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Solved and unsolved problems in generalized notions of Connes amenability

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Abstract. We survey the recent investigations on (bounded, sequential) approximate Connes amenability and pseudo-Connes amenability for dual Banach algebras. We will discuss the core problems concerning these notions and address the significance of any solutions to them to the development of the field.

Keywords: Connes amenability, pseudo-Connes amenability, approximate Connes amenability, injectivity.

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1. Introduction

The concept of amenability for Banach algebras was introduced by B. E. Johnson in 1972. Since his ground work [13] was published, the notion has proved to be of enormous importance in the theory of Banach algebras, operator algebras and abstract harmonic analysis. It reflects intrinsic features of many types of Banach algebras. For example, the group algebra $L^1(G)$ on a locally compact group G is amenable if and only if G is an amenable group [13]. However, it has been realized that in many instances amenability is too restrictive. Many efforts have been made in the literature to extend or to modify the concept of amenability. Weak amenability and operator amenability were introduced in [1] and [20], respectively. The notions of (bounded, sequential) approximate amenability/contractibility were also introduced in [9,11].

In [15], B.E. Johnson, R. V. Kadison, and J. Ringrose introduced a notion of amenability for von Neumann algebras which modifies Johnson's original definition for general

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Banach algebras [13], in the sense that it takes the dual space structure of von Neumann algebras into account. This notion of amenability later dubbed Connes amenability by A. Ya. Helemskii [12]. In [21], V. Runde extended the notion of Connes amenability to the larger class of dual Banach algebras. Examples of dual Banach algebras besides von Neumann algebras are, for example, the measure algebras $M(G)$ of locally compact groups G. In [23], it is proved that a locally compact group G is amenable if and only if $M(G)$ is Connes amenable, thus showing that the notion of Connes amenability is of interest also outside the framework of von Neumann algebras.

In this survey paper we will focus on generalized Connes amenability for dual Banach algebras, discussing solved and unsolved problems in this recently developed field.

Suppose that $\mathcal{A}$ is a Banach algebra and that E is a Banach $\mathcal{A}$-bimodule. A bounded linear operator $D : \mathcal{A} \longrightarrow E$ is a *derivation* if it satisfies $D(ab) = D(a) . b + a . D(b)$ for all $a, b \in \mathcal{A}$. A derivation D is *inner* if there is $x \in E$ such that $D(a) = ad_x(a) = a . x - x . a$ for $a \in \mathcal{A}$. A Banach algebra $\mathcal{A}$ is *amenable* if for any Banach $\mathcal{A}$-bimodule E, every derivation from $\mathcal{A}$ to E^*, the dual module of E, is inner. A derivation $D : \mathcal{A} \longrightarrow E$ is *approximately inner* if there exists a net $(x_\alpha)_\alpha$ in E, such that

$$D(a) = \lim_\alpha ad_{x_\alpha}(a) \quad (i.e.\ D(a) = \lim_\alpha a . x - x . a) \qquad (1)$$

in the norm topology of E. If in the above definition $(x_\alpha)_\alpha$ can been chosen so that $(ad_{x_\alpha})_\alpha$ is bounded as a net of operators from $\mathcal{A}$ into E (note that $(x_\alpha)_\alpha$ is not necessarily bounded in the case), then D is *boundedly approximately inner*. If $(x_\alpha)_\alpha$ can been chosen to be a sequence, then D is *sequentially approximately inner*. If the convergence of (1) is uniform in a on the unit ball of $\mathcal{A}$, then we say D *uniformly approximately inner*. If E is a dual $\mathcal{A}$-bimodule and the convergence of (1) is required in w^*-topology of E, we call D w^*-*approximately inner*. A Banach algebra $\mathcal{A}$ is (resp. *boundedly, sequentially, uniformly* or w^*-) approximately amenable, if every continuous derivation $D : \mathcal{A} \longrightarrow E^*$ is (resp. *boundedly, sequentially, uniformly* or w^*-) approximately inner for each Banach $\mathcal{A}$-bimodule E. These approximate versions of amenability were introduced by F. Ghahramani and R. Loy in [9].

Let $\mathcal{A}$ be a Banach algebra. It is known that the projective tensor product $\mathcal{A}\hat{\otimes}\mathcal{A}$ is a Banach $\mathcal{A}$-bimodule in the canonical way. There is a continuous linear $\mathcal{A}$-bimodule homomorphism $\pi : \mathcal{A}\hat{\otimes}\mathcal{A} \longrightarrow \mathcal{A}$ such that $\pi(a \otimes b) = ab$ for $a, b \in \mathcal{A}$. The reader may see [2] for more information. It is well-known that $\mathcal{A}$ is amenable if and only if there is a bounded net $(m_\alpha) \subseteq \mathcal{A}\hat{\otimes}\mathcal{A}$ such that $a . m_\alpha - m_\alpha . a \longrightarrow 0$ and $\pi(m_\alpha)a \longrightarrow a$ for each $a \in \mathcal{A}$ [14]. Such net (m_α) is called a *bounded approximate diagonal* for $\mathcal{A}$. Hence, amenability may be defined in terms of the existence of a bounded approximate diagonal. These characterization provide another way to generalize amenability; A Banach algebra $\mathcal{A}$ is *pseudo-amenable* if there is a net $(m_\alpha) \subseteq \mathcal{A}\hat{\otimes}\mathcal{A}$, called an *approximate diagonal* for $\mathcal{A}$, such that $a . m_\alpha - m_\alpha . a \longrightarrow 0$ and $\pi(m_\alpha)a \longrightarrow a$ for each $a \in \mathcal{A}$. Pseudo-amenability was introduced by F. Ghahramani and Y. Zhang in [11].

Some of the above generalized versions of amenability are equivalent. Some are the same if the Banach algebra has a bounded approximate identity.

Let $\mathcal{A}$ be a Banach algebra. A Banach $\mathcal{A}$-bimodule E is *dual* if there is a closed submodule E_* of E^* such that $E = (E_*)^*$. We call E_* the *predual* of E. A dual Banach $\mathcal{A}$-bimodule E is *normal* if the module actions of $\mathcal{A}$ on E are w^*-continuous. A Banach algebra is *dual* if it is dual as a Banach $\mathcal{A}$-bimodule. We write $\mathcal{A} = (\mathcal{A}_*)^*$ if we wish to stress that $\mathcal{A}$ is a dual Banach algebra with predual $\mathcal{A}_*$.

Definition 1.1 ([21]). A dual Banach algebra $\mathcal{A}$ is *Connes amenable* if every w^*-

continuous derivation from $\mathcal{A}$ into a normal, dual Banach $\mathcal{A}$-bimodule is inner.

The reader is referred to [22] for basic properties of Connes amenable dual Banach algebras. Let $\mathcal{A} = (\mathcal{A}_*)^*$ be a dual Banach algebra and let E be a Banach $\mathcal{A}$-bimodule. We write $\sigma wc(E)$ for the set of all elements $x \in E$ such that the maps

$$\mathcal{A} \longrightarrow E \quad , \quad a \longmapsto \begin{cases} a \, . \, x \\ x \, . \, a \end{cases} ,$$

are w^*-weak continuous. The space $\sigma wc(E)$ ia a closed submodule of E. It is shown in [24, Corollary 4.6], that $\pi^*(\mathcal{A}_*) \subseteq \sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*$. Taking adjoints, we can extend π to an $\mathcal{A}$-bimodule homomorphism $\pi_{\sigma wc}$ from $\sigma wc((\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$ to $\mathcal{A}$. A *σwc-virtual diagonal* for a dual Banach algebra $\mathcal{A}$ is an element $M \in \sigma wc((\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$ such that $a \, . \, M = M \, . \, a$ and $a\pi_{\sigma wc}(M) = a$ for $a \in \mathcal{A}$. It is known that Connes amenability of $\mathcal{A}$ is equivalent to existence of a σwc-virtual diagonal for $\mathcal{A}$ [24] .

The organization of the paper is as follow. Firstly, in section 2 we introduce some of the notions of amenability which generalize Connes amenability. We study (w^*-) approximate Connes amenability and pseudo-Connes amenability. The former is based on a property of derivations while the latter is established on existence of a kind of diagonal (not necessarily bounded) with certain properties.

In section 3 we will discuss the relations between these new notions of Connes amenability.

In section 4 we investigate the role of identities. we study direct sums and tensor products of dual Banach algebras regarding types of Connes amenability.

In section 5 we recall a generalization of the notion of injectivity for dual Banach algebras and study its relation to approximate Connes amenability.

Finally, in section 6 we verify these various notions of Connes amenability for certain dual Banach algebras, associated to locally compact groups.

In each sections, to complete the discussions, we raise some related open questions.

2. Generalized notions of Connes amenability

We start with basic definitions. In [8], motivated by the notion of approximate amenability, we defined the concept of approximate Connes amenability.

Definition 2.1 ([8]). A dual Banach algebra $\mathcal{A}$ is *approximately Connes-amenable* if for every normal, dual Banach $\mathcal{A}$-bimodule E every w^*-continuous derivation $D : \mathcal{A} \longrightarrow E$ is approximately inner.

In an analogous way, we may define other type of approximate Connes amenability; A dual Banach algebra $\mathcal{A}$ is (resp. *boundedly, sequentially, uniformly* or w^*-) approximately Connes amenable if for every normal, dual Banach $\mathcal{A}$-bimodule E every w^*-continuous derivation $D : \mathcal{A} \longrightarrow E$ is (resp. boundedly, sequentially, uniformly or w^*-) approximately inner. Throughout, if $\mathcal{A}$ is a Banach algebra we shall write $\mathcal{A}^\sharp$ for the forced unitization of $\mathcal{A}$. The adjoined identity element will usually denoted by e. Clearly, $\mathcal{A}$ is approximately Connes amenable in any of the above mods if and only if $\mathcal{A}^\sharp$ is.

Let $\mathcal{A}$ be a dual Banach algebra. We notice that $\mathcal{A}\hat{\otimes}\mathcal{A}$ is canonically mapped into $\sigma wc((\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$. Therefore there is an inherited w^*-topology on $\mathcal{A}\hat{\otimes}\mathcal{A}$ from $\sigma wc((\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$. Therefore in the sequel we can always speak of the w^*-topology on $\mathcal{A}\hat{\otimes}\mathcal{A}$ without ambiguity. This observation is the motivation of the following basic definition.

Definition 2.2 Suppose that $\mathcal{A}$ is a dual Banach algebra. A net (m_α) in $\mathcal{A}\hat{\otimes}\mathcal{A}$ is an *approximate* σwc*-diagonal* for $\mathcal{A}$ if for every $a \in \mathcal{A}$

(*i*) $a \,.\, m_\alpha - m_\alpha \,.\, a \xrightarrow{w^*} 0$ in $\sigma wc((\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$, and

(*ii*) $a\pi_{\sigma wc}(m_\alpha) \xrightarrow{w^*} a$ in $\mathcal{A}$.

The following is a characterization of Connes amenability in terms of virtual and approximate diagonals.

Proposition 2.3 ([16]). Suppose that $\mathcal{A}$ is a dual Banach algebra. Then $\mathcal{A}$ is Connes amenable if and only if There exists a bounded approximate σwc-diagonal for $\mathcal{A}$.

Pseudo-Connes amenability was introduced by the author in [16].

Definition 2.4 ([16]). A dual Banach algebra $\mathcal{A}$ is *pseudo-Connes amenable* if there exists an approximate σwc-diagonal for $\mathcal{A}$.

There are pseudo-Connes amenable dual Banach algebra which are not Connes amenable even approximately Connes amenable, for example, ℓ^1 with the pointwise multiplication.

3. The relations

By the principle of uniform boundedness we see that every sequentially approximately Connes amenable dual Banach algebra is boundedly approximately Connes amenable. For the converse, we have the following.

Theorem 3.1 ([17]). Suppose that $\mathcal{A}$ is boundedly approximately Connes amenable dual Banach algebra. If $\mathcal{A}$ is separable as a Banach space, then it is sequentially approximately Connes amenable.

To the author's knowledge, every approximately Connes amenable dual Banach algebra is boundedly approximately Connes amenable. So, we may raise the following question.

Question 3.1 Is there an approximately Connes amenable dual Banach algebra which is not boundedly approximately Connes amenable?

It is shown in [10] that approximate amenability and w^*-approximate amenability are the same notion. It is also shown in [10] and [19], independently, that uniform approximate amenability is equivalent to amenability. So, regarding our field, there are two major open questions.

Question 3.2 Are approximate Connes amenability and w^*-approximate Connes amenability the same notion?

Question 3.3 Is the notion of uniform approximate Connes amenability equivalent to Connes amenability?

It is known that there are pseudo-Connes amenable dual Banach algebras which are not (w^*-) approximately Connes amenable. Here are some relations between "pseudo" and "approximate".

Theorem 3.2 ([16]). Let $\mathcal{A}$ be a dual Banach algebra. Then

(*i*) $\mathcal{A}$ is w^*-approximately Connes amenable if and only if $\mathcal{A}^\sharp$ is pseudo-Connes amenable.

(*ii*) If $\mathcal{A}$ has an identity, then it is w^*-approximately Connes amenable if and only if it is pseudo-Connes amenable.

The existence of an identity in the above theorem cannot be removed. For example, ℓ^1 is pseudo-Connes amenable but it is not w^*-approximately Connes amenable. In general, $\mathcal{A}$ being pseudo-Connes amenable seems much weaker than $\mathcal{A}^\sharp$ being pseudo-Connes amenable. But if $\mathcal{A}$ has an identity, then they are equivalent.

Question 3.4 Dose w^*-approximate Connes amenability imply pseudo-Connes amenability?

The answer to above question is affirmative if the algebra has a central w^*-approximate identity [16]. In particular, it is true if the algebra is abelian.

4. Identities, direct sums and tensor products

From the definition it is easy to see that a pseudo-Connes amenable dual Banach algebra has a w^*-approximate identity, and an approximately Connes amenable dual Banach algebra has a left and a right approximate identity. In fact, all known approximately Connes amenable dual Banach algebras have an identity. However, we do not know this is true in general or not.

Question 4.1 Dose every approximately Connes amenable dual Banach algebra have a two-sided approximate identity? Does it have an identity?

It is interesting to mention that if $\mathcal{A} \oplus \mathcal{A}$ is approximately Connes amenable, then $\mathcal{A}$ must have a two-sided approximate identity.

A notable property of pseudo-Connes amenability is that the class is closed under ℓ^1-direct sum.

Theorem 4.1 ([16]). Suppose that $\mathcal{A}$ and $\mathcal{B}$ are pseudo-Connes amenable dual Banach algebras. Then, so is $\mathcal{A} \oplus^1 \mathcal{B}$.

Approximate Connes amenability lacks this property. But we conjecture that the class of approximately Connes amenable dual Banach algebras should be closed under taking finite direct sums. However, we only have a partial result.

Theorem 4.2 ([18]). Suppose that $\mathcal{A}$ and $\mathcal{B}$ are approximately Connes amenable dual Banach algebras and that one of $\mathcal{A}$ or $\mathcal{B}$ has an identity. Then $\mathcal{A} \oplus^1 \mathcal{B}$ is approximately Connes-amenable.

Question 4.2 Is $\mathcal{A} \oplus^1 \mathcal{B}$ approximately Connes amenable if both $\mathcal{A}$ and $\mathcal{B}$ are?

It was shown in [9] that, if $\mathcal{A}$ is approximately amenable and has bounded approximate identity and if $\mathcal{B}$ is amenable, then $\mathcal{A}\hat{\otimes}\mathcal{B}$ is approximately amenable. For dual Banach algebras $\mathcal{A}$ and $\mathcal{B}$, although, we know that $\mathcal{A}\hat{\otimes}\mathcal{B}$ is not, in general, a dual Banach algebra. In [7], M. Daws has introduced a type of tensor product of $\mathcal{A}$ and $\mathcal{B}$, denoted by $\mathcal{A}\tilde{\otimes}\mathcal{B}$, for which the product of two dual Banach algebras is again a dual Banach algebra. We do not know the relations between approximate Connes amenability/pseudo-Connes amenability of $\mathcal{A}\tilde{\otimes}\mathcal{B}$ and $\mathcal{A}$, $\mathcal{B}$.

Question 4.3 Let $\mathcal{A}$ and $\mathcal{B}$ be dual Banach algebras.

(i) What is the relations between approximate Connes amenability of $\mathcal{A}\tilde{\otimes}\mathcal{B}$ and that of $\mathcal{A}$, $\mathcal{B}$?

(ii) What is the relations between pseudo-Connes amenability of $\mathcal{A}\tilde{\otimes}\mathcal{B}$ and that of $\mathcal{A}$, $\mathcal{B}$?

5. Generalized notion of injectivity

The notion of injectivity for dual Banach algebras was introduced by M. Daws in [7]. A dual Banach algebra is injective if and only if it is Connes-amenable [7]. Let E be a Banach space, and let $\mathcal{F} \subseteq \mathcal{L}(E)$ be a subalgebra. A *quasi-expectation* for $\mathcal{F}$ is a *projection* $Q : \mathcal{L}(E) \longrightarrow \mathcal{F}^c$ such that $Q(STU) = SQ(T)U$ for $S, U \in \mathcal{F}^c$ and $T \in \mathcal{L}(E)$. When E is a reflexive Banach space, $E^* \hat{\otimes} E$ is the canonical predual for $\mathcal{L}(E)$. We recall that a unital dual Banach algebra $\mathcal{A}$ is *injective* if whenever $\pi : \mathcal{A} \longrightarrow \mathcal{L}(E)$ is a w^*-continuous, unital representation on a reflexive Banach space E, there is a quasi-expectation $Q : \mathcal{L}(E) \longrightarrow \pi(\mathcal{A})^c$ for $\pi(\mathcal{A})$.

Definition 5.1 ([18]). Let $\mathcal{F}$ be a subalgebra of $\mathcal{L}(E)$ for some Banach space E. An *approximate quasi-expectation* for $\mathcal{F}$ is a net of bounded linear maps $Q_\alpha : \mathcal{L}(E) \longrightarrow \mathcal{L}(E)$, such that

(i) Each Q_α is the identity map on $\mathcal{F}^c$,
(ii) $SQ_\alpha(T) - Q_\alpha(T)S \longrightarrow 0$, $(S \in \mathcal{F},$ uniformly for all $T \in ball\mathcal{L}(E))$, and
(iii) $Q_\alpha(STU) = SQ_\alpha(T)U \quad (S, U \in \mathcal{F}^c,\ T \in \mathcal{L}(E),$ and for all $\alpha)$.

In [18], the author extend the notion of injectivity to the approximate case.

Definition 5.2 ([18]). A (unital) dual Banach algebra $\mathcal{A}$ is *approximately injective* if whenever $\pi : \mathcal{A} \longrightarrow \mathcal{L}(E)$ is a (unital) w^*-continuous representation on a reflexive Banach space E, there is an approximate quasi-expectation for $\pi(\mathcal{A})$.

Theorem 5.1 ([18]). Suppose that $\mathcal{A}$ is a dual Banach algebra. Then $\mathcal{A}$ is approximately Connes-amenable if and only if $\mathcal{A}$ is approximately injective.

If we can define the notion of *pseudo-injectivity* for dual Banach algebras, then the following question comes up immediately.

Question 5.3 Are pseudo-injectivity and pseudo-Connes amenability the same notions for dual Banach algebras?

6. Algebras associated to locally compact groups

There is no difference between generalized Connes amenability and Connes amenability for measure algebras, von neumann algebras and $WAP(G)^*$, where $WAP(G)$ denotes the set of all weakly almost periodic functions on a locally compact group G.

Theorem 6.1 ([8]). Let G be a locally compact group. Then

(i) $M(G)$ is approximately Connes-amenable if and only if G is amenable.
(ii) $WAP(G)^*$ is approximately Connes-amenable if and only if G is amenable.

Theorem 6.2 ([8]). For a locally compact group G, consider the following:

(i) G is amenable.
(ii) $PM_p(G)$ is approximately Connes-amenable, for every $p \in (1, \infty)$.
(iii) $VN(G)$ is approximately Connes-amenable.
(iv) $PM_p(G)$ is approximately Connes-amenable, for one $p \in (1, \infty)$.

Then we have $(i) \Longrightarrow (ii) \Longrightarrow (iii) \Longrightarrow (iv)$,
and if G is inner amenable, $(iv) \Longrightarrow (i)$ holds.

Pseudo-Connes amenability of the above algebras, however, is still unknown.

Question 6.1 Let G be a locally compact group, and let X be $M(G)$, $WAP(G)^*$ or $VN(G)$. Does pseudo-Connes amenability of X yield amenability of G?

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