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The Calculation of Unit's Efficiency by Using the Interval $l_1$norm

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Abstract
Data Envelopment Analysis (DEA) is a technique for measuring the efficiency of decision making units. In all models of the DEA, for each unit under assessment, the numerical efficiency which may be less than or equal to one is obtained. Given the possible large number of efficiency units for evaluating units, we use various methods of ranking. l_1-norm is one of the methods of ranking. This method has been used for categorical data. In this paper, we assume data as interval and introducel_1-norm andrun it on a single example.

Keyword: DEA, Ranking, Interval data, l_1-norm

1. Introduction

DEA, which was developed to evaluate the relative efficiency of the decision-making units by Charnes et al in 1978 [1], is a non-parametric method and is based on the linear programming. In1957, Farrell [2] was the first to construct the production possibility set in a non-parametric method. Charnes et al developed Farrell's approach and presented a model called CCR. Then, in 1984 Banker et al [3] offered BCC model. Cooper et al [4] (1999) placed DEA Technique by the uncertain data. In 2004, Jahanshahloo et al [5] ranked DMUs by the norm1 method. In this paper, we intend to obtain the efficiency in a range of intervals and calculate the unit's efficiency by the interval l_1-norm method.

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Considering ranking is not completely specified in the interval efficiency, we attempt to rank decision-making units as well as interval data by Jahanshahloo et al's [6] method and determine the actual position of data in comparison with each other with respect to their distance from the efficiency boundary after removing the unit.
Furthermore, this paper will be as follows: InSection2, the necessary introductions for the next sections will be presented. In Section 3, ranking data by the norm1 method will be offered. InSection4, ranking interval data by the interval norm1method will be provided. In Section5, a numerical example will be offered to illustrate the method and in the final section we will have conclusions.

2. Background

Norm1 is on the basis that we remove the DMU under evaluation and we want to see the minimum distance from the boundary of the new PPS to their moved DMU.
We assume that there are n DMUs to be evaluated, indexed by j = 1,.., n.
In addition, each DMU is assumed to produce s different outputs from m different inputs.
Let the observed input and output vectors of DMU_J be $(j=1,\dots,n)\quad x_j=(x_{1j},\dots,x_{mj})$ and $(j=1,\dots,n)\quad y_j=(y_{1j},\dots,y_{sj})$ respectively, that all components of vectors x_j and y_j for all DMUs are non-negative and each DMU has at least one strictly positive input and output.
The production possibility sets T_C and T_V are defined as

$$T_c=\left\{(x,y)^t \middle| x\geq\sum_{j=1}^{n}\lambda_j x_j\,, y\leq\sum_{j=1}^{n}\lambda_j y_j\,,\lambda_j\geq 0\,, j=1,\dots,n\right\}$$

$$T_v=\left\{(x,y)^t \middle| x\geq\sum_{j=1}^{n}\lambda_j x_j\,, y\leq\sum_{j=1}^{n}\lambda_j y_j\,,\sum_{j=1}^{n}\lambda_j=1\,,\lambda_j\geq 0\,, j=1,\dots,n\right\}$$

In this section, we assume that the DMU_O is extreme efficient. By omitting $(x_o,y_o)^t$ from T_c,we define the production possibility setT_C'as

$$T_C'=\left\{(x,y)^t \middle| x\geq\sum_{\substack{j=1\\J\neq 0}}^{n}\lambda_j x_j\,, y\leq\sum_{\substack{j=1\\J\neq 0}}^{n}\lambda_j y_j\,,\lambda_j\geq 0\,, j=1,\dots,n\,, J\neq 0\right\}$$

To obtain the ranking score of DMU_O, we consider the following model:

$$
\begin{aligned}
Min \quad & \Gamma_C^O(x,y)=\sum_{i=1}^{m}\left|x_i-x_{io}\right|+\sum_{r=1}^{s}\left|y_r-y_{ro}\right| \\
s.t. \quad & \sum_{\substack{j=1\\ j\neq 0}}^{n}\lambda_j x_{ij}\le x_i \qquad (i=1,\ldots,m) \\
& \sum_{\substack{j=1\\ j\neq 0}}^{n}\lambda_j y_{rj}\ge y_r \qquad (r=1,..,s) \qquad\qquad (1) \\
& x_i\ge 0, \qquad (i=1,\ldots,m) \\
& y_r\ge 0, \qquad (r=1,\ldots,s) \\
& \lambda_j\ge 0, \qquad (j=1,\ldots,n,\ j\neq 0)
\end{aligned}
$$

where $x=(x_1,\ldots,x_m)$, $y=(y_1,\ldots,y_s)$ and $\lambda=(\lambda_1,\ldots,\lambda_{0-1},\lambda_{0+1},\ldots,\lambda_n)$ are the variables of the above model and $\Gamma_C^O(x,y)$ is the distance $(x_o,y_o)^t$from $(x,y)^t$by using l_1-norm.

It is obvious that the above model is non-linear. In order to convert this model to a linear model and state Theorem 1, the T_c'' set is defined as

$$T_c''=T_c'\cap\{(x,y)^t \mid x\ge x_o,\, y\le y_o\}$$

In Figure1, the polyhedral ROX, ZOX and UCBAX are T_C, T_c' and T_c'', respectively

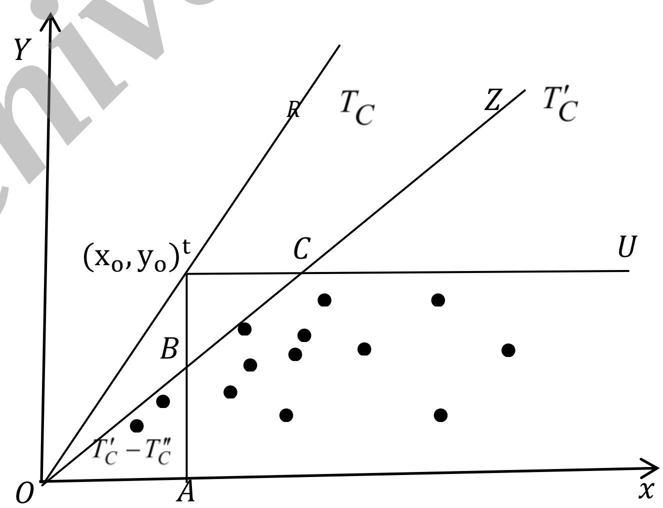

Fig.1.The polyhedralsare T_C, T_c' and T_c''

Theorem 1: Suppose $(x_o,y_o)^t\in T_c$is extreme efficient. For each $(\hat{x},\hat{y})^t\in T_c'-T_c''$ there exists at least are member of T_c'', say $(\bar{x},\bar{y})^t$, such that $\Gamma_C^O(\bar{x},\bar{y})<\Gamma_C^O(\hat{x},\hat{y})$

Now, suppose (x^*, y^*, λ^*) is an optimal solution of the model (1). Given Theorem 1, we find out that $(x^*, y^*)^t \in T_c''$. Therefore, for converting the model (1) into the linear form, we add the constraints $x \geq x_o$ and $y \leq y_o$ to the model (1). Therefore, we will have:

$$
\begin{aligned}
Min \quad & \Gamma_C^O(x,y) = \sum_{i=1}^{m} x_i - \sum_{r=1}^{s} y_r + \alpha \\
s.t. \quad & \sum_{\substack{j=1 \\ j\neq 0}}^{n} \lambda_j x_{ij} \leq x_i \qquad (i=1,\dots,m) \\
& \sum_{\substack{j=1 \\ j\neq 0}}^{n} \lambda_j y_{rj} \geq y_r \qquad (r=1,..,s) \qquad\qquad (2) \\
& x_i \geq x_{io}, \qquad (i=1,\dots,m) \\
& 0 \leq y_r \leq y_{ro}, \qquad (r=1,\dots,s) \\
& \lambda_j \geq 0, \qquad (j=1,\dots,n,\ j\neq 0)
\end{aligned}
$$

$\alpha = \sum_{r=1}^{s} y_{ro} - \sum_{i=1}^{m} x_{io}$ is a constant number.

3. Ranking by Using l_1-Norm for Interval Data

After presenting a certain mode of norm1, we want to express its interval mode.
Assume that in the following model the data are interval and the levels of inputs and outputs are known to lie within the bounded intervals, i.e. $x_{ij} \in \left[x_{ij}^l, x_{ij}^u\right]$ $(i=1,\dots,m)(j=1,\dots,n)$ and $y_{rj} \in \left[y_{rj}^l, y_{rj}^u\right]$ $(r=1,\dots,s)(j=1,\dots,n)$, with the upper and lower bounds of intervals given as constant and assumed strictly positive. So by the use of model 1, we consider the following model for interval data:

$$
\begin{aligned}
Min \quad & \Gamma_C^O(x,y) = \sum_{i=1}^{m} \left| x_i - \left[x_{io}^l, x_{io}^u\right] \right| + \sum_{r=1}^{s} \left| y_r - \left[y_{ro}^l, y_{ro}^u\right] \right| \\
s.t. \quad & \sum_{\substack{j=1 \\ j\neq 0}}^{n} \lambda_j \left[x_{ij}^l, x_{ij}^u\right] \leq x_i \qquad (i=1,\dots,m) \\
& \sum_{\substack{j=1 \\ j\neq 0}}^{n} \lambda_j \left[y_{rj}^l, y_{rj}^u\right] \geq y_r \qquad (r=1,..,s) \qquad\qquad (3) \\
& x_i \geq 0, \qquad (i=1,\dots,m) \\
& y_r \geq 0, \qquad (r=1,\dots,s) \\
& \lambda_j \geq 0, \qquad (j=1,\dots,n,\ j\neq 0)
\end{aligned}
$$

Using Model 3, we can obtain DMU's interval efficiency through models 4 and 5 as following.
The worst case for *DMUo* (unit under evaluation) is when *DMUo* in the worst case with the highest input (x_o^u), produces the lowest output (y_o^l) and the rest of *DMUs* j = 1. . . n and $(j \neq o)$ in their best conditions with lowest input (x_j^l),produces the highest output (y_j^u), so:

$$
\begin{aligned}
Min \quad & \Gamma_C^O(x,y)_L = \sum_{i=1}^{m}\left|x_i - x_{io}{}^u\right| + \sum_{r=1}^{s}\left|y_r - y_{ro}{}^l\right| \\
s.t. \quad & \sum_{\substack{j=1 \\ j\neq 0}}^{n} \lambda_j x_{ij}{}^l \le x_i \qquad (i=1,\dots,m) \\
& \sum_{\substack{j=1 \\ j\neq 0}}^{n} \lambda_j y_{rj}{}^u \ge y_r \qquad (r=1,..,s) \qquad\qquad (4) \\
& x_i \ge 0, \qquad (i=1,\dots,m) \\
& y_r \ge 0, \qquad (r=1,\dots,s) \\
& \lambda_j \ge 0, \qquad (j=1,\dots,n,\ j\neq 0)
\end{aligned}
$$

And also best case for *DMUo* (unit under evaluation) is when *DMUo* in the best case, i.e. with the lowest input (x_o^l) produces the highest output (y_o^u) and the rest of *DMUs* $(j=1,\dots,n)$ and $(j \neq o)$ in the worst conditions. i.e. with highest input (x_j^u) produces the lowest output (y_j^l), so

$$
\begin{aligned}
Min \quad & \Gamma_C^O(x,y)_U = \sum_{i=1}^{m}\left|x_i - x_{io}{}^l\right| + \sum_{r=1}^{s}\left|y_r - y_{ro}{}^u\right| \\
s.t. \quad & \sum_{\substack{j=1 \\ j\neq 0}}^{n} \lambda_j x_{ij}{}^u \le x_i \qquad (i=1,\dots,m) \\
& \sum_{\substack{j=1 \\ j\neq 0}}^{n} \lambda_j y_{rj}{}^l \ge y_r \qquad (r=1,..,s) \qquad\qquad (5) \\
& x_i \ge 0, \qquad (i=1,\dots,m) \\
& y_r \ge 0, \qquad (r=1,\dots,s) \\
& \lambda_j \ge 0, \qquad (j=1,\dots,n,\ j\neq 0)
\end{aligned}
$$

It is clear that the models 4 and 5 are non-linear, in order to convert these models to a linear model we will take the steps in norm1.

Thus, for converting the model to the linear form, we add $x_i \geq x_{oi}^u$ and $y_r \leq y_{or}^l$ constraints to the worst-case model and $x_i \geq x_{oi}^l$ and $y_r \leq y_{or}^u$ constraints are added to the best-case model. So we will have model 6 for the worst case:

$$
\begin{aligned}
Min \quad & \Gamma_C^O(x,y)_L = \sum_{i=1}^{m} x_i - \sum_{r=1}^{s} y_r + \sum_{r=1}^{s} y_{ro}^l - \sum_{r=1}^{m} x_{io}^u \\
s.t. \quad & \sum_{\substack{j=1 \\ j\neq 0}}^{n} \lambda_j x_{ij}{}^l \leq x_i \qquad (i=1,\ldots,m) \\
& \sum_{\substack{j=1 \\ j\neq 0}}^{n} \lambda_j y_{rj}{}^u \geq y_r \qquad (r=1,..,s) \qquad\qquad (6) \\
& x_i \geq x_{io}^u, \qquad (i=1,\ldots,m) \\
& 0 \leq y_r \leq y_{ro}^l, \qquad (r=1,\ldots,s) \\
& \lambda_j \geq 0, \qquad (j=1,\ldots,n,\ j\neq 0)
\end{aligned}
$$

And so we have model 7 for the best case:

$$
\begin{aligned}
Min \quad & \Gamma_C^O(x,y)_U = \sum_{i=1}^{m} x_i - \sum_{r=1}^{s} y_r + \sum_{r=1}^{s} y_{ro}^u - \sum_{r=1}^{m} x_{io}^l \\
s.t. \quad & \sum_{\substack{j=1 \\ j\neq 0}}^{n} \lambda_j x_{ij}{}^u \leq x_i \qquad (i=1,\ldots,m) \\
& \sum_{\substack{j=1 \\ j\neq 0}}^{n} \lambda_j y_{rj}{}^l \geq y_r \qquad (r=1,..,s) \qquad\qquad (7) \\
& x_i \geq x_{io}^l, \qquad (i=1,\ldots,m) \\
& 0 \leq y_r \leq y_{ro}^u, \qquad (r=1,\ldots,s) \\
& \lambda_j \geq 0, \qquad (j=1,\ldots,n,\ j\neq 0)
\end{aligned}
$$

Then it is implied that for each $x_{ij} \in \left[x_{ij}^l, x_{ij}^u\right]$ and $y_{rj} \in \left[y_{rj}^l, y_{rj}^u\right]$ we hav : $\Gamma_C^O(x,y) \in \left[\Gamma_C^O(x,y)_L, \Gamma_C^O(x,y)_U\right]$ where x and y are input and output matrices respectively.

4. Numerical Example:

In this paper, the performance of electronic services in 30 branches of Refah bank in 1389 will be assessed. The variables, which are to be studied, will be introduced In terms of two input and five output. Then we will solve a range of data in the form of gams software using the interval norm 1 method and finally after obtaining the upper and lower limits for the model, we will rank the bank branches by using the method mentioned in Jahanshahloo et al's paper.

In the following table, the input and output data are given.

Table 1:
The data of the inputs and outputs

	INPUT DATA		OUTPUT DATA										
	INPUT1		INPUT2		OUT PUT 1	OUTPUT2		OUT PUT 3	OUTPUT4		OUTPUT5	
	L	U	L	U		L	U		L	U	L	U
DMU1	1870202871	2122932989	10674980	12117545	404	25018.5	28399.4	6	115.63	131.3	39660.3	45019.8
DMU 2	2150449338	2441050600	3958581	4493525	765	28996	32914.4	42	209.98	238.4	40848.9	46369.1
DMU 3	2528052626	2869681359	28325497	32153268	981	64583.5	73311	45	1489.3	1691	33457.3	37978.5
DMU 4	1136253828	1289801643	32373066	36747805	829	16934.9	19223.4	96	23.125	26.25	397.75	451.5
DMU 5	1046802099	1188261842	10766374	12221290	220	20883.7	23705.9	154	294.15	333.9	306.175	347.6
DMU 6	1924404970	2184459695	97047882	110162462	261	67329.8	76428.5	134	202.58	230	44435.2	50439.9
DMU 7	2603221081	2955007713	67028730	76086667	682	75386.6	85574	2	71.225	80.85	265.475	301.4
DMU 8	1099459016	1248034559	51392589	58337534	717	34205.6	38828	87	185	210	16812.8	19084.8
DMU 9	1579341699	1792766253	9843792	11174034	1,975	24821.5	28175.7	52	220.15	249.9	46799.5	53123.7
DMU 10	1295118608	1470134636	17920821	20342554	1,707	32097.5	36435	78	22.2	25.2	1296.85	1472.1
DMU 11	1531862830	1738871320	31820945	36121073	192	43953.2	49892.9	121	229.4	260.4	283.05	321.3
DMU 12	988324309	1121881648	9099579	10329252	246	59022.4	66998.4	63	8.325	9.45	458.8	520.8
DMU 13	3548842299	4028415582	306866303	348334725	4,107	77140.4	87564.8	75	42.55	48.3	91.575	104.0
DMU 14	2115803803	2401723236	26299366	29853335	2,444	66706.4	75720.8	36	345.03	391.7	1012.88	1149.8
DMU15	1559981157	1770789422	8815641	10006944	1,648	37125.8	42142.8	22	333	378	21695.9	24627.8
DMU 16	1272494916	1444453688	131986299	149822287	329	44741.3	50787.5	210	537.43	610.1	28838.7	32735.9
DMU 17	1372412324	1557873449	18177087	20633451	284	55944.9	63505.1	64	226.63	257.3	148	168.0
DMU 18	2467588738	2801046676	16028088	18194046	1,460	39616.8	44970.5	553	430.13	488.3	97423.8	110589.2
DMU 19	849153943	963904476	74054238	84061568	436	15921.1	18072.6	394	27.75	31.5	35829	40670.7
DMU 20	952866320	1081632039	75507987	85711769	159	30848.8	35017.5	62	41.625	47.25	48200.8	54714.5
DMU 21	903175476	1025226216	37216645	42245922	229	30656.4	34799.1	197	562.4	638.4	58879	66835.7
DMU 22	1489083972	1690311536	40863258	46385320	2,169	13070.3	14836.5	18	130.43	148.1	182.225	206.9
DMU 23	2025705705	2299449719	58968501	66937218	555	27537.3	31258.5	135	43.475	49.35	81688.6	92727.6
DMU 24	2454634125	2786341439	21296048	24173892	789	54987.6	62418.3	22	15.725	17.85	428.275	486.2
DMU 25	1407453992	1597650477	147775663	167745349	1,384	28132	31933.7	174	91.575	104	40771.2	46280.9
DMU 26	1159940803	1316689560	282492102	320666714	1,586	50354.2	57158.9	334	87.875	99.75	80704.4	91610.4
DMU 27	1235117793	1402025603	28806079	32698793	838	10878	12348	292	42.55	48.3	69375.9	78751.1
DMU 28	1616696124	1835168573	26900139	30535293	922	4600.95	5222.7	523	80.475	91.35	138157	156827.0
DMU 29	1676040114	1902532021	108372683	123017642	1349	23381.2	26540.9	110	41.625	47.25	175.75	199.5
DMU 30	1258483488	1428548825	38474167	43673379	475	6518.48	7399.35	90	76.775	87.15	327.45	371.7

Finally, the upper and lower bounds obtained from solving model and the final Ranking have been presented.

Table 2:
Final ranking of DMUs

	Norm1		Ranking
	EFF(u)	EFF(l)	
DMU1	0.00	0.00	30 to 24
DMU2	0.00393	0.00239	20
DMU3	0.1843	0.16087	1
DMU4	0.0013	0.00	23
DMU5	0.00951	0.00307	16
DMU6	0.01819	0.00	9
DMU7	0.01399	0.00	10
DMU8	0.00142	0.00	22
DMU9	0.02386	0.00989	8
DMU10	0.00847	0.00	13
DMU11	0.00	0.00	30 to 24
DMU12	0.02469	0.00726	7
DMU13	0.07579	0.0606	3
DMU14	0.05189	0.03774	5
DMU15	0.01004	0.00058	12
DMU16	0.00836	0.00	15
DMU17	0.00346	0.00	19
DMU18	0.08156	0.06972	2
DMU19	0.01182	0.00648	17
DMU20	0.00187	0.00	21
DMU21	0.01937	0.00973	11
DMU22	0.00913	0.00109	14
DMU23	0.00	0.00	30 to 24
DMU24	0.00	0.00	30 to 24
DMU25	0.00	0.00	30 to 24
DMU26	0.0424	0.01576	6
DMU27	0.0046	0.00	18
DMU28	0.07727	0.03719	4
DMU29	0.00	0.00	30 to 24
DMU30	0.00	0.00	30 to 24

5. Conclusion

In many DEA models, the efficiency score is given within the range of (0, 1], and the DMU is called efficient if its score equals 1. We can know superior and inferior items of each DMU by analyzing an optimal solution. However, relative ranking of the DMU is not necessarily the best ranking because an optimal solution is obtained independently of ranking. Moreover, not only score but ranking also has a key role as evaluation. In this paper, we ranked the units using interval norm1 method, and learned that this method is useful for ranking extreme efficiency units.

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