*Journal of*
*Linear and Topological Algebra*
*Vol.* 02, *No.* 01, 2013, 25- 33

# $G$-Frames, $g$-orthonormal bases and $g$-Riesz bases

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**Abstract.** $G$-Frames in Hilbert spaces are a redundant set of operators which yield a representation for each vector in the space. In this paper we investigate the connection between $g$-frames, $g$-orthonormal bases and $g$-Riesz bases. We show that a family of bounded operators is a $g$-Bessel sequences if and only if the Gram matrix associated to its defines a bounded operator.



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**Keywords:** $G$-frames, $G$-Bessel sequences, $G$-orthonormal bases.

**2010 AMS Subject Classification**: 42C15, 46C99, 94A12.

## 1. Introduction

Let $\mathcal{H}, \mathcal{K}$ be separable Hilbert spaces, let $I, J$ and every $J_i$ denote the countable (or finite) index sets. Let $\{W_j\}_{j\in J}$ is a sequence of closed subspaces of $\mathcal{K}$ and let $B(\mathcal{H}, W_j)$ denote the algebra of all bounded linear operators from $\mathcal{H}$ to $W_j$. Recall that a family $\mathcal{F} = \{f_j\}_{j\in J}$ is called a frame for $\mathcal{H}$ if there exist constants $0 < A \leqslant B < \infty$ such that,

$$A\|f\|^2 \leqslant \sum_{j\in J} | < f, f_j > |^2 \leqslant B\|f\|^2 \quad \forall f \in \mathcal{H}. \tag{1}$$

Frames have many nice properties which make them very useful in the characterization of function spaces, signal processing and many other fields. Gabor [5], in 1946 introduced a technique for signal processing which eventually led to wavelet theory. Later in 1952, Duffin and Schaeffer [3] in the context of nonharmonic Fourier series introduced

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ISSN: 2252−0201
http://jlta.iauctb.ac.ir

frame theory for Hilbert spaces. In 1986, Daubechies, Grassman and Meyer [2] showed that Duffin and Schaeffers definition was an abstraction of Gabors concept. Sun in [6] introduced g-frames and g-Riesz bases in a complex Hilbert space and discussed some properties of them. We refer to [1, 4, 7] for an introduction to the frames and $g$-frames and its applications.

**Definition 1.1** A family $\Lambda = \{\Lambda_j \in B(\mathcal{H}, W_j)| \ j \in J\}$ is called a generalized frame, or simply a $g$-frame for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$ if there are two positive constant $C$ and $D$ such that

$$C\|f\|^2 \leqslant \sum_{j\in J} \|\Lambda_j f\|^2 \leqslant D\|f\|^2 \quad \forall f \in \mathcal{H}. \tag{2}$$

The real numbers $0 < C \leqslant D < \infty$ are called the lower and upper $g$-frame bounds, respectively. We call this family a $g$-frame for $\mathcal{H}$ with respect to $\mathcal{K}$ whenever $\mathcal{K} = W_j$ for all $j \in J$. The family $\Lambda$ is called a $C$-tight $g$-frame if $C = D$ and if $C = D = 1$, it's called a Parseval $g$-frame, the $\sup\{rank(\Lambda_j) : j \in J\}$ is called the multiplicity of the $g$-frame. If $\|\Lambda_i\| = \|\Lambda_j\| = \lambda$ for all $i, j \in J$, then the $g$-frame is called $\lambda$-uniform. If we only have the upper bound, we call $\Lambda$ a $g$-Bessel sequence for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$ with $g$-Bessel bound $D$. The family $\Lambda$ is called

$(i)$ A $g$-complete set for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$ if $\mathcal{H} = \overline{\text{span}}\{\Lambda_j^*(W_j)\}_{j\in J}$.

$(ii)$ A $g$-orthonormal system for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$, if:

$$< \Lambda_i^* g, \Lambda_j^* g' >= \delta_{ij} < g, g' > \quad \forall i, j \in J, \ g \in W_i, g' \in W_j.$$

$(iii)$ A $g$-orthonormal basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$, if it is a $g$-orthonormal system for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$ and $\{\Lambda_j^*(e_{ij})\}_{j\in J, i\in J_j}$ is a basis for $\mathcal{H}$, where $\{e_{ij}\}_{i\in J_j}$ is an orthonormal basis for $W_j$ for all $j \in J$.

**Notation 1.2** Let $\Lambda = \{\Lambda_j\}_{j\in J}$ be a $g$-frame for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$. The representation space associated to $\Lambda$ denotes by

$$\Big(\sum_{j\in J} \oplus W_j\Big)_{\ell^2} = \Big\{\{g_j\}_{j\in J}| \quad g_j \in W_j \ \text{ and } \ \sum_{j\in J} \|g_j\|^2 < \infty\Big\}. \tag{3}$$

which is a Hilbert space with inner product as follows:

$$< \{g_j\}_{j\in J}, \{g'_k\}_{k\in J} >= \sum_{j\in J} < g_j, g'_j > \quad \forall \{g_j\}_{j\in J}, \{g'_j\}_{j\in J} \in \big(\sum_{j\in J} \oplus W_j\big)_{\ell^2}.$$

Moreover, if $\{e_{ij}\}_{i\in J_j}$ is an orthonormal basis for $W_j$ for all $j \in J$, then $\{u_{ij}\}_{j\in J, i\in J_j}$ is called the standard orthonormal basis of $\big(\sum_{j\in J} \oplus W_j\big)_{\ell^2}$ where $u_{ij} = \{\delta_{kj} e_{ij}\}_{k\in J}$ and $\delta_{kj}$ is the Kronecker delta.

**Definition 1.3** The synthesis operator of a $g$-frame $\Lambda = \{\Lambda_j\}_{j\in J}$ is defined by

$$\Theta_\Lambda : \big(\sum_{j\in J} \oplus W_j\big)_{\ell^2} \longrightarrow \mathcal{H} \quad \text{with} \quad \Theta_\Lambda(\{g_j\}_{j\in J}) = \sum_{j\in J} \Lambda_j^* g_j \tag{4}$$

The associated adjoint operator given by

$$\Theta_\Lambda^* : \mathcal{H} \longrightarrow \big(\sum_{j\in J} \oplus W_j\big)_{\ell^2} \quad \text{with} \quad \Theta_\Lambda^*(f) = \{\Lambda_j f\}_{j\in J}. \tag{5}$$

is called the analysis operator. By composing $\Theta_\Lambda$ and $\Theta_\Lambda^*$ we obtain the $g$-frame operator

$$S_\Lambda : \mathcal{H} \longrightarrow \mathcal{H} \quad \text{with} \quad S_\Lambda(f) = \Theta_\Lambda \Theta_\Lambda^*(f) = \sum_{j\in J} \Lambda_j^* \Lambda_j(f) \tag{6}$$

which is a bounded, invertible, and positive operator. This provides the reconstruction formula

$$f = \sum_{j\in J} \Lambda_j^* \tilde{\Lambda}_j f = \sum_{j\in J} \tilde{\Lambda}_j^* \Lambda_j f \tag{7}$$

where $\tilde{\Lambda}_j = \Lambda_j S_\Lambda^{-1}$. The family $\tilde{\Lambda} = \{\tilde{\Lambda}_j\}_{j\in J}$ is also a $g$-frame for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$ with $g$-frame bounds $\frac{1}{D}$ and $\frac{1}{C}$ respectively.

The well-known relations between a frame and associated analysis and synthesis operator also holds in $g$-frames situation.

**Theorem 1.4** Let $\Lambda_j \in B(\mathcal{H}, W_j)$ for all $j \in J$. Then the following are equivalent:

(*i*) $\Lambda = \{\Lambda_j\}_{j\in J}$ is a $g$-frame for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$.
(*ii*) The synthesis operator $\Theta_\Lambda$ is bounded, linear and onto.
(*iii*) The analysis operator $\Theta_\Lambda^*$ is injective with closed range.

**Proof.** This claim holds in an analogous way as in frame theory. ■

A family $\Lambda = \{\Lambda_j\}_{j\in J}$ is called a $g$-frame sequence for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$ if $\Lambda$ is a $g$-frame for $\overline{\text{span}}\{\Lambda_j^*(W_j)\}_{j\in J}$ with respect to $\{W_j\}_{j\in J}$. The definition shows that if $\Lambda$ is a $g$-frame for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$ then $\Lambda$ is $g$-complete set for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$. Theorem 1.4 leads to a statement about $g$-frame sequence.

**Corollary 1.5** A sequence $\Lambda = \{\Lambda_j\}_{j\in J}$ is a $g$-frame sequence for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$ if and only if

$$\Theta_\Lambda : \big(\sum_{j\in J} \oplus W_j\big)_{\ell^2} \longrightarrow \mathcal{H} \quad , \quad \Theta_\Lambda(\{g_j\}_{j\in J}) = \sum_{j\in J} \Lambda_j^* g_j,$$

is a well-defined bounded operator with closed range.

**Proof.** This follows immediately from Theorem 1.4. ■

## 2. *G*-Frames, *g*-orthonormal bases and *g*-Riesz bases

If $\Lambda = \{\Lambda_j\}_{j\in J}$ is a $g$-Bessel sequence for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$, then the Gram matrix associated to $\Lambda$ is defined by

$$\Theta_\Lambda^* \Theta_\Lambda = \{< \Lambda_j^* e_{ij}, \Lambda_n^* e_{mn} >\}_{j,n\in J, i\in J_j, m\in J_n}. \tag{8}$$

**Theorem 2.1** Let $\Lambda_j \in B(\mathcal{H}, W_j)$ for all $j \in J$, then the following are equivalent:

(*i*) $\Lambda = \{\Lambda_j\}_{j\in J}$ is $g$-Bessel sequence with bound $B$ for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$.
(*ii*) The Gram matrix associated to $\Lambda$ defines a bounded operator on $\big(\sum_{j\in J}\oplus W_j\big)_{\ell^2}$, with norm at most $B$.

**Proof.** $(i) \Rightarrow (ii)$ Let $\{g_j\}_{j\in J} \in \big(\sum_{j\in J}\oplus W_j\big)_{\ell^2}$, we show that $\Theta_\Lambda(\{g_j\}_{j\in J})$ is well-defined. Fix $I \subseteq J$ with $|I| < \infty$, we have

$$\Big\|\sum_{i\in I}\Lambda_i^* g_i\Big\|^2 = \sup_{\|f\|=1}\big| < f, \sum_{i\in I}\Lambda_i^* g_i > \big|^2 = \sup_{\|f\|=1}\big|\sum_{i\in I} < \Lambda_i f, g_i > \big|^2 \leqslant \sup_{\|f\|=1}\Big(\sum_{i\in I}\|\Lambda_i f\|^2\Big)\Big(\sum_{i\in I}\|g_i\|^2\Big) \leqslant B\sum_{i\in I}\|g_i\|^2.$$

It follows that $\sum_{j\in J}\Lambda_j^* g_j$ is weakly unconditionally Cauchy and hence unconditionally convergent in $\mathcal{H}$. Thus $\Theta_\Lambda(\{g_j\}_{j\in J})$ is well-defined. Clearly $\Theta_\Lambda^*\Theta_\Lambda$ is bounded and $\|\Theta_\Lambda^*\Theta_\Lambda\| \leqslant B$.

$(ii) \Rightarrow (i)$ suppose that $\{g_j\}_{j\in J} \in \big(\sum_{j\in J}\oplus W_j\big)_{\ell^2}$, then we have

$$\sum_{j\in J}\|\sum_{k\in J}\Lambda_j\Lambda_k^* g_k\|^2 \leqslant B^2\sum_{j\in J}\|g_j\|^2.$$

Given arbitrary $I \subseteq J$ with $|I| < \infty$, we have

$$\Big\|\sum_{i\in I}\Lambda_i^* g_i\Big\|^4 = \big| < \sum_{i\in I}\Lambda_i^* g_i, \sum_{k\in I}\Lambda_k^* g_k > \big|^2 = \big|\sum_{i\in I} < g_i, \sum_{k\in I}\Lambda_i\Lambda_k^* g_k > \big|^2 \leqslant \Big(\sum_{i\in I}\|g_i\|^2\Big)\Big(\sum_{i\in I}\|\sum_{k\in I}\Lambda_i\Lambda_k^* g_k\|^2\Big) \leqslant B^2\Big(\sum_{i\in I}\|g_i\|^2\Big)^2.$$

It follows that $\Theta_\Lambda(\{g_j\}_{j\in J}) = \sum_{j\in J}\Lambda_j^* g_j$ is convergent and $\|\Theta_\Lambda\| \leqslant \sqrt{B}$. Hence for all $f \in \mathcal{H}$ we obtain

$$\sum_{j\in J}\|\Lambda_j f\|^2 = \|\Theta_\Lambda^* f\|^2 \leqslant B\|f\|^2.$$

■

**Theorem 2.2** Let $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ be $g$-Bessel sequences for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}, \{V_j\}_{j\in J}$ then the series $\sum_{j\in J}\Gamma_j^*\Lambda_j f$ converges unconditionally for all $f \in \mathcal{H}$.

**Proof.** Since $\{\Lambda_j\}_{j\in J}$ is a $g$-Bessel sequence for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$, hence $\{\Lambda_j f\}_{j\in J} \in \ell^2(\mathcal{K}, J)$ for all $f \in \mathcal{H}$. Fix $I \subset J$ with $|I| < \infty$, and let $B$ be the $g$-Bessel bound of $\{\Gamma_j\}_{j\in J}$. Then for all $f \in \mathcal{H}$ we have

$$\begin{aligned}\Big\|\sum_{i\in I}\Gamma_i^*\Lambda_i f\Big\|^2 &= \sup_{\|g\|=1}\Big| < \sum_{i\in I}\Gamma_i^*\Lambda_i f, g > \Big|^2\\ &= \sup_{\|g\|=1}\Big|\sum_{i\in I} < \Lambda_i f, \Gamma_i g > \Big|^2\\ &\leqslant \sup_{\|g\|=1}\Big(\sum_{i\in I}\|\Lambda_i f\|^2\Big)\Big(\sum_{i\in I}\|\Gamma_i g\|^2\Big)\\ &\leqslant B\sum_{i\in I}\|\Lambda_i f\|^2.\end{aligned}$$

It follows that $\sum_{j\in J}\Gamma_j^*\Lambda_j f$ is weakly unconditionally Cauchy and hence unconditionally convergent in $\mathcal{H}$. ■

**Proposition 2.3** Let $\{\Xi_j\}_{j\in J}, \{\Xi'_j\}_{j\in J}$ be $g$-orthonormal bases for $\mathcal{H}$,$\mathcal{U}$ with respect to $\{W_j\}_{j\in J}$ respectively, and let $T : \mathcal{H} \to \mathcal{U}$ be a bounded linear operator such that $\Xi'_j T = \Xi_j$ for all $j \in J$. Then $T$ is a unitary operator.

**Proof.** For each $f \in \mathcal{H}$ we have

$$\|Tf\|^2 = \sum_{j\in J}\|\Xi'_j Tf\|^2 = \sum_{j\in J}\|\Xi_j f\|^2 = \|f\|^2,$$

which implies that $T$ is an isometry operator. Further for every $g \in \mathcal{U}$ we compute

$$T^*g = \sum_{j\in J} T^*\Xi'^*_j\Xi'_j g = \sum_{j\in J}\Xi^*_j\Xi'_j g.$$

This yields

$$\begin{aligned}\|T^*g\|^2 &= < \sum_{j\in J}\Xi^*_j\Xi'_j g, \sum_{k\in J}\Xi^*_k\Xi'_k g >\\ &= \sum_{j\in J}\sum_{k\in J}\delta_{jk} < \Xi'_j g, \Xi'_k g >\\ &= \sum_{j\in J}\|\Xi'_j g\|^2 = \|g\|^2.\end{aligned}$$

Thus $T$ is a co-isometry, which finishes the proof. ■

**Theorem 2.4** Let $\Xi = \{\Xi_j\}_{j\in J}$ be a $g$-orthonormal system for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$. Then the following conditions are equivalent:

(*i*) $\Xi$ is a $g$-orthonormal basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$.
(*ii*) $f = \sum_{j\in J}\Xi^*_j\Xi_j f \qquad \forall f \in \mathcal{H}$.
(*iii*) $\|f\|^2 = \sum_{j\in J}\|\Xi^*_j\Xi_j f\|^2 \qquad \forall f \in \mathcal{H}$.
(*iv*) $\|f\|^2 = \sum_{j\in J}\|\Xi_j f\|^2 \quad \forall f \in \mathcal{H}$.
(*v*) $< f, g >= \sum_{j\in J} < \Xi_j f, \Xi_j g > \qquad \forall f, g \in \mathcal{H}$.
(*vi*) If $\Xi_j f = 0$ for all $j \in J$ then $f = 0$.

(*vii*) $\mathcal{H} = \overline{\text{span}}\{\Xi_j^*\Xi_j(\mathcal{H})\}_{j\in J}$.
(*viii*) $\mathcal{H} = \overline{\text{span}}\{\Xi_j^*(W_j)\}_{j\in J}$.

**Proof.** $(i) \Rightarrow (ii)$ By the assumptions $\{\Xi_j^* e_{ij}\}_{j\in J, i\in J_j}$ is an orthonormal basis for $\mathcal{H}$. Therefore for all $f \in \mathcal{H}$ we have

$$< \sum_{j\in J} \Xi_j^*\Xi_j f, f > = \sum_{j\in J} \|\Xi_j f\|^2 = \sum_{j\in J}\sum_{i\in J_j} | < f, \Xi_j^* e_{ij} > |^2 = \|f\|^2 =< f, f > .$$

From this $(ii)$ follows.

$(ii) \Rightarrow (iii)$ Since $\Xi$ is a $g$-orthonormal system for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$, hence for all $f \in \mathcal{H}$ and $j \in J$ we have $(\Xi_j^*\Xi_j)^2 f = \Xi_j^*\Xi_j f$. This yields

$$\|f\|^2 =< \sum_{j\in J} \Xi_j^*\Xi_j f, f >= \sum_{j\in J} \|\Xi_j^*\Xi_j f\|^2,$$

which implies $(iii)$. The implications $(iii) \Rightarrow (iv) \Rightarrow (v) \Rightarrow (vi)$ are clear. To prove $(vi) \Rightarrow (vii)$ assume that $f \perp \overline{\text{span}}\{\Xi_j^*\Xi_j(\mathcal{H})\}_{j\in J}$, hence $\|\Xi_j f\|^2 =< f, \Xi_j^*\Xi_j f >= 0$ and so $\Xi_j f = 0$ for all $j \in J$, it shows that $f = 0$ and thus $(vii)$ follows. Also the implication $(vii) \Rightarrow (viii)$ is obvious. To prove $(viii) \Rightarrow (i)$ suppose that

$$\mathcal{A} = \Big\{f \in \mathcal{H} : \sum_{j\in J} \Xi_j^*\Xi_j f = f\Big\}.$$

It is obvious that $\mathcal{A}$ is a closed subspace of $\mathcal{H}$. It follows by assumption that for every $i, j \in J$ and $f \in \mathcal{H}$ we have $< \Xi_j^*\Xi_j f, \Xi_i^*\Xi_i f >= \delta_{ij} < \Xi_j f, \Xi_i f >$, which implies that $\Xi_j^*\Xi_j\Xi_i^*\Xi_i = \delta_{ij}\Xi_j^*\Xi_i$. Thus $\Xi_j^*\Xi_j f \in \mathcal{A}$. Now suppose $f \in \mathcal{A}^\perp$, then for all $j \in J$ we compute $\|\Xi_j f\|^2 =< \Xi_j^*\Xi_j f, f >= 0$, hence $\Xi_j f = 0$. Let $j \in J$ and $g \in W_j$ then $< f, \Xi_j^* g >=< \Xi_j f, g >= 0$. It follows that $f \perp \overline{\text{span}}\{\Xi_j^*(W_j)\}_{j\in J} = \mathcal{H}$ and so $f = 0$, therefore $\mathcal{H} = \mathcal{A}$. For every $f \in \mathcal{H}$ we further have

$$f = \sum_{j\in J} \Xi_j^*\Xi_j f = \sum_{j\in J}\sum_{i\in J_j} < f, \Xi_j^* e_{ij} > \Xi_j^* e_{ij}.$$

From this the result follows. ■

Let $\{Z_j\}_{j\in J}$ be a family of closed subspaces in $\mathcal{H}$, then $\{Z_j\}_{j\in J}$ is called an orthonormal fusion basis for $\mathcal{H}$ if $\mathcal{H} = \bigoplus_{j\in J} Z_j$.

**Corollary 2.5** Let $\Xi = \{\Xi_j\}_{j\in J}$ be a $g$-orthonormal basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$, and let $V_j = \Xi_j^*\Xi_j(\mathcal{H})$ for all $j \in J$. Then $\{V_j\}_{j\in J}$ is an orthonormal fusion basis for $\mathcal{H}$.

**Proof.** This claim follows immediately from the fact that for each $i, j \in J$ we have

$$\Xi_j^*\Xi_j\Xi_i^*\Xi_i = \delta_{ij}\Xi_j^*\Xi_i.$$

■

**Corollary 2.6** Let $\Xi = \{\Xi_j\}_{j\in J}$ be a Parseval $g$-frame of co-isometries for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$. Then $\Xi$ is a $g$-orthonormal basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$.

**Proof.** Fix $i \in J$, since $\Xi$ is a Parseval $g$-frame for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$. Thus for every $g \in W_i$, we have

$$\|\Xi_i^* g\|^2 = \sum_{j\in J} \|\Xi_j \Xi_i^* g\|^2 = \|\Xi_i^* g\|^2 + \sum_{\substack{j\in J\\ j\neq i}} \|\Xi_j \Xi_i^* g\|^2.$$

Hence $\sum_{\substack{j\in J\\ j\neq i}} \|\Xi_j \Xi_i^* g\|^2 = 0$. So $\Xi_j \Xi_i^* g = 0$ for all $j \neq i$. This shows that $\Xi = \{\Xi_j\}_{j\in J}$ is a $g$-orthonormal system for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$. Now the result follows from the Theorem 2.4. ■

**Definition 2.7** Let $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ be sequences for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$ and $\{V_j\}_{j\in J}$ respectively. Then

(*i*) $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ are said to be biorthogonal for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}, \{V_j\}_{j\in J}$ if

$$< \Lambda_i^* g, \Gamma_j^* g' >= \delta_{ij} < g, g' > \qquad \forall i, j \in J,\ g \in W_i, g' \in V_j.$$

(*ii*) $\{\Lambda_j\}_{j\in J}$ is called a $g$-Riesz basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$ if it's $g$-complete set for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$ and there exist constants $0 < A \leqslant B < \infty$ such that for any finite subset $I \subset J$ and $g_i \in W_i, (i \in I)$ we have

$$A \sum_{i\in I} \|g_i\|^2 \leqslant \Big\| \sum_{i\in I} \Lambda_i^* g_i \Big\|^2 \leqslant B \sum_{i\in I} \|g_i\|^2. \tag{9}$$

**Theorem 2.8** Let $\Lambda = \{\Lambda_j\}_{j\in J}$ be a $g$-Riesz basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$, then there exists a sequence $\Gamma = \{\Gamma_j\}_{j\in J}$ for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$ such that

$$f = \sum_{j\in J} \Gamma_j^* \Lambda_j f \qquad \forall f \in \mathcal{H}. \tag{10}$$

$\Gamma$ is also a $g$-Riesz basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$ and $\Lambda, \Gamma$ are biorthogonal for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$. Moreover the series (10) converges unconditionally for all $f \in \mathcal{H}$.

**Proof.** By [6, Corollary 3.4] there is a $g$-orthonormal basis $\{\Xi_j\}_{j\in J}$ for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$ and a bounded invertible operator $T$ on $\mathcal{H}$ such that $\Lambda = \{\Lambda_j\}_{j\in J} = \{\Xi_j T\}_{j\in J}$. Put $\Gamma_j = \Xi_j (T^{-1})^*$ for all $j \in J$. Obviously $\Gamma = \{\Gamma_j\}_{j\in J}$ is a $g$-Riesz basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$ and we have

$$< \Lambda_i^* g, \Gamma_j^* g' >=< T^* \Xi_j^* g, T^{-1} \Xi_j^* g' >= \delta_{ij} < g, g' > \qquad \forall i, j \in J,\ g \in W_i, g' \in W_j$$

which implies that $\Lambda, \Gamma$ are biorthogonal for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$. Moreover, for all $f \in \mathcal{H}$ we observe that

$$\sum_{j\in J} \Gamma_j^* \Lambda_j f = \sum_{j\in J} T^{-1} \Xi_j^* \Xi_j T f = T^{-1} T f = f.$$

Since every $g$-Riesz basis is a $g$-Bessel sequence thus, convergent unconditionally of the above series follows by Theorem 2.2. ■

Let $\Lambda = \{\Lambda_j\}_{j\in J}$ and $\Gamma = \{\Gamma_j\}_{j\in J}$ be $g$-Bessel sequences for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}, \{V_j\}_{j\in J}$ respectively. Then $\Gamma$ is called a dual $g$-frame of $\Lambda$ for $\mathcal{H}$ with respect to $\{V_j\}_{j\in J}, \{W_j\}_{j\in J}$ if

$$f = \sum_{j\in J} \Gamma_j^* \Lambda_j f \qquad \forall f \in \mathcal{H}.$$

It is easy to check that $\Gamma$ is a dual $g$-frame of $\Lambda$ for $\mathcal{H}$ with respect to $\{V_j\}_{j\in J}, \{W_j\}_{j\in J}$ if and only if $\Theta_\Gamma \Theta_\Lambda^* = Id_{\mathcal{H}}$, in this case $\Lambda$ and $\Gamma$ are also $g$-frames for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}, \{V_j\}_{j\in J}$. Since $\Theta_\Lambda \Theta_\Gamma^* = Id_{\mathcal{H}}$, hence $\Lambda$ is also a dual $g$-frame of $\Gamma$ for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}, \{V_j\}_{j\in J}$.

The following example shows that the dual $g$-frame of a $g$-orthonormal basis is not unique.

***Example* 2.9** Fix some $n \in \mathbb{N}, 1 \leqslant j \leqslant n$ and define $W_j \subset \mathbb{C}^{n+1}$, by $W_j = \text{span}\{\sum_{k=1}^{j+1} e_k\}$, where $\{e_i\}_{i=1}^{n+1}$ is the standard orthonormal basis for $\mathbb{C}^{n+1}$. Also define

$$\Xi_j : \mathbb{C}^n \to W_j \quad \text{with} \quad \Xi_j(\{z_i\}_{i=1}^n) = \frac{z_j}{\sqrt{j+1}} \sum_{k=1}^{j+1} e_k.$$

Then $\Xi = \{\Xi_j\}_{j=1}^n$ is a $g$-orthonormal basis for $\mathbb{C}^n$ with respect to $\{W_j\}_{j=1}^n$. Therefore by Theorem 2.4, $\Xi$ is a dual $g$-frame of itself for $\mathbb{C}^n$ with respect to $\{W_j\}_{j=1}^n$. Now if for each $1 \leqslant j \leqslant n$, we define $V_j = \text{span}\{e_j\}$ and

$$\Gamma_j : \mathbb{C}^n \to V_j \quad \text{and} \quad \Gamma_j(\{z_i\}_{i=1}^n) = \sqrt{j+1} z_j e_j.$$

Then for all $z \in \mathbb{C}^n$ we have $z = \sum_{j=1}^n \Gamma_j^* \Xi_j z$, that is $\Gamma = \{\Gamma_j\}_{j=1}^n$ is a dual $g$-frame of $\Xi = \{\Xi_j\}_{j=1}^n$ for $\mathbb{C}^n$ with respect to $\{V_j\}_{j=1}^n, \{W_j\}_{j=1}^n$ respectively.

**Proposition 2.10** Let $\Lambda = \{\Lambda_j\}_{j\in J}$ be a $g$-frame for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$, then there exists a $g$-orthonormal basis $\{\Xi_j\}_{j\in J}$ for $\left(\sum_{j\in J} \oplus W_j\right)_{\ell^2}$ with respect to $\{W_j\}_{j\in J}$ such that $\Xi_j \Theta_\Lambda^* = \Lambda_j$ for all $j \in J$.

**Proof.** For all $j \in J$ define $\Xi_j : \left(\sum_{j\in J} \oplus W_j\right)_{\ell^2} \to W_j$ by $\Xi_j(\{g_k\}_{k\in J}) = g_j$, then $\Xi_j^* g = \{\delta_{kj} \pi_{W_k} g\}_{k\in J}$ for all $g \in \mathcal{K}$, where $\delta_{kj}$ is the Kronecker delta. First of all, $\{\Xi_j\}_{j\in J}$ is a $g$-orthonormal system for $\mathcal{H}$ with respect to $\{W_j\}_{j\in J}$. To see this, let $g \in W_j, g' \in W_i$ and $i, j \in J$. Then we have

$$\begin{aligned} < \Xi_j^* g, \Xi_i^* g' > &= \sum_{k\in J} \delta_{kj} \delta_{ki} < \pi_{W_k} g, \pi_{W_k} g' > \\ &= \delta_{ji} < \pi_{W_j} g, \pi_{W_j} g' >= \delta_{ji} < g, g' > . \end{aligned}$$

On the other hand for any $g = \{g_k\}_{k\in J} \in \left(\sum_{j\in J} \oplus W_j\right)_{\ell^2}$ we compute

$$\sum_{j\in J} \|\Xi_j g\|^2 = \sum_{j\in J} \|g_j\|^2 = \|g\|^2.$$

By Theorem 2.4 $\{\Xi_j\}_{j\in J}$ is a $g$-orthonormal basis for $\left(\sum_{j\in J} \oplus W_j\right)_{\ell^2}$ with respect to $\{W_j\}_{j\in J}$. It is easy to check that $\Xi_j \Theta_\Lambda^* = \Lambda_j$ for all $j \in J$. Now the conclusion follows.■

## Acknowledgements

The author would like to thank from the anonymous reviewers for carefully reading of the manuscript and giving useful comments, which will help to improve the paper.

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