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OD-characterization of $S_4(4)$ and its group of automorphisms

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Abstract. Let G be a finite group and $\pi(G)$ be the set of all prime divisors of $|G|$. The prime graph of G is a simple graph $\Gamma(G)$ with vertex set $\pi(G)$ and two distinct vertices p and q in $\pi(G)$ are adjacent by an edge if an only if G has an element of order pq. In this case, we write $p \sim q$. Let $|G| = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where $p_1 < p_2 < \ldots < p_k$ are primes. For $p \in \pi(G)$, let $deg(p) = |\{q \in \pi(G)|p \sim q\}|$ be the degree of p in the graph $\Gamma(G)$, we define $D(G) = (deg(p_1), deg(p_2), \ldots, deg(p_k))$ and call it the degree pattern of G. A group G is called k-fold OD-characterizable if there exist exactly k non-isomorphic groups S such that $|G| = |S|$ and $D(G) = D(S)$. Moreover, a 1-fold OD-characterizable group is simply called an OD-characterizable group. Let $L = S_4(4)$ be the projective symplectic group in dimension 4 over a field with 4 elements. In this article, we classify groups with the same order and degree pattern as an almost simple group related to L. Since $Aut(L) \cong Z_4$ hence almost simple groups related to L are L, $L : 2$ or $L : 4$. In fact, we prove that L, $L : 2$ and $L : 4$ are OD-characterizable.

Keywords: Finite simple group, OD-characterization, group of lie type

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1. Introduction

Let G be a finite group. Denote by $\pi(G)$ the set of all prime divisors of the order of G. The prime graph $\Gamma(G)$ of a finite group G is a simple graph with vertex set $\pi(G)$ in which two distinct vertices p and q are joined by an edge if and only if G has an element of order pq.

Definition 1.1 Let G be a finite group and $|G| = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where $p_1 < p_2 < \ldots < p_k$. For $p \in \pi(G)$, let $deg(p) = |\{q \in \pi(G)|p \sim q\}|$ be the degree of p in the graph $\Gamma(G)$,

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we define $D(G) = (deg(p_1), deg(p_2), \ldots, deg(p_k))$, which is called the degree pattern of G.

Given a finite group G, denote by $h_{OD}(G)$ the number of isomorphism classes of finite groups S such that $|G| = |S|$ and $D(G) = D(S)$. In terms of the function h_{OD}, groups G are classified as follows:

Definition 1.2 A group G is called k-fold OD-characterizable if $h_{OD}(G) = k$. Moreover, a 1-fold OD-characterizable group is simply called an OD-characterizable.

Definition 1.3 A group G is said to be an almost simple group if and only if $S \trianglelefteq G \trianglelefteq Aut(S)$ for some non-abelian simple group S.

2. Preliminaries

For any group G, let $\omega(G)$ be the set of orders of elements in G, where each possible order element occurs once in $\omega(G)$ regardless of how many elements of that order G has. This set is closed and partially ordered by divisibility, hence it is uniquely determined by its maximal elements. The set of maximal elements of $\omega(G)$ is denoted by $\mu(G)$. The number of connected components of $\Gamma(G)$ is denoted by $t(G)$. Let $\pi_i = \pi_i(G)$, $1 \leqslant i \leqslant t(G)$, be the ith connected component of $\Gamma(G)$. For a group of even order we let $2 \in \pi_1(G)$. We denote by $\pi(n)$ the set of all primes divisors of n, where n is a natural number. Then $|G|$ can be expressed as a product of $m_1, m_2, \ldots, m_{t(G)}$, where m_i's are positive integers with $\pi(m_i) = \pi_i$. The numbers m_i's, $1 \leqslant i \leqslant t(G)$, are called the order components of G. We write $OC(G) = \{m_1, m_2, \ldots, m_{t(G)}\}$ and call it the set of order components of G. The set of prime graph components of G is denoted by $T(G) = \{\pi_i(G) | i = 1, 2, \ldots, t(G)\}$.

Definition 2.1 Let n be a natural number. We say that a finite simple group G is a K_n-group if $|\pi(G)| = n$.

3. Elementary Results

Definition 3.1 A group G is called a 2-Frobenius group, if there exists a normal series $1 \lhd H \lhd K \lhd G$, such that K and G/H are Frobenius groups with kernels H and K/H, respectively.

Lemma 3.2 [3]Let G be a 2-Frobenius group of even order which has a normal series $1 \lhd H \lhd K \lhd G$, such that K and G/H are Frobenius groups with kernels H and K/H, respectively. Then

(a) $t(G) = 2$ and $T(G) = \{\pi_1(G) = \pi(H) \cup \pi(G/K), \pi_2(G) = \pi(K/H)\}$.
(b) G/K and K/H are cyclic groups, $|G/K| \mid |Aut(K/H)|$, and $(|G/K|, |K/H|) = 1$.
(c) H is a nilpotent group and G is a solvable group.

The following lemmas are useful when dealing with a Frobenius group.

Lemma 3.3 [5], [7]Let G be a Frobenius group with complement H and kernel K. Then the following assertions hold:

(a) K is a nilpotent group;
(b) $|K| \equiv 1(mod|H|)$;
(c) Every subgroup of H of order pq, with p, q (not necessarily distinct)primes, is cyclic. In particular, every Sylow Subgroup of H of odd order is cyclic and a

2-Sylow subgroup of H is either cyclic or a generalized quaternion group. If H is a non-solvable group, then H has a subgroup of index at most 2 isomorphic to $Z \times SL(2,5)$, where Z has cyclic Sylow p-subgroups and $\pi(Z) \cap \{2,3,5\} = \varnothing$. In particular, $15, 20 \notin \omega(H)$. If H is solvable and $O(H) = 1$, then either H is a 2-group or H has a subgroup of index at most 2 isomorphic to $SL(2,3)$.

Lemma 3.4 [3]Let G be a Frobenius group of even order where H and K are Frobenius complement and Frobenius kernel of G, respectively. Then $t(G) = 2$ and $T(G) = \{\pi(H), \pi(K)\}$.

Let G be a finite group with disconnected prime graph. The structure of G is given in [8] which is stated as a lemma here.

Lemma 3.5 Let G be a finite group with disconnected prime graph. Then G satisfies one of the following conditions:

a) $s(G) = 2$, $G = KC$ is a Frobenius group with kernel K and complement C, and the two connected components of G are $\Gamma(K)$ and $\Gamma(C)$. Moreover K is nilpotent, and here $\Gamma(K)$ is a complete graph.
b) $s(G) = 2$ and G is a 2-Frobeuius group, i.e. , $G = ABC$ where $A, AB \trianglelefteq G$, $B \trianglelefteq BC$, and AB, BC are Frobenius groups.
c) There exists a non-abelian simple group P such that $P \leqslant \overline{G} = \frac{G}{N} \leqslant Aut(P)$ for some nilpotent normal $\pi_1(G)$-subgroup N of G and $\frac{\overline{G}}{P}$ is a $\pi_1(G)$-group. Moreover, $\Gamma(P)$ is disconnected and $s(P) \geqslant s(G)$.

If a group G satisfies condition(c) of the above lemma we may write $P = B/N$, $B \leqslant G$, and $\frac{\overline{G}}{P} = G/B = A$, hence in terms of group extensions $G = N \cdot P \cdot A$, where N is a nilpotent normal $\pi_1(G)$-subgroup of G and A is a $\pi_1(G)$-group.

Theorem 3.6 [6]The following assertions are equivalent:

(a) G is a Frobenius group with kernel K and complement H.
(b) $G = HK$ such that $K \triangleleft G$ and $H < G$ and H act on K without fixed point.

By [2] the outer automorphism group of $S_4(4)$ is isomorphic to Z_4, hence we have the following lemma:

Lemma 3.7 If G is an almost simple group related to $L = S_4(4)$, then G is isomorphic to one of the following groups: L, $L : 2$ or $L : 4$.

4. Main Results

Theorem 4.1 If G is a finite group such that $D(G) = D(M)$ and $|G| = |M|$, where M is an almost simple group related to $L = S_4(4)$, then the following assertions hold:

(a) If $M = L$, then L is OD-characterizable.
(b) If $M = L : 2$, then $L : 2$ is OD-characterizable.
(c) If $M = L : 4$, then $L : 4$ is OD-characterizable.

Proof. We break the proof into a number of separate cases:
Case 1: If $M = L$, then $G \cong L$. This follows from [1].
Case 2: If $M = L : 2$, then $G \cong L : 2$.
If $M = L : 2$, by [2], we have $\mu(L : 2) = \{8, 10, 12, 15, 17\}$ from which we deduce that $D(L : 2) = (2, 2, 2, 0)$. The prime graph of $L : 2$ has the following form:

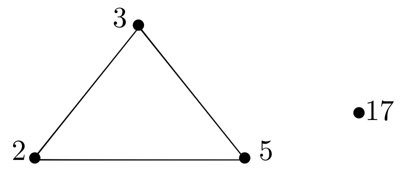

Figure 1: The prime graph of $S_4(4):2$

As $|G| = |L:2| = 2^9 \cdot 3^2 \cdot 5^2 \cdot 17$ and $D(G) = D(L:2) = (2,2,2,0)$, then $\Gamma(G) = \Gamma(M) = \{2 \sim 3, 2 \sim 5, 3 \sim 5; 17\}$. Thus G has a disconnected prime graph with $s(G) = 2$. Now, We show that G is neither a Frobenius group nor 2-Frobenius group. If G be a Frobenius group, then by Lemma 3.4(a), $G = KC$, with Frobenius kernel K and Frobenius complement C with connected components $\Gamma(K)$ and $\Gamma(C)$. $\Gamma(K)$ is a graph with vertex $\{17\}$ and $\Gamma(C)$ with vertices $\{2,3,5\}$. By Lemma 3.2(b), $|K| \mid (|C|-1)$. Since $|K| = 17$ and $|C| = 2^9.3^2.5^2$ then $17 \nmid (2^9.3^2.5^2 - 1)$ a contradiction. If G be a 2-Frobenius group, then, there is a normal series $1 \lhd H \lhd K \lhd G$ such that K and G/H are Frobenius groups with kernels H and K/H, respectively. By Lemma 3.1(a), we have $T(G) = \{\pi_1(G) = \pi(H) \cup \pi(G/K), \pi_2(G) = \pi(K/H)\}$. Therefore, $|K/H| = 17$. Also, by Lemma 3.1(b), we have $G/K \leqslant Aut(K/H) \cong Z_{16}$, hence $|G/K| \mid 2^4$, which implies that $\{3,5,17\} \subseteq \pi(K)$ from which we deduce that $5 \in \pi(H)$. Let $H_5 \in Syl_5(H)$ and $G_{17} \in Syl_{17}(G)$. Then $H_5 char H \trianglelefteq G$. By nilpotency of H, we have $H_5 \lhd G$ and H_5 act on G_{17} without fixed point, since $5 \nsim 17$ in $\Gamma(G)$. Therefore, by Theorem 3.1, $H_5.G_{17}$ is a Frobenius group. So, $|G_{17}| \mid (|H_5|-1)$, i.e., $17 \mid (5^i - 1)$, $i = 1$ or 2, a contradiction.

Now by Lemma 3.4(c), there exists a non-abelian simple group P such that $P \leqslant \overline{G} = G/N \leqslant Aut(P)$ for some nilpotent normal $\{2,3,5\}$-subgroup N of G and $\overline{G}/P$ is a $\{2,3,5\}$-group.

$17 \in \pi(P)$. Since $\overline{G}/P$ is a $\{2,3,5\}$-group and $17 \mid |G|$, therefore, we have $17 \mid |P|$, i.e., $P \in \mathfrak{S}_{17}$, which implies that $\pi(P) \subseteq \{2,3,5,17\}$. Using [9], we list the possibilities for P in the following table.

Table 1: Simple groups in $\mathfrak{S}_p$, $p \leqslant 17, p \neq 7, 11, 13$.

P	$\|P\|$	$\|out(P)\|$
$L_2(17)$	$2^4.3^2.17$	2
$L_2(16)$	$2^4.3.5.17$	4
$S_4(4)$	$2^8.3^2.5^2.17$	4

If $P \cong L_2(17)$ we get $L_2(17) \leqslant G/N \leqslant Aut(L_2(17))$. It follows that $|N| = 2^5 \cdot 5^2$ or $|N| = 2^4 \cdot 5^2$. Let $N_5 \in Syl_5(N)$ and $G_{17} \in Syl_{17}(G)$. Then $N_5 char N \trianglelefteq G$. By the nilpotency of N, which implies that $N_5 \trianglelefteq G$ and N_5 act on G_{17} without fixed point, since $5 \nsim 17$ in $\Gamma(G)$. Therefore, by Theorem 3.1, $N_5.G_{17}$ is a Frobenius group. So, $|G_{17}| \mid (|N_5|-1)$, i.e., $17 \mid (5^i - 1)$, $i = 1$ or 2, a contradiction.

If $P \cong L_2(16)$ we get $L_2(16) \leqslant G/N \leqslant Aut(L_2(16))$. It follows that $|N| = 2^5 \cdot 3 \cdot 5$ or $|N| = 2^3 \cdot 3 \cdot 5$. Let $N_5 \in Syl_5(N)$ and $G_{17} \in Syl_{17}(G)$. Then $N_5 char N \trianglelefteq G$. By the nilpotency of N, which implies that $N_5 \trianglelefteq G$ and N_5 act on G_{17} without fixed point, since $5 \nsim 17$ in $\Gamma(G)$. Therefore, by Theorem 3.1, $N_5.G_{17}$ is a Frobenius group. So, $|G_{17}| \mid (|N_5|-1)$, i.e., $17 \mid (5-1)$ a contradiction.

Therefore, $P \cong S_4(4)$. We have $S_4(4) \leqslant G/N \leqslant Aut(S_4(4))$. It follows that $|N| = 2$ or $|N| = 1$.

If $|N| = 1$, then $G \cong S_4(4) : 2$.

If $|N| = 2$, then $G/C_G(N) \leqslant Aut(N) = 1$, therefore $|G/C_G(N)| = 1$, hence $G = C_G(N)$ and $N \leqslant Z(G)$. Let $G_{17} \in Syl_{17}(G)$. Then $N.G_{17}$ is a subgroup of G, therefore, $N.G_{17}$ has an element of order 2.17, which implies that $2 \sim 17$ in $\Gamma(G)$, a contradiction.

Case 3: If $M = L : 4$, then $G \cong L : 4$.

If $M = L : 4$, by [2], we have $\mu(M) = \{12, 15, 16, 17, 20\}$ from which we deduce that $D(L : 4) = (2, 2, 2, 0)$. The prime graph of $L : 4$ has the following form:

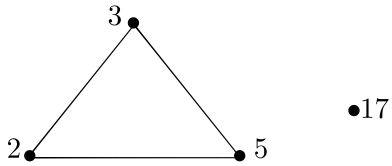

Figure 2: The prime graph of $S_4(4) : 4$

As $|G| = |L : 4| = 2^{10} \cdot 3^2 \cdot 5^2 \cdot 17$ and $D(G) = D(L : 4) = (2, 2, 2, 0)$, then $\Gamma(G) = \Gamma(M) = \{2 \sim 3, 2 \sim 5, 3 \sim 5; 17\}$. Thus G has a disconnected prime graph with $s(G) = 2$. Now, we show that G is neither a Frobenius group nor 2-Frobenius group. If G be a Frobenius group, then by Lemma 3.4(a), $G = KC$, with Frobenius kernel K and Frobenius complement C with connected components $\Gamma(K)$ and $\Gamma(C)$. $\Gamma(K)$ is a graph with vertex $\{17\}$ and $\Gamma(C)$ with vertices $\{2, 3, 5\}$. By Lemma 3.2(b), $|K| \mid (|C| - 1)$. Since $|K| = 17$ and $|C| = 2^{10}.3^2.5^2$ then $17 \nmid (2^{10}.3^2.5^2 - 1)$ a contradiction. If G be a 2-Frobenius group, then, there is a normal series $1 \lhd H \lhd K \lhd G$ such that K and G/H are Frobenius groups with kernels H and K/H, respectively. By Lemma 3.1(a), we have $T(G) = \{\pi_1(G) = \pi(H) \cup \pi(G/K), \pi_2(G) = \pi(K/H)\}$. Therefore, $|K/H| = 17$. Also, by Lemma 3.1(b), we have $G/K \leqslant Aut(K/H) \cong Z_{16}$, hence $|G/K| \mid 2^4$, which implies that $\{3, 5, 17\} \subseteq \pi(K)$ from which we deduce that $5 \in \pi(H)$. Let $H_5 \in Syl_5(H)$ and $G_{17} \in Syl_{17}(G)$. Then $H_5 char H \unlhd G$. By nilpotency of H, we have $H_5 \lhd G$ and H_5 act on G_{17} without fixed point, since $5 \nsim 17$ in $\Gamma(G)$. Therefore, by Theorem 3.1, $H_5.G_{17}$ is a Frobenius group. So, $|G_{17}| \mid (|H_5| - 1)$, i.e., $17 \mid (5^i - 1)$, $i = 1$ or 2, a contradiction.

Now by Lemma 3.4(c), there exists a non-abelian simple group P such that $P \leqslant \overline{G} = G/N \leqslant Aut(P)$ for some nilpotent normal $\{2, 3, 5\}$-subgroup N of G and $\overline{G}/P$ is a $\{2, 3, 5\}$-group.

Similarly to case 2, we deduce that $P \cong S_4(4)$. We have $S_4(4) \leqslant G/N \leqslant Aut(S_4(4))$. It follows that $|N| = 4$, 2 or 1.

If $|N| = 1$, then $G \cong S_4(4) : 4$.

If $|N| = 2$, then $G/C_G(N) \leqslant Aut(N) = 1$, therefore $|G/C_G(N)| = 1$, hence $G = C_G(N)$ and $N \leqslant Z(G)$. Let $G_{17} \in Syl_{17}(G)$. Then $N.G_{17}$ is a subgroup of G, therefore, $N.G_{17}$ has an element of order 2.17, which implies that $2 \sim 17$ in $\Gamma(G)$, a contradiction.

If $|N| = 4$, then $G/C_G(N) \leqslant Aut(N) \cong Z_2$. Thus, $|G/C_G(N)| = 1$ or 2. If $|G/C_G(N)| = 1$, then, we have $N \leqslant Z(G)$. Let $G_{17} \in Syl_{17}(G)$. Then $N.G_{17}$ is a subgroup of G, therefore, $N.G_{17}$ has an element of order 2.17, which implies that $2 \sim 17$ in $\Gamma(G)$, a contradiction. If $|G/C_G(N)| = 2$, then $N < C_G(N)$ and $1 \neq C_G(N)/N \unlhd G/N \cong L$. Therefore, from simplicity L we deduce that $G = C_G(K)$, a contradiction. ■

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