biorthogonal for  $\mathcal{H}$  with respect to  $\{W_j\}_{j\in J}, \{V_j\}_{j\in J}$  if

$$<\Lambda_i^*g, \Gamma_j^*g'>=\delta_{ij}< g, g'> \quad \forall i, j \in J, g \in W_i, g' \in V_j.$$

(*ii*)  $\{\Lambda_j\}_{j\in J}$  is called a g-Riesz basis for  $\mathcal{H}$  with respect to  $\{W_j\}_{j\in J}$  if it's g-complete set for  $\mathcal{H}$  with respect to  $\{W_j\}_{j\in J}$  and there exist constants  $0 < A \leq B < \infty$  such that for any finite subset  $I \subset J$  and  $g_i \in W_i, (i \in I)$  we have

$$A\sum_{i\in I} \|g_i\|^2 \leqslant \|\sum_{i\in I} \Lambda_i^* g_i\|^2 \leqslant B\sum_{i\in I} \|g_i\|^2.$$
(9)

**Theorem 2.8** Let  $\Lambda = {\Lambda_j}_{j \in J}$  be a *g*-Riesz basis for  $\mathcal{H}$  with respect to  ${W_j}_{j \in J}$ , then there exists a sequence  $\Gamma = {\Gamma_j}_{j \in J}$  for  $\mathcal{H}$  with respect to  ${W_j}_{j \in J}$  such that

$$f = \sum_{j \in J} \Gamma_j^* \Lambda_j f \quad \forall f \in \mathcal{H}.$$
 (10)

 $\Gamma$  is also a g-Riesz basis for  $\mathcal{H}$  with respect to  $\{W_j\}_{j\in J}$  and  $\Lambda, \Gamma$  are biorthogonal for  $\mathcal{H}$  with respect to  $\{W_j\}_{j\in J}$ . Moreover the series (10) converges unconditionally for all  $f \in \mathcal{H}$ .

**Proof.** By [6, Corollary 3.4] there is a *g*-orthonormal basis  $\{\Xi_j\}_{j\in J}$  for  $\mathcal{H}$  with respect to  $\{W_j\}_{j\in J}$  and a bounded invertible operator T on  $\mathcal{H}$  such that  $\Lambda = \{\Lambda_j\}_{j\in J} = \{\Xi_j T\}_{j\in J}$ . Put  $\Gamma_j = \Xi_j (T^{-1})^*$  for all  $j \in J$ . Obviously  $\Gamma = \{\Gamma_j\}_{j\in J}$  is a *g*-Riesz basis for  $\mathcal{H}$  with respect to  $\{W_j\}_{j\in J}$  and we have

$$<\Lambda_i^*g, \Gamma_j^*g'> = < T^*\Xi_j^*g, T^{-1}\Xi_j^*g'> = \delta_{ij} < g, g'> \quad \forall i, j \in J, g \in W_i, g' \in W_j$$

which implies that  $\Lambda, \Gamma$  are biorthogonal for  $\mathcal{H}$  with respect to  $\{W_j\}_{j \in J}$ . Moreover, for all  $f \in \mathcal{H}$  we observe that

$$\sum_{j\in J} \Gamma_j^* \Lambda_j f = \sum_{j\in J} T^{-1} \Xi_j^* \Xi_j T f = T^{-1} T f = f.$$

Since every g-Riesz basis is a g-Bessel sequence thus, convergent unconditionally of the above series follows by Theorem 2.2.  $\blacksquare$ 

Let  $\Lambda = {\Lambda_j}_{j \in J}$  and  $\Gamma = {\Gamma_j}_{j \in J}$  be g-Bessel sequences for  $\mathcal{H}$  with respect to  ${W_j}_{j \in J}, {V_j}_{j \in J}$  respectively. Then  $\Gamma$  is called a dual g-frame of  $\Lambda$  for  $\mathcal{H}$  with respect to  ${V_j}_{j \in J}, {W_j}_{j \in J}$  if

$$f = \sum_{j \in J} \Gamma_j^* \Lambda_j f \qquad \forall f \in \mathcal{H}.$$

It is easy to check that  $\Gamma$  is a dual g-frame of  $\Lambda$  for  $\mathcal{H}$  with respect to  $\{V_j\}_{j\in J}, \{W_j\}_{j\in J}$ if and only if  $\Theta_{\Gamma}\Theta^*_{\Lambda} = Id_{\mathcal{H}}$ , in this case  $\Lambda$  and  $\Gamma$  are also g-frames for  $\mathcal{H}$  with respect to  $\{W_j\}_{j\in J}, \{V_j\}_{j\in J}$ . Since  $\Theta_{\Lambda}\Theta^*_{\Gamma} = Id_{\mathcal{H}}$ , hence  $\Lambda$  is also a dual g-frame of  $\Gamma$  for  $\mathcal{H}$  with respect to  $\{W_j\}_{j\in J}, \{V_j\}_{j\in J}, \{V_j\}_{j\in J}$ .

The following example shows that the dual g-frame of a g-orthonormal basis is not unique.

**Example 2.9** Fix some  $n \in \mathbb{N}, 1 \leq j \leq n$  and define  $W_j \subset \mathbb{C}^{n+1}$ , by  $W_j = \operatorname{span}\{\sum_{k=1}^{j+1} e_k\}$ , where  $\{e_i\}_{i=1}^{n+1}$  is the standard orthonormal basis for  $\mathbb{C}^{n+1}$ . Also define

$$\Xi_j: \mathbb{C}^n \to W_j \quad \text{with} \quad \Xi_j(\{z_i\}_{i=1}^n) = \frac{z_j}{\sqrt{j+1}} \sum_{k=1}^{j+1} e_k.$$

Then  $\Xi = \{\Xi_j\}_{j=1}^n$  is a g-orthonormal basis for  $\mathbb{C}^n$  with respect to  $\{W_j\}_{j=1}^n$ . Therefore by Theorem 2.4,  $\Xi$  is a dual g-frame of itself for  $\mathbb{C}^n$  with respect to  $\{W_j\}_{j=1}^n$ . Now if for each  $1 \leq j \leq n$ , we define  $V_j = \operatorname{span}\{e_j\}$  and

$$\Gamma_j : \mathbb{C}^n \to V_j \quad \text{and} \quad \Gamma_j(\{z_i\}_{i=1}^n) = \sqrt{j+1}z_j e_j.$$

Then for all  $z \in \mathbb{C}^n$  we have  $z = \sum_{j=1}^n \Gamma_j^* \Xi_j z$ , that is  $\Gamma = {\Gamma_j}_{j=1}^n$  is a dual *g*-frame of  $\Xi = {\Xi_j}_{j=1}^n$  for  $\mathbb{C}^n$  with respect to  ${V_j}_{j=1}^n$ ,  ${W_j}_{j=1}^n$  respectively.

**Proposition 2.10** Let  $\Lambda = {\Lambda_j}_{j \in J}$  be a *g*-frame for  $\mathcal{H}$  with respect to  ${W_j}_{j \in J}$ , then there exists a *g*-orthonormal basis  ${\Xi_j}_{j \in J}$  for  $(\sum_{j \in J} \oplus W_j)_{\ell^2}$  with respect to  ${W_j}_{j \in J}$  such that  $\Xi_j \Theta_{\Lambda}^* = \Lambda_j$  for all  $j \in J$ .

**Proof.** For all  $j \in J$  define  $\Xi_j : \left(\sum_{j \in J} \oplus W_j\right)_{\ell^2} \to W_j$  by  $\Xi_j(\{g_k\}_{k \in J}) = g_j$ , then  $\Xi_j^* g = \{\delta_{kj} \pi_{W_k} g\}_{k \in J}$  for all  $g \in \mathcal{K}$ , where  $\delta_{kj}$  is the Kronecker delta. First of all,  $\{\Xi_j\}_{j \in J}$  is a *g*-orthonormal system for  $\mathcal{H}$  with respect to  $\{W_j\}_{j \in J}$ . To see this, let  $g \in W_j, g' \in W_i$  and  $i, j \in J$ . Then we have

$$<\Xi_{j}^{*}g, \Xi_{i}^{*}g' > = \sum_{k \in J} \delta_{kj}\delta_{ki} < \pi_{W_{k}}g, \pi_{W_{k}}g' > \\ = \delta_{ji} < \pi_{W_{j}}g, \pi_{W_{j}}g' > = \delta_{ji} < g, g' > .$$

On the other hand for any  $g = \{g_k\}_{k \in J} \in \left(\sum_{j \in J} \oplus W_j\right)_{\ell^2}$  we compute

$$\sum_{j \in J} \|\Xi_j g\|^2 = \sum_{j \in J} \|g_j\|^2 = \|g\|^2.$$

By Theorem 2.4  $\{\Xi_j\}_{j\in J}$  is a *g*-orthonormal basis for  $(\sum_{j\in J} \oplus W_j)_{\ell^2}$  with respect to  $\{W_j\}_{j\in J}$ . It is easy to check that  $\Xi_j \Theta^*_{\Lambda} = \Lambda_j$  for all  $j \in J$ . Now the conclusion follows.

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