Journal of
Linear and Topological Algebra
Vol. 02*, No.* 03*,* 2013, 149- 154

On the Finite Groupoid $G(n)$

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Abstract. In this paper we study the existence of commuting regular elements, verifying the notion left (right) commuting regular elements and its properties in the groupoid $G(n)$. Also we show that $G(n)$ contains commuting regular subsemigroup and give a necessary and sufficient condition for the groupoid $G(n)$ to be commuting regular.

Keywords: Commuting regular semigroup, semigroup, groupoid.

2010 AMS Subject Classification: 15A27, 20M16, 20L05.

1. Introduction

We use S and G to denote a semigroup and a groupoid, respectively. An element x of a semigroup S is called regular if there exists y in S such that, $x = xyx$ [3]. Two elements x and y of a semigroup S are commuting regular if for some $z \in S$, $xy = yxzyx$ [2]. A semigroup S is called commuting regular if and only if for each $x, y \in S$ there exists an element z of S such that $xy = yxzyx$ [1]. In [2] Pourfaraj showed that the existence of commuting regular elements for the loop ring $Z_t[L_n(m)]$ when t is an even perfect number or t is the form of $2^i p$ or $3^i p$, where p is an odd prime or in general, when $t = p_1^i p_2$ (p_1 and p_2 are distinct odd primes). Define a binary operation * on $G = Z_n \cup \{e\}$ as follows,

1) $a * a = a$ for all $a \in G$.
2) $a * e = e * a = a$ for all $a \in G$.
3) $a * b = ta + ub \ (mod n)$, where $t, u \in Z_n$ are fix elements and $a, b \in G$ $(a \neq b)$, $Z_n = \{0, 1, 2, ..., n-1\}, n \geqslant 3$ and $e \notin Z_n$.

The properties of these groupoids denote by $G(n)$ has been studied in [5].

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Print ISSN: 2252-0201
Online ISSN: 2345-5934

2. Commuting Regular Elements

Definition 2.1 Two elements a and b of a groupoid G are called left commuting regular if for some $c_1 \in G$, $ab = ((ba)c_1)(ba)$. Similarly, they are called right commuting regular if for some $c_2 \in G$, $ab = (ba)(c_2(ba))$. Finally, two elements x and y are commuting regular if they are both left and right commuting regular. [see 4]

Definition 2.2 A groupoid G is called left commuting regular groupoid if for each $a, b \in G$ there exists $c_1 \in G$ such that , $ab = ((ba)c_1)(ba)$. Similarly, right commuting regular groupoid is defined. A groupoid G is called commuting regular groupoid if G is both a left and right commuting regular groupoid.[see 4]

Example **2.3** The groupoid $G(3)$ where $t = 1$ and $u = 2$ is given by the following table

$*$	e	0	1	2
e	e	0	1	2
0	0	e	2	1
1	1	1	e	2
2	2	2	1	e

We have:

$$(2 * 1) * (0 * (2 * 1)) = 1 * (0 * 1) = 1 * 2.$$

So,1 and 2 are right commuting regular. On the other hand,

$$1 * 2 \neq ((2 * 1) * 0) * (2 * 1)$$
$$1 * 2 \neq ((2 * 1) * 1) * (2 * 1)$$
$$1 * 2 \neq ((2 * 1) * 2) * (2 * 1)$$
$$1 * 2 \neq ((2 * 1) * e) * (2 * 1).$$

Thus, 1 and 2 aren't left commuting regular. 2 and 2 are commuting regular,

$$2 * 2 = (2 * 2) * e * (2 * 2).$$

Proposition 2.4 Let the $G(n)$ be a groupoid, where $n = tu-1$. Suppose that $a, b \in G(n)$ and pair of elements $\{b * a, c_1, (b * a) * c_1\}$ and $\{b * a, c_2, (b * a) * c_2\}$ are distinct. Then a and b are commuting regular elements, where $b \equiv au\ (mod\ n)$, $c_1 \equiv -bt^3 - b\ (mod\ n)$ and $c_2 \equiv -au^3 - a\ (mod\ n)$.

Proof We consider two follows case:

Case1) If $a * b = b * a$ then:

$$a * b = (b * a) * (a * b) * (b * a)$$

Case2) If $a * b \neq b * a$ then:

$$
\begin{aligned}
((b*a)*c_1)*(b*a) = &\\
&= ((bt+au)*c_1)*(bt+au)\\
&= ((bt+au)t+c_1u)*(bt+au)\\
&= bt^3+aut^2+c_1tu+btu+au^2\\
&= bt^3+at+bt^3-b+b+bu\ (since\ tu\equiv 1\ (mod\ n)\ and\ b\equiv au\ (modn))\\
&= at+bu\\
&= a*b
\end{aligned}
$$

Similarly,

$$a*b=(b*a)*(c_2*(b*a)).$$

Proposition 2.5 Let the $G(n)$ be a groupoid, where $n\equiv tu+1$. Suppose that $a,b\in G(n)$ and pair of elements in $\{b*a, c_1, (b*a)*c_1\}$ and $\{b*a, c_2, (b*a)*c_2\}$ are distinct. Then a and b are commuting regular elements, where, $b\equiv au\ (mod\ n)$, $c_1\equiv -2at+bt^3-b\ (mod\ n)$ and $c_2\equiv -2at-2bu+au^3-a\ (mod\ n)$.

Example 2.6 Let $G(20)$ where $t=3$ and $u=7$, then $a=11$ and $b=17$ are commuting regular elements:

$$((17*11)*4)*(17*11)=(17*11)*(16*(17*11))=11*17.$$

Note that $17\equiv 11\times 7\ (mod\ 20)$.

Proposition 2.7 Let $G(n)$ be a groupoid, where $t\equiv -u\ (mod\ n)$, then $a,b\in G(n)$ are commuting regular elements, where $at\equiv bt\ (mod\ n)$.

Proof Since $at\equiv bt\ (mod\ n)$ and $t\equiv -u\ (mod\ n)$:

$$-au\equiv -bu\ (mod\ n).$$

So in $G(n)$,

$$a*b=at+bu=bt+au=b*a.$$

And therefore:

$$a*b=(b*a)*(a*b)*(b*a).$$

So a and b are commuting regular.

Proposition 2.8 Let $G(n)$ be a groupoid, where $n=(t-u)k,\ k\in Z$, if for some $a,b\in G(n)$, $a-b\equiv k\ (mod\ n)$, then a and b are commuting regular elements.

Proof We have $a-b=\dfrac{n}{t-u}\ (mod\ n)$, so

$$(a-b)(t-u)\equiv 0\ (mod\ n)$$

Therefore, in $G(n)$:

$$at-au-bt+bu=0$$

$$at+bu=bt+au$$

$$a * b = b * a$$

So:

$$a * b = (b * a) * (a * b) * (b * a)$$

Proposition 2.9 Let $G(n)$ be a groupoid, then $a, b \in G(n)$ are commuting regular elements where $at \equiv au\ (mod\ n)$ and $bt \equiv bu\ (mod\ n)$.
Proof We have $a * b = at + bu = bt + au = b * a$ So

$$a * b = (b * a) * (a * b) * (b * a)$$

Thus a and b are commuting regular elements.
Proposition 2.10 Let $G(n)$ be a groupoid, where $t + u = n$. Suppose that $a \in G(n)$ and $k \in Z$. Then a and ka are commuting regular elements, where $au \equiv -au\ (mod\ n)$.
Proof Since $t \equiv -u\ (mod\ n)$, for all $a \in G(n)$ we have $at \equiv -au\ (mod\ n)$ and by $au \equiv -au\ (mod\ n)$, $at \equiv au\ (mod\ n)$. So $kat \equiv kau\ (mod\ n)$. Now by the proposition 2.9, a and ka are commuting regular elements.

3. Commuting Regular Groupoids

Proposition 3.1 The groupoid $G(n)$ for all $a \in G(n)$ contains the commuting regular subgroupoid $\{e, a\}$.
Proof The subgroupoid $\{e, a\}$ given by the following table,

$*$	e	a
e	e	a
a	a	e

$$e * a = (a * e) * a * (a * e)$$
$$a * a = (a * a) * e * (a * a)$$
$$e * e = (e * e) * e * (e * e)$$

Proposition 3.2 Let $G(n)$ be a groupoid, where $n = 2u$, $u^2 \equiv u\ (mod\ n)$ and $t = 1$. Then for every a in $G(n)$, $\{e, a, a + u\}$ is a commuting regular groupoid.
Proof Let $b = a + u$. If, we have:

$$x * x = e\ ,\ x * e = e * x = x$$

Also,

$$au = \begin{cases} 0\ a\ is\ even\ (mod\ n), \\ u\ a\ is\ odd\ \ (mod\ n), \end{cases}$$

$$a * b = b * a \equiv a + u + au \equiv \begin{cases} b \quad if\ a\ is\ even\ (mod\ n) \\ a \quad if\ a\ is\ odd\ \ (mod\ n) \end{cases}$$

So $\{e, a, b\}$ is groupoid.
For all $x, y \in \{e, a, b\}$ we have $x * y = y * x$. So

$$x * y = (y * x) * (x * y) * (y * x)$$

Thus $\{e, a, b\}$ is a commuting regular groupoid.
Example 3.3 Let $G(n)$ be a groupoid, where $n = 6$, $u = 3$ and $t = 1$ is given by the following table,

$*$	e	0	1	2	3	4	5
e	e	0	1	2	3	4	5
0	0	e	3	0	3	0	3
1	1	1	e	1	4	1	4
2	2	2	5	e	5	2	5
3	3	3	0	3	e	3	0
4	4	4	1	4	1	e	1
5	2	5	2	5	2	5	e

$\{e, 0, 3\}$, $\{e, 1, 4\}$ and $\{e, 2, 5\}$ are commuting regular groupoids.
Proposition 3.4 Let $G(n)$ be a groupoid, where $t = 0$, $n = 2u$ and u is an odd element. Therefore groupoid $G(n)$ contains commuting regular and commutative groupoids $G_1 = \{e, 1, 3, ..., n-1\}$ and $G(2) = \{e, 0, 2, ..., n-2\}$. In particular, if $u^2 \equiv u \ (mod\ n)$, then G_1 and G_2 are commuting regular and commutative semigroup.
Proof For all $a, b \in G_1 - \{e\}$, if $a \neq b$ we have $a * b = b * a = u$. So, we have:

$$a * b = (b * a) * (a * b) * (b * a)$$

In particular, if $u^2 = u \ (mod\ n)$ for all $a, b, c \in G_1$ we have:

$$(a * b) * c = bu * c = cu$$

$$a * (b * c) = a * cu = cu^2$$

Therefore G_1 is a semigroup. The proof for G_2 is the same as above.
Corollary 3.5 Let $G(n)$ be a groupoid, where $u = 0, n = 2t$ and t is odd element. Then groupoid $G(n)$ contains commuting regular and commutative groupoids $G_1 = \{e, 1, 3, ..., n-1\}$ and $G(2) = \{e, 0, 2, ..., n-2\}$. In particular, if $t^2 \equiv t \ (mod\ n)$ then G_1 and G_2 are commuting regular and commutative semigroup.
Proposition 3.6 Let $G(n)$ be a groupoid, where $t = 0$, $n = 3u$ and $u = 3k + 1$ for some $k \in Z$. Then groupoid $G(n)$ contains commuting regular and commutative groupoids $G_1 = \{e, 2, 5, ..., n-1\}$, $G(2) = \{e, 1, 4, ..., n-2\}$ and $G_3 = \{e, 0, 3, ..., n-3\}$. Inparticular, if $u^2 \equiv u \ (mod\ n)$, then G_1, G_2 and G_3 are commuting regular and commutative semigroups.
Theorem 3.7 Let $G(n)$ be a groupoid, where $t = 0$, $n = mu$ and $u = mk + 1$, for some $m, k \in Z$. Then groupoid $G(n)$ contains commuting regular and commutative groupoids. Inparticular, if $u^2 \equiv u \ (mod\ n)$ then $G(n)$ contains commuting regular and commutative semigroups.
Example 3.8 Let $G(n)$ be a groupoid, where $t = 0$, $u = 5$ and $n = 10$ is given in the following table,

$*$	e	0	1	2	3	4	5	6	7	8	9
e	e	0	1	2	3	4	5	6	7	8	9
0	0	e	5	0	5	0	5	0	5	0	5
1	1	0	e	0	5	0	5	0	5	0	5
2	2	0	5	e	5	0	5	0	5	0	5
3	3	0	5	0	e	0	5	0	5	0	5
4	4	0	5	0	5	e	5	0	5	0	5
5	5	0	5	0	5	0	e	0	5	0	5
6	6	0	5	0	5	0	5	e	5	0	5
7	7	0	5	0	5	0	5	0	e	0	5
8	8	0	5	0	5	0	5	0	5	e	5
9	9	0	5	0	5	0	5	0	5	0	e

Clearly, the semigroups $\{e, 0, 2, 4, 6, 8\}$, $\{e, 1, 3, 5, 7, 9\}$ are comuting regular and commutative.

Theorem 3.9 Let $G(n)$ be a groupoid, where $t = u$. If $t^2 \equiv t\ (mod\ n)$ then $G(n)$ is a commuting regular and commutative semigroup.

Proof Let $a, b \in G(n) - \{e\}$,

1) If $a \neq b$, then a and b are commuting regular elements [4, Theorem 3.8].

2) If $a = b$ then $a * b = b * a = e$, so $a * b = (b * a) * e * (b * a)$,

3) If $b = e$ then $a * e = e * a = a$, so $a * e = (e * a) * a * (e * a)$.

On the other hand,

$$a * (b * c) = a * (bt + ct) = at + bt^2 + ct^2$$

$$(a * b) * c = (at + bt) * c = at^2 + bt^2 + ct.$$

So, the groupoid $G(n)$ is a semigroup.

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