

On the nonnegative inverse eigenvalue problem of tridiagonal matrices

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Abstract. In this paper, at first for a given set of real or complex numbers σ with nonnegative summation, we introduce some special conditions that with them there is no nonnegative tridiagonal matrix in which σ is its spectrum. In continue we present some conditions for existence such nonnegative tridiagonal matrices.

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1. Introduction

The nonnegative inverse eigenvalue problem (NIEP) asks for necessary and sufficient conditions on a list $\sigma = (\lambda_1, \lambda_2, \dots, \lambda_n)$ of complex numbers in order that it be the spectrum of a nonnegative matrix. In terms of n the NIEP solve only for $n \leq 5$ [1,2,3,4,5].

The problem of constructing a symmetrical tridiagonal matrix from certain spectral information is important in many applications, such as vibration theory, structural design, control theory, and it has attracted the attention of many authors [7,8,9]. In this paper we discuss about inverse eigenvalue problem for nonnegative tridiagonal matrices.

The spectral radius of nonnegative matrix A denoted by $\rho(A)$. There is a right and a left eigenvector associated with the Perron eigenvalue with nonnegative entries. In addition s_k the k -th power sum of the eigenvalues λ_i and in the list σ , λ_1 is the Perron element.

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Some necessary conditions on the list of complex number $\sigma = (\lambda_1, \lambda_2, \dots, \lambda_n)$ to be the spectrum of a nonnegative matrix are listed below.

- (1) The Perron eigenvalue $\max\{|\lambda_i|; \lambda_i \in \sigma\}$ belongs to σ (Perron-Frobenius theorem).
- (2) $s_k = \sum_{i=1}^n \lambda_i^k \geq 0$.
- (3) $s_k^m \leq n^{m-1} s_{km}$ for $k, m = 1, 2, \dots$ (JLL inequality)[2,6].

We recall below Theorem 2.1 of [5] that is similar to Lemma 5 of [3] and by using this Theorem, we construct a $n \times n$ nonnegative tridiagonal matrix for a given set which satisfies in the special conditions in a recursive method for $n \leq 5$.

Theorem 1.1 Let B be a $m \times m$ nonnegative matrix, $M_1 = \{\mu_1, \mu_2, \dots, \mu_m\}$ be its eigenvalues and μ_1 be Perron eigenvalue of B . Also assume that A is a $n \times n$ nonnegative matrix in following form $A = \begin{pmatrix} A_1 & a \\ b^T & \mu_1 \end{pmatrix}$, where A_1 is a $(n-1) \times (n-1)$ matrix, a and b are arbitrary vectors in \mathbb{R}^{n-1} and $M_2 = \{\lambda_1, \lambda_2, \dots, \lambda_m\}$ is the set of eigenvalues of A . Then there exist a $(m+n-1) \times (m+n-1)$ nonnegative matrix such that $M = \{\mu_2, \dots, \mu_m, \lambda_1, \lambda_2, \dots, \lambda_m\}$ is its eigenvalues.

In section 2 of this paper we show that for a given set of real or complex numbers with nonnegative summation that satisfies in following conditions:

$$\alpha_1 = \sum_{1 \leq i < j \leq n} \lambda_i \lambda_j \leq 0, \quad (1.1)$$

$$\alpha_2 = \sum_{1 \leq i < j < k \leq n} \lambda_i \lambda_j \lambda_k > 0, \quad (1.2)$$

there is no nonnegative tridiagonal matrix that σ is spectrum.

In section 3 for $n \leq 5$ and for set of real numbers σ we introduce some necessary conditions for existence nonnegative tridiagonal matrix that realizes σ . We also present some cases that there is no solution of problem.

2. Absence solution

Theorem 2.1 Let $\sigma = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ be a set of complex numbers that satisfies in (1.1) and (1.2) and following conditions

- (1) $\lambda_1 > 0$
- (2) $\sum_{i=1}^n \lambda_i \geq 0$
- (3) $\lambda_1 > |\lambda_i|, i = 2, \dots, n$.

Then there is no any nonnegative tridiagonal matrix with spectrum σ .

Proof. If λ_i for $i = 1, 2, \dots, n$ are the eigenvalues of $n \times n$ matrix, then the its characteristic polynomials is as follows

$$p(\lambda) = \lambda^n - \left(\sum_{i=1}^n \lambda_i \right) \lambda^{n-1} + \left(\sum_{1 \leq i < j \leq n} \lambda_i \lambda_j \right) \lambda^{n-2} - \left(\sum_{1 \leq i < j < k \leq n} \lambda_i \lambda_j \lambda_k \right) \lambda^{n-3} + \dots \quad (2.2)$$

We continue proof by reductio ad absurdum. Assume that there exist the $n \times n$ nonnegative tridiagonal matrix $A = (a_{ij})_{n \times n}$ as

$$\begin{pmatrix} a_{11} & a_{12} & & & & \\ a_{21} & a_{22} & a_{23} & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & a_{n-1,n} \\ & & & & a_{n,n-1} & a_{n,n} \end{pmatrix}$$

such that λ_i for $i = 1, 2, \dots, n$ are its spectrum. Therefore the characteristics polynomial of A is

$$P_A(\lambda) = \lambda^n - (\sum_{i=1}^n a_{ii})\lambda^{n-1} + (\sum_{1 \leq i < j \leq n} a_{ii}a_{jj} - \sum_{i=1}^{n-1} a_{i,i+1}a_{i+1,i})\lambda^{n-2} - (\sum_{1 \leq i < j < k \leq n} a_{ii}a_{jj}a_{kk} - \sum_{i=1}^n \sum_{\substack{j=1 \\ i \neq j}}^{n-1} a_{ii}a_{j,j+1}a_{j+1,j})\lambda^{n-3} + \dots \tag{2.3}$$

By relations (1.1) and (1.2) we have

$$\sum_{i=1}^{n-1} a_{i,i-1}a_{i+1,i} > \sum_{1 \leq i < j \leq n} a_{ii}a_{jj}, \tag{2.4}$$

$$\sum_{1 \leq i < j < k \leq n} a_{ii}a_{jj}a_{kk} > \sum_{i=1}^n \sum_{\substack{j=1 \\ i \neq j}}^{n-1} a_{ii}a_{j,j+1}a_{j+1,j}. \tag{2.5}$$

With lose of generality we assume that

$$\left. \begin{array}{l} a_{33} = \min \{a_{11}, a_{33}\} \qquad \qquad \qquad \text{for } n = 3 \\ a_{44} = \min \{a_{22}, a_{44}\}, a_{33} = \min \{a_{11}, a_{33}\} \qquad \qquad \text{for } n = 4 \\ a_{55} = \min \{a_{33}, a_{55}\}, a_{44} = \min \{a_{22}, a_{44}\}, a_{33} = \min \{a_{11}, a_{33}\} \text{ for } n = 5 \\ \vdots \\ a_{n,n} = \min \{a_{n-2,n-2}, a_{n,n}\}, a_{n-1,n-1} = \\ \min \{a_{n-3,n-3}, a_{n-1,n-1}\}, \dots, a_{33} = \min \{a_{11}, a_{33}\}. \end{array} \right\} \tag{2.6}$$

If we replace the relation (2.6) to the right hand side of (2.5) we can reach to the relation

that includes the right hand side of relation (2.4). i.e.

$$\begin{aligned} \sum_{1 \leq i < j < k \leq n} a_{ii} a_{jj} a_{kk} &> \sum_{i=1}^n \sum_{\substack{j=1 \\ i \neq j}}^{n-1} a_{ii} a_{j,j+1} a_{j+1,j} = \\ &a_{11} a_{23} a_{32} + a_{11} a_{34} a_{43} + \dots + a_{11} a_{n-1,n} a_{n,n-1} + \\ &a_{22} a_{34} a_{43} + a_{22} a_{45} a_{54} + \dots + a_{22} a_{n-1,n} a_{n,n-1} + \\ &a_{33} a_{12} a_{21} + a_{33} a_{45} a_{54} + \dots + a_{33} a_{n-1,n} a_{n,n-1} + \dots + a_{n-1,n-1} a_{12} a_{21} + \\ &a_{n-1,n-1} a_{23} a_{32} + \dots + a_{n-1,n-1} a_{n-3,n-2} a_{n-2,n-3} + a_{n,n} a_{12} a_{21} + a_{n,n} a_{23} a_{32} + \\ &\dots + a_{n,n} a_{n-3,n-2} a_{n-2,n-3} + a_{n,n} a_{n-2,n-1} a_{n-1,n-2} > \\ &a_{33} \left(\sum_{i=1}^{n-1} a_{i,i-1} a_{i+1,i} \right) + a_{44} \left(\sum_{i=1}^{n-1} a_{i,i-1} a_{i+1,i} \right) + \\ &a_{55} \left(\sum_{i=1}^{n-1} a_{i,i-1} a_{i+1,i} \right) + \dots + a_{n,n} \left(\sum_{i=1}^{n-1} a_{i,i-1} a_{i+1,i} \right) > \\ &a_{33} \left(\sum_{1 \leq i < j \leq n} a_{ii} a_{jj} \right) + a_{44} \left(\sum_{1 \leq i < j \leq n} a_{ii} a_{jj} \right) + a_{55} \left(\sum_{1 \leq i < j \leq n} a_{ii} a_{jj} \right) + \dots + a_{n,n} \\ &\left(\sum_{1 \leq i < j \leq n} a_{ii} a_{jj} \right) \end{aligned}$$

After simplifying of the above relations we reach to the following relation

$$\begin{aligned} &a_{33}^2 (a_{11} + a_{22} + a_{44} + \dots + a_{n,n}) + a_{44}^2 (a_{11} + \\ &a_{22} + a_{33} + a_{55} + \dots + a_{n,n}) + \dots + a_{n,n}^2 (a_{11} + a_{22} + a_{33} + \\ &\dots + a_{n-2,n-2} + a_{n-1,n-1}) + a_{33} a_{44} (a_{11} + a_{22} + a_{55} + \\ &a_{66} + \dots + a_{n,n}) + a_{33} a_{55} (a_{11} + a_{22} + a_{44} + a_{66} + a_{77} + \dots + a_{n,n}) + a_{33} a_{n,n} (a_{11} + \\ &a_{22} + a_{44} + \dots + a_{n-1,n-1}) + a_{44} a_{55} (a_{11} + a_{22} + a_{66} + a_{77} + \dots + a_{n,n}) + a_{44} a_{66} (a_{11} + \\ &a_{22} + a_{55} + a_{77} + \dots + a_{n,n}) + \dots + a_{44} a_{n,n} (a_{11} + a_{22} + \\ &a_{55} + a_{66} + \dots + a_{n-1,n-1}) + a_{55} a_{66} (a_{11} + \\ &a_{22} + a_{77} + a_{88} + \dots + a_{n,n}) + a_{55} a_{77} (a_{11} + a_{22} + a_{66} + a_{88} + a_{99} + \dots + a_{n,n}) + \\ &\dots + a_{55} a_{n,n} (a_{11} + a_{22} + a_{66} + a_{77} + \dots + a_{n-1,n-1}) + \dots + a_{n-1,n-1} a_{n,n} (a_{11} + a_{22}) < 0. \end{aligned}$$

And this means the summation of sum of nonnegative numbers is strictly negative and this is impossible. \square

Corollary 2.2 Let $\sigma = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ is a set of real numbers that holds in (2.1) and for $i = 2, 3, \dots, n$ we have $\lambda_i < 0$, then there is no nonnegative tridiagonal matrix that σ is its spectrum.

Proof. If $\lambda_i < 0$ for $i = 2, 3, \dots, n$, it is obvious the relations (1.1) and (1.2) are hold, and therefore by Theorem 2.1 proof is complete. \square

3. Existence and construction

In this section we study existence (with construction) or absence of nonnegative tridiagonal matrix of order maximum 5, for a given set of real numbers σ with $|\sigma| = n \leq 5$ that σ is its spectrum.

• The case $n = 2$

Theorem 3.1 Let $\sigma = \{\lambda_1, \lambda_2\}$ be a set of real numbers such that satisfies relation (2.1) then σ is the set of eigenvalues of a nonnegative tridiagonal matrix.

Proof. σ has only one of following cases:

(a) If $\lambda_2 \geq 0$, then $A = \text{diag}(\lambda_1, \lambda_2)$ is a solution of problem.

(b) If $\lambda_2 < 0$, then the matrix

$$A = \begin{pmatrix} 0 & -\lambda_1\lambda_2 \\ 1 & \lambda_1 + \lambda_2 \end{pmatrix}, \quad (3.1)$$

solves the problem. \square

• **The case $n = 3$**

Theorem 3.2 Let $\sigma = \{\lambda_1, \lambda_2, \lambda_3\}$ be a set of real numbers that satisfies relation (2.1). Then we have the following cases:

- (a) If $\lambda_2, \lambda_3 \geq 0$, then σ is the set of eigenvalues of a nonnegative tridiagonal matrix.
- (b) If $\lambda_2, \lambda_3 < 0$, then there is no nonnegative tridiagonal matrix with spectrum σ .
- (c) If $\lambda_2 < 0$ and $\lambda_3 \geq 0$, then σ is the set of eigenvalues of a nonnegative tridiagonal matrix.

Proof.

- (a) $B = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$ is a solution of our problem.
- (b) Corollary 2.2
- (c) The nonnegative tridiagonal matrix

$$B = \begin{pmatrix} A & 0 \\ 0^T & \lambda_3 \end{pmatrix}, \quad (3.2)$$

is a solution of our problem where A is matrix (3.1) and o is zero vector with dimension 2×1 . \square

• **The case $n = 4$**

Theorem 3.3 Let $\sigma = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ be a set of real numbers that satisfies relation (2.1). Then we have the following cases:

- (a) If $\lambda_2, \lambda_3, \lambda_4 \geq 0$, then σ is the set of eigenvalues of a nonnegative tridiagonal matrix.
- (b) If $\lambda_2, \lambda_3, \lambda_4 < 0$, then there is no nonnegative tridiagonal matrix with spectrum σ .
- (c) If $\lambda_2 < 0$ and $\lambda_3, \lambda_4 \geq 0$, then σ is the set of eigenvalues of a nonnegative tridiagonal matrix.
- (d) If $\lambda_2, \lambda_3 \leq 0$, $\lambda_4 > 0$ and at least for one of the eigenvalues λ_2 and λ_3 , for example λ_3 , we have $\lambda_3 + \lambda_4 \geq 0$ then σ is the set of eigenvalues of a nonnegative tridiagonal matrix.
- (e) If $\lambda_2, \lambda_3 \leq 0$, $\lambda_4 > 0$ and we have $\lambda_2 + \lambda_4 \leq 0$, $\lambda_3 + \lambda_4 \leq 0$ and relation (1.2) then there is no nonnegative tridiagonal matrix with spectrum σ .

Proof.

- (a) $C = \text{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ is a solution of our problem.
- (b) Corollary 2.2
- (c) The nonnegative tridiagonal matrix

$$C = \begin{pmatrix} B & o \\ o^T & \lambda_4 \end{pmatrix}, \quad (3.3)$$

is a solution of our problem where B is matrix (3.2) and o is zero vector with dimension of 3×1 .

(d) The nonnegative tridiagonal matrix

$$C = \begin{pmatrix} 0 & -\lambda_1\lambda_2 & 0 & 0 \\ 1 & \lambda_1 + \lambda_2 & 0 & 0 \\ 0 & 0 & 0 & -\lambda_3\lambda_4 \\ 0 & 0 & 1 & \lambda_3 + \lambda_4 \end{pmatrix}, \quad (3.4)$$

is a solution of our problem.

(e) If $\lambda_2 + \lambda_4 \leq 0$, $\lambda_3 + \lambda_4 \leq 0$ and $\lambda_2, \lambda_3 \leq 0$, $\lambda_4 > 0$ then $\lambda_2, \lambda_3 < 0$ and $\alpha_1 = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4 = \lambda_1(\lambda_3 + \lambda_4) + \lambda_2(\lambda_3 + \lambda_4) + \lambda_1\lambda_2 + \lambda_3\lambda_4 < 0$ and if we have relation (1.2) then by Theorem 2.1 proof is complete. \square

• **The case $n = 5$**

Theorem 3.4 Let $\sigma = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$ be a set of real numbers that satisfies relation (2.1). Then we have the following cases:

(a) If $\lambda_2, \lambda_3, \lambda_4, \lambda_5 \geq 0$, then σ is the set of eigenvalues of a nonnegative tridiagonal matrix.

(b) If $\lambda_2, \lambda_3, \lambda_4, \lambda_5 < 0$, then there is no nonnegative tridiagonal matrix with spectrum σ .

(c) If $\lambda_2 < 0$ and $\lambda_3, \lambda_4, \lambda_5 \geq 0$, then σ is the set of eigenvalues of a nonnegative tridiagonal matrix.

(d) If $\lambda_2, \lambda_3 < 0$ and $\lambda_4, \lambda_5 \geq 0$ and at least for one of the eigenvalues λ_2 and λ_3 , for example λ_3 , we have $\lambda_3 + \lambda_4 \geq 0$ or $\lambda_3 + \lambda_5 \geq 0$ then σ is the set of eigenvalues of a nonnegative tridiagonal matrix.

(e) If $\lambda_2, \lambda_3 < 0$, $\lambda_4, \lambda_5 \geq 0$ and we have $\lambda_2 + \lambda_4 \leq 0$, $\lambda_3 + \lambda_4 \leq 0$ and $\lambda_2 + \lambda_5 \leq 0$, $\lambda_3 + \lambda_5 \leq 0$ and relation (1.2) then there is no nonnegative tridiagonal matrix with spectrum σ .

(f) If $\lambda_2, \lambda_3, \lambda_4 < 0$, $\lambda_5 \geq 0$ and we have $\lambda_2 + \lambda_5 < 0$, $\lambda_3 + \lambda_5 < 0$, $\lambda_4 + \lambda_5 < 0$ then there is no nonnegative tridiagonal matrix with spectrum σ .

Proof.

(a) $D = \text{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$ is a solution of our problem.

(b) Corollary 2.2

(c) The nonnegative tridiagonal matrix

$$D = \begin{pmatrix} C & o \\ o^T & \lambda_5 \end{pmatrix}, \quad (3.5)$$

is a solution of our problem where C is matrix (3.3) and o is zero vector with dimension of 4×1 .

(d) The nonnegative tridiagonal matrix

$$D = \begin{pmatrix} 0 & -\lambda_1\lambda_2 & 0 & 0 & 0 \\ 1 & \lambda_1 + \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\lambda_3\lambda_4 & 0 \\ 0 & 0 & 1 & \lambda_3 + \lambda_4 & 0 \\ 0 & 0 & 0 & 0 & \lambda_5 \end{pmatrix}, \quad (3.6)$$

is a solution of our problem.

(e) If $\lambda_2 + \lambda_4 \leq 0$, $\lambda_3 + \lambda_4 \leq 0$ and $\lambda_2 + \lambda_5 \leq 0$, $\lambda_3 + \lambda_5 \leq 0$ then $\alpha_1 = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_1\lambda_5 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_2\lambda_5 + \lambda_3\lambda_4 + \lambda_3\lambda_5 + \lambda_4\lambda_5 = \lambda_1(\lambda_3 + \lambda_4) + \lambda_2(\lambda_3 + \lambda_4) + \lambda_1(\lambda_2 + \lambda_5) + \lambda_5(\lambda_2 + \lambda_4) + \lambda_3(\lambda_4 + \lambda_5) < 0$ and if we have relation (1.2) then by theorem 2.1 proof is complete.

(f) The relations (1.1) and (1.2) are hold because

$$\alpha_1 = \sum_{1 \leq i < j \leq 5} \lambda_i \lambda_j = \lambda_1(\lambda_2 + \lambda_5) + \lambda_3(\lambda_1 + \lambda_4) + \lambda_4(\lambda_1 + \lambda_2) + \lambda_5(\lambda_2 + \lambda_4) + \lambda_3(\lambda_2 + \lambda_5) < 0$$

and

$$\alpha_2 = \sum_{1 \leq i < j < k \leq 5} \lambda_i \lambda_j \lambda_k = \lambda_1\lambda_2(\lambda_3 + \lambda_5) + \lambda_1\lambda_4(\lambda_2 + \lambda_5) + \lambda_1\lambda_3(\lambda_4 + \lambda_5) + \lambda_2\lambda_3(\lambda_4 + \lambda_5) + \lambda_4\lambda_5(\lambda_2 + \lambda_3) > 0$$

Therefore by Theorem 2.1 proof is complete. \square

4. The conjecture

In this section we introduce a conjecture that is proved above with some conditions:

If relation (1.2) is not hold, again there is no any nonnegative tridiagonal matrix for (e) of The case $n = 4$ and The case $n = 5$.

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