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A new approximation to the solution of the linear matrix equation AXB = C

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Abstract. It is well-known that the matrix equations play a significant role in several applications in science and engineering. There are various approaches either direct methods or iterative methods to evaluate the solution of these equations. In this research article, the homotopy perturbation method (HPM) will employ to deduce the approximated solution of the linear matrix equation in the form $\mathbf{AXB} = \mathbf{C}$. Furthermore, the conditions will be explored to check the convergence of the homotopy series. Numerical examples are also adapted to illustrate the properties of the modified method.

Keywords: Matrix equation; homotopy perturbation method; diagonally dominant matrix; convergence; iterative method.

2010 AMS Subject Classification: 15A24, 15A57.

1. Introduction

The linear matrix equations have a major role in numerous applications in science and engineering such as evaluation of implicit numerical schemes for partial differential equations, decoupling techniques for ordinary differential equations, image restoration, signal processing, filtering, model reduction, block-diagonalization of matrices, computation of the matrix functions, and control theory [1, 2, 4, 6].

It is known that the linear matrix equation in the form $\mathbf{AXB} = \mathbf{C}$ is important in linear system theory which has been actively studied since the 1960s. For instance, Liao

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and Lei [8] derived a least squares solution with the minimum norm to this matrix equation. Liu [14] employed the matrix rank method to provide the necessary and sufficient condition for solving linear matrix equation. With concern this problem, Navarra et al. [16] proposed a representation of the modified solution to this equation by utilizing the generalized singular value decomposition and the canonical correlation decomposition. Furthermore, Yuan et al. proposed a least squares Hermitian solution with the least norm over the skew field of a quaternion [26]. J. Ding et. al [5] proposed numerical solutions to the linear matrix equations in the form $\mathbf{A}_1\mathbf{X}\mathbf{B}_1 = \mathbf{F}_1$ and $\mathbf{A}_2\mathbf{X}\mathbf{B}_2 = \mathbf{F}_2$. They introduced two iterative algorithms to attain the solutions. Moreover, they mentioned that for any initial value, the iterative solutions obtained by the proposed algorithms converge to their true values. For more information, on can refer to [3, 15, 21–23, 25].

As we know, Doctor Liao in a fundamental article [7] introduced the basic ideas of homotopy to propose a general method for nonlinear problems on 1992. During past two decades, this approach has been successfully applied to approximate many types of nonlinear problems. Following him, an analytic approach based on the same theory in 1998, which is so called "homotopy perturbation method" (HPM), is provided by Doctor He [9–11], as well as the recent developments. In the most cases, using HPM, gives a very rapid convergence of the solution series, and usually only a few iterations leading to very accurate solutions.

In terms of linear algebra, Keramati [12] first applied a HPM to solve linear system of equations $\mathbf{Ax} = \mathbf{b}$. He demonstrated that the splitting matrix of this method is only the identity matrix. However, this method does not converge for some systems when the spectrum radius is greater than one. In order to solve this issue, Liu [13] was added the auxiliary parameter and the auxiliary matrix to the homotopy method. He has adjusted the Richardson method, the Jacobi method, and the Gauss-Seidel method to choose the splitting matrix. Yusufoglu [27] has also considered a similar technique with different auxiliary parameter, and he has obtained the exact solution of the linear system of equations. Following him, Noor [17] has proposed a technique that was more flexible than the method in [27]. Moreover, Edalatpanah and Rashidi [18] focused on modification of (HPM) for solving systems of linear equations by choosing an auxiliary matrix to increase the rate of convergence. More recently, Saeidian et. al [24] proposed an iterative scheme to solve linear systems equations based on the concept of homotopy. They have shown that their modified method presents more cases of convergence.

According to our knowledge, nevertheless the homotopy perturbation method was not adjusted to solve matrix equations. In this work, an improvement of the HPM to find an approximation to the solution of the matrix equations will be given. We also investigate the necessary and sufficient conditions for convergence of the method. Finally, some numerical examples are carried out.

2. Main results

Consider the linear matrix equation

$$\mathbf{AXB} = \mathbf{C}, \tag{1}$$

wherein, $\mathbf{A} = [a_{ij}] \in \mathbb{R}^{m\times n}, \mathbf{B} = [b_{ij}] \in \mathbb{R}^{m\times q}, \mathbf{C} = [c_{ij}] \in \mathbb{R}^{p\times q}$. In the following theorem, first the existence and uniqueness of the solution of (1) are pointed out.

Theorem 2.1 [19] The equation (1) has a solution if and only if $\mathbf{AA^{\dagger}BC^{\dagger}C} = \mathbf{B}$. in which case the general solution is in the form

$$\mathbf{X} = \mathbf{A}^{+}\mathbf{C}\mathbf{B}^{+} + \mathbf{Y} - \mathbf{A}^{+}\mathbf{AYBB}^{+}, \tag{2}$$

where, $\mathbf{Y} \in \mathbb{R}^{n\times p}$ is arbitrary matrix. The solution of (1) is unique if $\mathbf{BB}^{\dagger} \otimes \mathbf{A}^{\dagger}\mathbf{A} = \mathbf{I}$.

Remark 1 *It should be remarked that whenever the system (1) is non-square, because of disagreement between dimensions of the matrices, the homotopy function cannot be constructed. Thus, we have focused on solving linear matrix equation when the system (1) is square.*

Now, we would like to solve the equation (1) via HPM. A general type of homotopy method for solving $\mathbf{AXB} - \mathbf{C} = \mathbf{0}$ can be described as follows. The homotopy $H(\mathbf{U}, p)$ can be defined by

$$H(\mathbf{U}, p) = (1-p)F(\mathbf{U}) + pL(\mathbf{U}) = \mathbf{0}, \tag{3}$$

where,

$$H(\mathbf{U}, 0) = \mathbf{AUB} - \mathbf{C} = F(\mathbf{U}), \tag{4}$$

$$H(\mathbf{U}, 1) = \mathbf{U} - \mathbf{W}_0 = L(\mathbf{U}). \tag{5}$$

In this case, HPM utilizes the homotopy parameter p as an expanding parameter to obtain

$$\mathbf{U} = \sum_{i=0}^{\infty} p^i \mathbf{U}_i = \mathbf{U}_0 + p\mathbf{U}_1 + p^2\mathbf{U}_2 + \cdots, \tag{6}$$

and it gives an approximation to the solution of (1) as following

$$\mathbf{V} = \lim_{p\to 1}\left(\sum_{i=0}^{\infty} p^i \mathbf{U}_i\right) = \sum_{i=0}^{\infty} \mathbf{U}_i = \mathbf{U}_0 + \mathbf{U}_1 + \mathbf{U}_2 + \cdots. \tag{7}$$

By substituting (4) and (5) in (3), and by equating the terms with the identical power of p, we attain

$$\begin{cases} p^0 : \mathbf{U}_0 - \mathbf{W}_0 = \mathbf{0}, \\ p^1 : -\mathbf{U}_0 + \mathbf{U}_1 - \mathbf{W}_0 + \mathbf{AU}_0\mathbf{B} - \mathbf{C} = \mathbf{0}, \\ p^2 : \mathbf{U}_2 - \mathbf{U}_1 + \mathbf{AU}_1\mathbf{B} = \mathbf{0}, \\ \qquad\vdots \\ p^i : \mathbf{U}_{i+1} - \mathbf{U}_i + \mathbf{AU}_i\mathbf{B} = \mathbf{0}, \quad i = 1, 2, \cdots. \end{cases} \tag{8}$$

In other words, after application and simplification of its reciprocal, we obtain

$$\begin{cases} p^0 : \mathbf{U}_0 = \mathbf{W}_0, \\ p^1 : \mathbf{U}_1 = \mathbf{U}_0 + \mathbf{W}_0 - \mathbf{A}\mathbf{U}_0\mathbf{B} + \mathbf{C}, \\ p^2 : \mathbf{U}_2 = \mathbf{U}_1 - \mathbf{A}\mathbf{U}_1\mathbf{B}, \\ \qquad \vdots \\ p^i : \mathbf{U}_{i+1} = \mathbf{U}_i - \mathbf{A}\mathbf{U}_i\mathbf{B}, \quad i = 1, 2, \cdots . \end{cases} \tag{9}$$

Now, taking $\mathbf{U}_0 = \mathbf{W}_0 = \mathbf{0}$, it can be found that

$$\begin{cases} p^0 : \mathbf{U}_0 = \mathbf{W}_0, \\ p^1 : \mathbf{U}_1 = \mathbf{C}, \\ p^2 : \mathbf{U}_2 = -(\mathbf{A}\mathbf{U}_1\mathbf{B} - \mathbf{U}_1), \\ \qquad \vdots \\ p^i : \mathbf{U}_{i+1} = (-1)^i(\mathbf{A}\mathbf{U}_i\mathbf{B} - \mathbf{U}_i), \quad i = 1, 2, \cdots . \end{cases} \tag{10}$$

In fact, by using vector operator and Keronecker product, the matrix equation (11) could be transformed to linear systems as follows:

$$\begin{cases} \mathrm{vec}(\mathbf{U}_1) = \mathrm{vec}(\mathbf{C}), \\ \mathrm{vec}(\mathbf{U}_2) = -((\mathbf{B}^t \otimes \mathbf{A}) - \mathbf{I}_{n^2})\mathrm{vec}(\mathbf{U}_1), \\ \qquad \vdots \\ \mathrm{vec}(\mathbf{U}_{i+1}) = (-1)^i((\mathbf{B}^t \otimes \mathbf{A}) - \mathbf{I}_{n^2})^i\mathrm{vec}(\mathbf{U}_i). \quad i = 1, 2, \cdots . \end{cases} \tag{11}$$

Hence, according to the property of the vector operator, the solution is in the form

$$\mathrm{vec}(\mathbf{V}) = \mathrm{vec}\left(\sum_{i=0}^{\infty} \mathbf{U}_i\right) = \sum_{i=0}^{\infty} \mathrm{vec}(\mathbf{U}_i).$$

Consequently, the approximated solution of equation (1) is obtained by

$$\mathrm{vec}(\mathbf{V}) \approx \sum_{k=0}^{\infty}(-1)^k \left((\mathbf{B}^t \otimes \mathbf{A}) - \mathbf{I}_{n^2}\right)^k \mathrm{vec}(\mathbf{C}). \tag{12}$$

Now, an important issue that must be proven is to show the convergency of the new method for finding the solution of linear matrix equation. In the following theorem, we will show that the series (12) is converges to the solution of (1). Before this, we emphasize that $\rho(\mathbf{W})$ is denoted the spectral radius of $\mathbf{W}$, and defined by $\rho(\mathbf{W}) = \max_{\lambda_i \in \sigma(\mathbf{W})} |\lambda_i|$.

Theorem 2.2 The sequence $S_m = \sum_{k=0}^{m}(-1)^k \left((\mathbf{B}^t \otimes \mathbf{A}) - \mathbf{I}_{n^2}\right)^k \mathrm{vec}(\mathbf{C})$ is a Cauchy sequence if the following condition is satisfied:

$$\rho\left((\mathbf{B}^t \otimes \mathbf{A}) - \mathbf{I}_{n^2}\right) < 1. \tag{13}$$

Proof. It is straightforward that

$$S_{m+q} - S_m = \sum_{k=1}^{q} (-1)^{m+k} \left((\mathbf{B}^t \otimes \mathbf{A}) - \mathbf{I}_{n^2} \right)^{m+k} \mathrm{vec}(\mathbf{C}).$$

Taking norm and setting $\tau = \|(\mathbf{B}^t \otimes \mathbf{A}) - \mathbf{I}_{n^2}\|$, we can write

$$\|S_{m+q} - S_m\| \leqslant \|\mathrm{vec}(\mathbf{C})\| \sum_{k=1}^{q} \tau^{m+k} = \|\mathrm{vec}(\mathbf{C})\| \left(\frac{\tau^q - 1}{\tau - 1} \right) \tau^{m+1}$$

Now, if $\tau < 1$ then we have $\lim_{m\to\infty} \tau^m = 0$. Subsequently, we have

$$\lim_{m\to\infty} \|S_{m+p} - S_m\| = 0$$

Therefore, S_m is a Cauchy sequence which the convergence is established. ■

Corollary 2.3 The sequence $S_m = \sum_{k=0}^{m} (-1)^k \left((\mathbf{B}^t \otimes \mathbf{A}) - \mathbf{I}_{n^2} \right)^k \mathrm{vec}(\mathbf{C})$ is convergent if we have

$$\|(\mathbf{B}^t \otimes \mathbf{A}) - \mathbf{I}_{n^2}\| < 1. \tag{14}$$

Proof. The relation $\rho\left((\mathbf{B}^t \otimes \mathbf{A}) - \mathbf{I}_{n^2} \right) \leqslant \|(\mathbf{B}^t \otimes \mathbf{A}) - \mathbf{I}_{n^2}\|$ completes the proof. ■

Now, consider matrix $\mathbf{Q} = \mathrm{diag}\left(\frac{1}{q_{11}}, \ldots, \frac{1}{q_{n^2,n^2}} \right)$, where $[q_{ij}]$ are the elements of matrix $(\mathbf{B}^t \otimes \mathbf{A})$. If the matrix $(\mathbf{B}^t \otimes \mathbf{A})$ is strictly row diagonally dominant (SRDD), then we have

$$\rho((\mathbf{B}^t \otimes \mathbf{A}) - \mathbf{I}_{n^2}) < 1.$$

Because, if assume $\mathbf{T} = (\mathbf{B}^t \otimes \mathbf{A}) - \mathbf{I}_{n^2}$, then it can be easily observed that

$$[t_{ij}] = \begin{cases} 0, & i = j, \\ \frac{q_{ij}}{q_{ii}}, & i \neq j. \end{cases}$$

Since $(\mathbf{A} \oplus \mathbf{B}^t)$ is SRDD, it is clear that $|q_{ii}| > \sum_{j=1, j\neq i}^{n} |q_{ij}|$. Hence,

$$\|\mathbf{T}\|_\infty = \|(\mathbf{B}^t \otimes \mathbf{A}) - \mathbf{I}_{n^2}\|_\infty = \sum_{j=1}^{n} |t_{ij}| = \sum_{j=1, i\neq j}^{n} |t_{ij}| = \sum_{j=1, j\neq i}^{n} |\frac{q_{ij}}{q_{ii}}| < 1.$$

Therefore, it is concluded that

$$\rho((\mathbf{B}^t \otimes \mathbf{A}) - \mathbf{I}_{n^2}) \leqslant \|(\mathbf{B}^t \otimes \mathbf{A}) - \mathbf{I}_{n^2}\|_\infty < 1.$$

From the other point of view, if $\rho((\mathbf{B}^t \otimes \mathbf{A}) - \mathbf{I}_{n^2}) > 1$, by pre-multiplying the both sides of equation (1) by matrix $\mathbf{Q}$, and using convex homotopy function once again, it can be easily verified that

$$\text{vec}(\mathbf{U}) = \sum_{k=0}^{\infty}(-1)^k \left(\mathbf{Q}(\mathbf{B}^t \otimes \mathbf{A}) - \mathbf{I}_{n^2}\right)^k \mathbf{Q}\text{vec}(\mathbf{C}). \tag{15}$$

However, we have to respond this important question that "Does the matrix $(\mathbf{B}^t \otimes \mathbf{A})$ is strictly diagonally dominant when $\mathbf{A}$ and $\mathbf{B}^t$ are strictly diagonally dominant matrices?" The answer of this question is negative. Firstly, note that the structure of $(\mathbf{B}^t \otimes \mathbf{A})$ is as following:

$$(\mathbf{B}^t \otimes \mathbf{A}) = (b_{ij})^t \mathbf{A} = \begin{pmatrix} b_{11}\mathbf{A} & b_{21}\mathbf{A} & \dots & b_{n1}\mathbf{A} \\ b_{12}\mathbf{A} & b_{22}\mathbf{A} & \dots & b_{n2}\mathbf{A} \\ \vdots & \vdots & \ddots & \vdots \\ b_{1n}\mathbf{A} & b_{2n}\mathbf{A} & \dots & b_{nn}\mathbf{A} \end{pmatrix}. \tag{16}$$

Secondly, the following matrices can be considered as a counterexample:

$$\mathbf{A} = \begin{pmatrix} 0.2 & 0.1 \\ 0.3 & 0.7 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 4 & 1 \\ 2 & 8 \end{pmatrix}.$$

In this part, we will show that the series $S_m = \sum_{k=0}^{m}(-1)^k \left(\mathbf{Q}(\mathbf{B}^t \otimes \mathbf{A}) - \mathbf{I}_{n^2}\right)^k \mathbf{Q}.\text{vec}(\mathbf{C})$ is converges. Thus, first we need the following lemma.

Lemma 2.4 [20] Let $\mathbf{A}, \mathbf{M}, \mathbf{N}$ are three matrices satisfying $\mathbf{A} = \mathbf{M} - \mathbf{N}$. The pair of matrices $\mathbf{M}, \mathbf{N}$ is a regular splitting of $\mathbf{A}$, if $\mathbf{M}$ is nonsingular, and $\mathbf{M}^{-1}$ and $\mathbf{N}$ are nonnegative.

Theorem 2.5 [20] Let $\mathbf{M}, \mathbf{N}$ are regular splitting of a matrix $\mathbf{A}$. Then $\rho(\mathbf{M}^{-1}\mathbf{N}) < 1$ if and only if $\mathbf{A}$ is nonsingular and $\mathbf{A}^{-1}$ is nonnegative.

Theorem 2.6 Let $\mathbf{Q}$ is nonsingular matrix such that $\mathbf{Q}$ and $\mathbf{Q}^{-1} - (\mathbf{B}^t \otimes \mathbf{A})$ are nonnegative. Then the sequence

$$\text{vec}(\mathbf{U}) = \sum_{k=0}^{\infty}(-1)^k \left(\mathbf{Q}(\mathbf{B}^t \otimes \mathbf{A}) - \mathbf{I}_{n^2}\right)^k \mathbf{Q}\text{vec}(\mathbf{C}), \tag{17}$$

converges if $(\mathbf{B}^t \otimes \mathbf{A})$ is nonsingular and $(\mathbf{B}^t \otimes \mathbf{A})^{-1}$ is nonnegative.

Proof. Suppose that $\mathbf{Q}$ is a singular matrix such that $\mathbf{Q}$ and $\mathbf{Q}^{-1} - (\mathbf{B}^t \otimes \mathbf{A})$ are nonnegative. By employing Theorem 2.3, it can be obtain that $\rho(\mathbf{Q}(\mathbf{B}^t \otimes \mathbf{A}) - \mathbf{I}_{n^2}) < 1$ as both $\mathbf{Q}^{-1}$ and $\mathbf{Q}^{-1} - (\mathbf{B}^t \otimes \mathbf{A})$ are a regular splitting of $(\mathbf{B}^t \otimes \mathbf{A})$. This implies that the series

$$S_m = \sum_{k=0}^{m}(-1)^k \left(\mathbf{Q}(\mathbf{B}^t \otimes \mathbf{A}) - \mathbf{I}_{n^2}\right)^k \mathbf{Q}\text{vec}(\mathbf{C}). \tag{18}$$

convergent. ■

3. Numerical experiments

In this section, we support the theory which has been developed with some numerical implementations. All the computations have been done using MATLAB 2014(Ra), and all matrices that have been supposed are strictly diagonally dominant. Moreover, the residual error of the approximations are measured by

$$\mathrm{Res}(\mathbf{U}) = \frac{\|\mathbf{U} - \mathbf{X}\|_\infty}{\|\mathbf{X}\|_\infty}, \tag{19}$$

whenever, $\mathbf{U}$ is approximated solution by HPM.

Test 1. For the first example, the solution of the equation (1) using HPM is approximated, where 4×4 matrices $\mathbf{A}$,$\mathbf{B}$ and $\mathbf{C}$ are as follows:

$$\mathbf{A} = \begin{pmatrix} 10 & -1 & -1 & -1 \\ -1 & 10 & -1 & -1 \\ -1 & -1 & 10 & -1 \\ -1 & -1 & -1 & 10 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 12 & 2 & 2 & 2 \\ 2 & 12 & 2 & 2 \\ 2 & 2 & 12 & 2 \\ 2 & 2 & 2 & 12 \end{pmatrix},$$

$$\mathbf{C} = \begin{pmatrix} -1128 & 82 & 82 & 82 \\ 82 & -1128 & 82 & 82 \\ 82 & 82 & -1128 & 82 \\ 82 & 82 & 82 & -1128 \end{pmatrix}.$$

It is observed that the matrix $(\mathbf{B}^t \otimes \mathbf{A})$ is strictly row diagonally dominant. In addition, the spectral radius is computed as $\rho\left((\mathbf{B}^t \otimes \mathbf{A}) - \mathbf{I}_{n^2}\right) = 0.65 < 1$. Hence, employing eight terms of equation (18), the approximated solution will compute by considering long precision as follows:

$$\mathbf{x} = \begin{pmatrix} -9.999999980744782 & 0.999999993672738 & 0.999999993672738 & 0.999999993672737 \\ 0.999999993672738 & -9.999999980744777 & 0.999999993672739 & 0.999999993672738 \\ 0.999999993672738 & 0.999999993672738 & -9.999999980744777 & 0.999999993672738 \\ 0.999999993672738 & 0.999999993672738 & 0.999999993672738 & -9.999999980744775 \end{pmatrix},$$

while the exact solution is

$$\mathbf{X} = \begin{pmatrix} 10 & -1 & -1 & -1 \\ -1 & 10 & -1 & -1 \\ -1 & -1 & 10 & -1 \\ -1 & -1 & -1 & 10 \end{pmatrix}.$$

It can be seen that the approximation has a good agreement with the exact solution. In this case the residual error is 2.9413×10^{-9}. In Figure 1., the exact solution is compared by approximated solution.

Test 2. This numerical example made considering the solution of the matrix equation (1) where the following $n \times n$ matrices are assumed.

$$\mathbf{A} = \mathrm{Heptadiag}_n\,(3, 2, 1, 50, 1, 2, 3)\,,$$

$$\mathbf{B} = \mathrm{Heptadiag}_n\,(-1, 0, 1, 25, 1, 0, -1)\,.$$

The matrix $\mathbf{X} = \mathrm{Hendecadiag}_n\,(4, 3, 2, 1, 0, 30, 0, 1, 2, 3, 4)$, is supposed as the exact solution, and further the matrix $\mathbf{C}$ is calculated by $\mathbf{C} = \mathbf{AXB}$. The arrangement of the

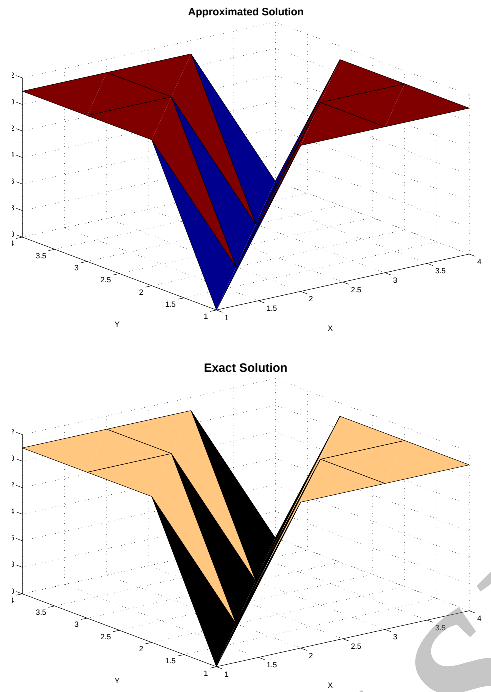

Figure 1.: Comparison of the exact and approximated solution in Test 1.

matrices **A,B,X** can be seen in Figure 2. Now, the approximated solution of the matrix

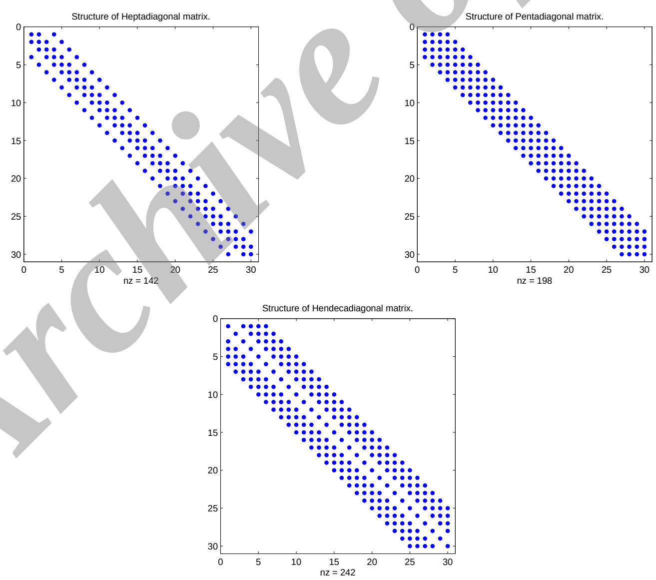

Figure 2.: The structure of matrices for $n = 30$ in Test 2.

equation is evaluated by considering eight terms of (18). The CPU time and estimation errors are measured by increasing dimension of the matrices. Results are reported in Figure 3. It is obvious that by increasing dimension of matrices, the CPU time will be enhanced normally and error estimation will be soared gradually.

Test 3. In this example, we would like to compare the results of the homotopy series by an iterative scheme. For this purpose, we select a recursive method proposed by J.

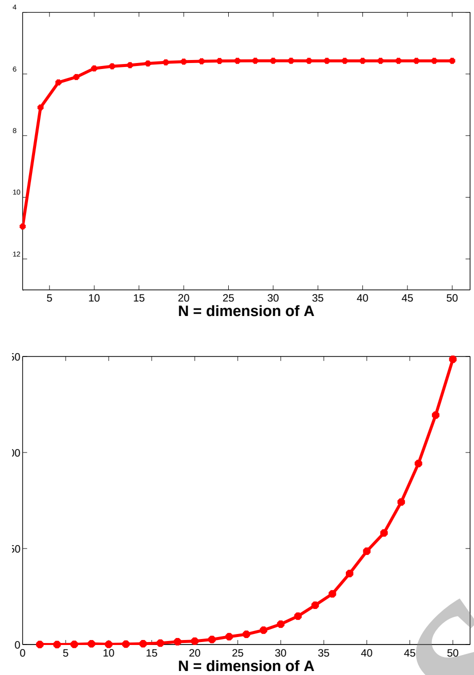

Figure 3.: The behavior of the method for residual error and CPU time in Test 2.

Ding et. al in [5] for solving the matrix equation (1) as following:

$$\mathbf{X}_{k+1} = \mathbf{X}_k + \mu \mathbf{A}^T(\mathbf{C} - \mathbf{A}\mathbf{X}_k\mathbf{B})\mathbf{B}^T, \tag{20}$$

whenever, $\mathbf{X}_0$ is suitable initial matrix. Moreover, they shown that the parameter μ can be chosen according the following relation:

$$0 < \mu \leqslant \frac{2}{\|\mathbf{A}\|^2\|\mathbf{B}\|^2}, \tag{21}$$

and it is proven that $\lim_{k\to\infty} \mathbf{X}_k = \mathbf{X}$. Now, for solving matrix equation (1), three following matrices

$$\mathbf{A} = \begin{pmatrix} 40 & 5 \\ 0 & -12 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 30 & -2 \\ -1 & 10 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 2610 & 720 \\ -696 & -192 \end{pmatrix},$$

are considered. It is pointed out that the exact solution is as following

$$\mathbf{X} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}.$$

Now, we employ the iteration (20) considering $\mu = 6.77 \times 10^{-7}$ and $\mathbf{X}_0 = \mathbf{1}_{2\times 2}$. After 24 iteration we obtain the following solution with the residual error 5.99×10^{-8}:

$$\mathbf{X} = \begin{pmatrix} 2.000000238546721 & 2.000000018535255 \\ 2.000000011333342 & 2.000000022148962 \end{pmatrix}.$$

Once again, we apply homotopy series with eight terms and we obtain the following solution with the residual error 3.06×10^{-8}:

$$\mathbf{X} = \begin{pmatrix} 2.000000011851852 & 2.000000110617284 \\ 1.999999996049383 & 1.999999996049383 \end{pmatrix}.$$

It can be observed that the homotopy series has very acceptable accuracy for computing the solution. Moreover, in the most numerical implementation, the time consumption is less than iterative method because this method is not iterative scheme.

4. Conclusion

In this work, the homotopy perturbation method is utilized to approximate the linear matrix equation $\mathbf{AXB} = \mathbf{C}$. Numerical experiments reveal that by considering more terms of the approximation, error will be decreased. Furthermore, if the matrix $(\mathbf{B}^t \otimes \mathbf{A})$ becomes more strictly diagonally dominant, the speed of method convergence will increase. Apparently, by growing up dimension of the matrices, the error of the approximation will be increased.

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