A generalization of Bertrand's test

A. A. Tabatabai Adnani[a,*], A. Reza[a] and M. Morovati[b]

[a]*Islamic Azad University, Central Tehran Branch, Tehran, Iran.*;
[b]*School of Automotive Engineering, Iran University of Science and Technology, Tehran, Iran.*

Abstract. One of the most practical routine tests for convergence of a positive series makes use of the ratio test. If this test fails, we can use Rabbe's test. When Rabbe's test fails the next sharper criteria which may sometimes be used is the Bertrand's test. If this test fails, we can use a generalization of Bertrand's test and such tests can be continued infinitely. For simplicity, we call ratio test, Rabbe's test, Bertrand's test as the Bertrand's test of order 0, 1 and 2, respectively. In this paper, we generalize Bertrand's test in order k for natural $k > 2$. It is also shown that for any k, there exists a series such that the Bertrand's test of order fails, but such test of order $k + 1$ is useful, furthermore we show that there exists a series such that for any k, Bertrand's test of order k fails. The only prerequisite for reading this article is a standard knowledge of advanced calculus.

© 2013 IAUCTB. All rights reserved.

Keywords: Bertrand's test, Convergence test, Series test.

2010 AMS Subject Classification: 40A05, 40A30.

1. Introduction

In recent decade, the theory of series is widely used in applied mathematics and so many other applied sciences such as environmental engineering and mechanical engineering. The theory of series is studied by many authors. The article Wen et al. [2] studied the convergence tests on constant Dirichlet series. In continue for showing its application the authors gave a statement to show the effectiveness of their convergence tests. In 2003 Chen et al. [3] showed the application of Laplace-transformed power series techniques in solving two-dimensional advection and dispersion equation in cylindrical coordinates. In this article, for evaluating its robustness and accuracy, the authors compared the solution

---

*Corresponding author.
E-mail address: a.t.adnani@gmail.com (A. A. Tabatabai Adnani).

with a numerical solution and showed that it is valid for a Peclet number up to 60. Wonzy [4] introduced and applied the z transformation and its method for summation of the slowly convergence series. He showed that this solving algorithm can be easily applied to the case of generalized or basic hypergeometric series and some orthogonal polynomial expansions. At the end he gave some examples for showing numerical results. Liflyand et al. [5] decreased the monotonicity assumption for the sequence of terms of the series. For this aim they investigated and analyzed several classical tests for convergence and divergence of numerical series. In this article, the sharpness of the obtained results on corresponding classes of sequences and functions was verified as well. Bartoszewicz et al. [6] studied algebrability of non-absolutely convergent series. They showed that c-algebrable is the set of all complex series which are non-absolutely convergent. Moricz [7] showed a quantitative version of Dirichlet-Jordan test for double fourier series and in continue he proved his estimation in a larger generality. Consider a series with the positive terms,

$$I = \sum_{n=1}^{\infty} a_n \tag{1}$$

And the limit,

$$\lim_{n \to \infty} \frac{a_n}{a_{n+1}} = L_0 \tag{2}$$

Then if $L_0 > 1$, I is convergent and if $L_0 < 1$, I is divergent. If $L_0 = 1$, the test fails, in this case we can use Rabbe's test. For this test consider the limit,

$$\lim_{n \to \infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = L_1 \tag{3}$$

This test is similar to the ratio test, i.e.; if $L_1 > 1$, then I is convergent and if $L_1 < 1$, then I is divergent and if $L_1 = 1$, the test fails. In this case we can use Bertrand's test, i.e.; assume that,

$$\lim_{n \to \infty} \ln n\left[n\left(\frac{a_n}{a_{n+1}} - 1\right) - 1\right] = L_2 \tag{4}$$

exists (finite or infinite) then

  (*i*) If $1 < L_2 \leq +\infty$, I converges.
  (*ii*) If $-\infty \leq L_2 < 1$, I diverges.
  (*iii*) If $L_2 = 1$, Σa_n may either converges or diverges, and the test fails.

In the case (*iii*) we can generalize the test, namely Bertrand's generalized test in order 3, 4,..., k. In fact we can use Bertrand's test of order k if the tests of orders 0, 1, 2, ..., $k - 1$ fail and such tests can be used infinitely many times.

2.   Generalized Tests

To introduce the generalized Bertrand's test we need to use some notations and results. We define $\lambda_0(x) = x$, and for integer $k > 0$, $\lambda_k(x) = \ln \lambda_{k-1}(x)$ where, x is positive and

real number. It is clear that for sufficiently large x the functions $\lambda_k(x)$ are positive and increasing. Put $p_1(x) = x$, and for $k > 1$, $p_k(x) = \lambda_{k-1}(x)p_{k-1}(x)$. It may be proved that the functions $p_k(x)$ are increasing. Furthermore,

$$\lambda'_k(x) = \frac{1}{p_k(x)}. \tag{5}$$

To show (5), note that $\lambda'_1(x) = (\ln x)' = \frac{1}{x} = \frac{1}{p_1(x)}$, and if $\lambda'_s(x) = \frac{1}{p_s(x)}$ for $s \geq 1$ then

$$\lambda'_{s+1}(x) = (\ln \lambda_s(x))' = \frac{\lambda'_s(x)}{\lambda_s(x)} = \frac{1}{\lambda_s(x)p_s(x)} = \frac{1}{p_{s+1}(x)}.$$

Since $p_{k+1}(x) = \lambda_k(x)p_k(x)$, by (5) we get,

$$p'_{k+1}(x) = \lambda'_k(x)p_k(x) + \lambda_k(x)p'_k(x) = 1 + \lambda_k(x)p'_k(x) \tag{6}$$

The equation (6) shows that $p'_k(x)$ is increasing, for, if $x < y$ then $\lambda_k(x) \leq \lambda_k(y)$. Now we may use an induction method.

Lemma 2.1 For every integers $t \geq 0$ and $k \geq 1$, we have,

$$\lim_{x \to +\infty} (\ln x)^t p''_k(x) = 0 \tag{7}$$

Proof. For $k = 1, 2$ the lemma is clear, assume that (3) is true. For $k \geq 2$, (6) yields,

$$p''_{k+1}(x) = [p'_{k+1}(x)]' = [1 + \lambda_k(x)p'_k(x)]' = \lambda'_k(x)p'_k(x) + \lambda_k(x)p''_k(x) \tag{8}$$

$$= \frac{p'_k(x)}{p_k(x)} + \lambda_k(x)p''_k(x) = (\ln p_k(x))' + \lambda_k(x)p''_k(x) = (\lambda_1(x) + \lambda_2(x) +$$

$$...+\lambda_k(x))'+\lambda_k(x)p''_k(x) = \frac{1}{p_1(x)}+\frac{1}{p_2(x)}+...+\frac{1}{p_k(x)}+\lambda_k(x)p''_k(x)+...+\lambda_k(x))'+\lambda_k(x)p''_k(x)$$

$$= \frac{1}{p_1(x)} + \frac{1}{p_2(x)} + ... + \frac{1}{p_k(x)} + \lambda_k(x)p''_k(x)$$

Therefore, for every t we get,

$$\lim_{x \to +\infty} (\ln x)^t p''_{k+1}(x)$$
$$= \lim_{x \to +\infty} \frac{(\ln x)^t}{p_1(x)} + \lim_{x \to +\infty} \frac{(\ln x)^t}{p_2(x)} + ... + \lim_{x \to +\infty} \frac{(\ln x)^t}{p_k(x)} + \lim_{x \to +\infty} [(\ln x)^t \lambda_k(x)p''_k(x)] \tag{9}$$

In (9) the first $k - th$ limits are zero, the last one by the inequality,

$$0 < (\ln x)^t \lambda_k(x)p''_k(x) \leq (\ln x)^{t+1} p''_k(x) \tag{10}$$

and (7) is zero. In (7) for $t = 0$ we get the following limit for every $k \geq 1$,

$$\lim_{x \to +\infty} p''_k(x) = 0 \tag{11}$$

∎

Note that in this paper the Kummer's test is used as a key theorem.

Theorem 2.2 Let $\sum a_n$ be a positives series, and let p_n be a sequence of positive constants such that,

$$\lim_{n \to \infty} [p_n \frac{a_n}{a_{n+1}} - p_{n+1}] = \lim_{n \to \infty} U_n = L \tag{12}$$

exists and is positive $(0 < L < +\infty)$, then $\sum a_n$ convergences, if the limit (12) exists and is negative $(-\infty < L < 0)$ and if $\sum \frac{1}{p_n}$ diverges, then $\sum a_n$ diverges [1].

Definition 2.3 We denote the index sequence of generalized Bertrand's test of order k for $k \geq 0$ by $F(k, n)$ and define it by induction as follows,

$$F(0, n) = \frac{a_n}{a_{n+1}}, \quad F(1, n) = n\left(\frac{a_n}{a_{n+1}} - 1\right), \quad F(2, n) = \ln n[n(\frac{a_n}{a_{n+1}} - 1) - 1]$$

if $F(k, n) = \lambda_{k-1}(n)[F(k-1, n) - 1]$ For $k \geq 1$ then $F(k+1, n) = \lambda_k(n)[F(k, n) - 1]$, also we define L_k for $k \geq 0$ as the, $\lim_{n \to \infty} F(k, n) = L_k$ if the limit exists (finite or infinite).

The main aim of this paper is the following theorem.

Theorem 2.4 Let $L_0 = L_1 = ... = L_{k-1} = 1$ for $k \geq 1$ then:

(i) If $1 < L_k \leq +\infty$, $\sum a_n$ converges.
(ii) If $-\infty \leq L_k < 1$, $\sum a_n$ diverges.
(iii) If $L_k = 1$, $\sum a_n$ may either converge or diverge, and the test fails.

Proof At first we claim that,

$$F(k, n) = U(k, n) + T(k, n) \tag{13}$$

where,

$$U(k, n) = p_k(n) \frac{a_n}{a_{n+1}} - p_k(n + 1), \tag{14}$$

and,

$$T(k, n) = p_k(n + 1) - p_k(n) - p'_k(n) + 1 \tag{15}$$

if $k = 1$ then $T(1, n) = 1$ and $U(1, n) = n\frac{a_n}{a_{n+1}} - (n + 1)$. Therefore (14) and (15) are valid for $k = 1$ Now assume that (14) and (15) are true for $k \geq 1$, then,

$$F(k + 1, n) = \lambda_k(n)[F(k, n) - 1] = \lambda_k(n)[U(k, n) + T(k, n) - 1]$$
$$= \lambda_k(n)[p_k(n)\frac{a_n}{a_{n+1}} - p_k(n + 1)] + \lambda_k(n)[p_k(n + 1) - p_k(n) - p'_k(n) + 1] - \lambda_k(n)$$

$$= U(k + 1, n) + [p_{k+1}(n + 1) - p_{k+1}(n) - \lambda_k(n)p'_k(n)] \tag{16}$$

By the second term of (16) we have,

$$
\begin{aligned}
&p_{k+1}(n+1) - p_{k+1}(n) - \lambda_k(n)p_k'(n) \\
&= p_{k+1}(n+1) - p_{k+1}(n) - p_{k+1}'(n) + 1 \\
&= T(k+1, n)
\end{aligned}
$$

Therefore, $F(k+1, n) = U(k+1, n) + T(k+1, n)$. Now we show that, $\lim\limits_{n\to\infty} T(k, n) = 1$. By the mean value theorem we have, $p_k(n+1) - p_k(n) = p_k'(\theta)$, for some θ where $n < \theta < n+1$. Since $p_k'(n)$ is increasing the inequality the following holds,

$$
0 < p_k'(\theta) - p_k'(n) < p_k'(n+1) - p_k'(n) \tag{17}
$$

Again by the mean value theorem there exists $n < \theta' < n+1$ such that,

$$
p_k'(n+1) - p_k'(n) = p_k''(\theta') \tag{18}
$$

(15), (17) and (18) yield,

$$
0 < T(k, n) - 1 < p_k''(\theta') \tag{19}
$$

From (19) and (11) we deduce that,

$$
\lim\limits_{n\to\infty} T(k, n) = 1.
$$

If in Kummer's test we put $p_n = p_k(n)$ then $U_n = U(k, n)$. If $\lim\limits_{n\to\infty} U(k, n) = L$, then,

$$
\lim\limits_{n\to\infty} F(k, n) = \lim\limits_{n\to\infty} U(k, n) + \lim\limits_{n\to\infty} T(k, n) = L + 1 = L_k. \tag{20}
$$

So by Kummer's test (theorem 2.1) (i) and (ii) are proved. For (iii) (when $L_k = 1$) it suffices to show that for any $k \geqslant 0$, there exists two series $\sum a_n$ and $\sum b_n$ such that $L_k = 1$ for these series and $\sum a_n$ converges but $\sum b_n$ diverges. Assume that $\alpha \in \mathbb{R}, \alpha > 0$ and $n > \exp(\exp(...(\exp e)...)$, where the exponential function are composed by itself $(k+1)$-times. We define the sequence a_n by the recursion formula, $a_1 = a_2 = ... = a_N = 1$ and $p_{k+1}(N) > 0, n \geqslant N$

$$
\frac{a_n}{a_{n+1}} = 1 + \frac{1}{p_1(n)} + \frac{1}{p_2(n)} + ... + \frac{1}{p_k(n)} + \frac{\alpha}{p_{k+1}(n)}
$$

It is clear that, $L_0 = L_1 = ... = L_k = 1, L_{k+1} = \alpha$. By Bertrand's test of order $k+1$ if $\alpha > 1$, $\sum a_n$ converges and if $\alpha < 1$, $\sum a_n$ diverges, and part (iii) is proved. Finally we show that there exists a series such that the generalized Bertrand's test of any order fails for it. If $\sigma(n)$ be the largest natural number such that $\lambda_{\sigma(n)-1}(n) \geqslant 1$, then $\sigma(n)$ is an increasing unbounded function of n, for example

$$
\sigma(1) = \sigma(2) = 1, \quad \sigma(3) = \sigma(4) = \sigma(5) = 2, \quad \sigma(27) = 3, \quad \sigma(3^{27}) = 4,
$$

it follows that $\lim\limits_{n\to\infty} \sigma(n) = +\infty$ We define a_n, by the recursion formula,

$$a_1 = 1, \frac{a_n}{a_{n+1}} = 1 + \frac{1}{p_1(n)} + \frac{1}{p_2(n)} + ... + \frac{1}{p_{\sigma(n)}(n)}. \tag{21}$$

We now show that $a_n \to 0$. Let, $T_n = \frac{1}{p_1(n)} + \frac{1}{p_2(n)} + ... + \frac{1}{p_{\sigma(n)}(n)}$ and (21) can be rewrites $\frac{a_n}{a_{n+1}} = 1 + T_n$. Since $T_n > \frac{1}{n}$, then $\sum_{n=1}^{\infty} T_n = +\infty$. Therefore, $\frac{1}{a_{n+1}} = \prod\limits_{k=1}^{n} (1 + T_k) \to +\infty$. This yields $a_n \to 0$. If $k \geq 0$ be an integer and $k + 2 \leq \sigma(n)$, then,

$$F(k, n) = 1 + \frac{1}{\lambda_k(n)} + \frac{1}{\lambda_k(n)\lambda_{k+1}(n)} + ... + \frac{1}{\lambda_k(n)\lambda_{k+1}(n)...\lambda_{\sigma(n)-1}(n)}. \tag{22}$$

Since $\lambda_{\sigma(n)-1}(n) \geq 1$ we have,

$$\lambda_k(n) \geq \exp(\exp(...(\exp e))...) \tag{23}$$

where the number of exponential functions is $\sigma(n) - 1 - k$. Since $\exp e > \exp 2$ then $\exp(\exp e) > \exp 3$, therefore,

$$\exp(\exp(...(\exp e))...) > \exp(\sigma(n) - k - 1). \tag{24}$$

At last from (22), (23) and (24) we have,

$$0 < F(k, n) - 1 < \frac{\sigma(n) - k}{\lambda_k(n)} \leq e \frac{\sigma(n) - k}{\exp(\sigma(n) - k)} \tag{25}$$

In (25) let $n \to \infty$ then, $\sigma(n) - k \to +\infty$ and $\frac{\sigma(n)-k}{\exp(\sigma(n)-k)} \to 0$. Therefore,

$$\lim_{n\to\infty} F(k, n) = 1 \, for \, k = 0, 1, 2, ... \tag{26}$$

acknowledgements

References

[1] J. M. H. Olmsted, Advanced Calculus, *Prentice Hall.* (1961).
[2] J. Wen, T. Han, C. Gao, Convergence tests on constant Dirichlet series, *Computers and Mathematics with Applications.* **62** (2011) 3472-3489.
[3] J. S. Chen, C. W. Liu, C. M. Liao, Two-dimensional Laplace-transformed power series solution for solute transport in a radially convergent flow field, *Advances in Water Resources.* **26** (2003) 1113-1124.
[4] P. Wonzy, Efficient algorithm for summation of some slowly convergent series, *Applied Numerical Mathematics.* **60** (2010) 1442-1453.
[5] E. Liflyand, S. Tikhonov, M. Zeltser, Extending tests for convergence of number series, *Journal of Mathematical Analysis and Applications.* **377** (2011) 194-206.
[6] A. Bartoszewicz, S. Glab, T. Poreda, On algebrability of nonabsolutely convergent series, *Linear Algebra and its Applications.* **435** (2011) 1025-1028.

[7] F. Moricz, A Quantitative Version of the Dirichlet-Jordan Test for Double Fourier Series, *Journal of Approximation Theory.* **71**(1992) 344-358.