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Sensitivity Analysis and Finding the Stability Region with Adding DMUs in DEA

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Abstract

One of the important issues in data envelopment analysis (DEA) is sensitivity analysis. Heretofore the existent studies have considered the data modification of inputs and outputs in one or multiple DMUs. In this paper the number of DMUs is increased and a stability region is obtained in T_v by applying defining hyperplanes in which if the added DMU (only one DMU) is in this region then all of the extreme efficient units will be remained on the frontier. Then it is shown that the obtained region is the largest stability region. Finally the mentioned stability region for a number of DMUs is obtained and the results are reported.

Keywords: Data Envelopment Analysis; Sensitivity Analysis; Efficiency; Hyperplane; Frontier

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Introduction

Data envelopment analysis (DEA) is a nonparametric approach for evaluating the performance of different organizations which uses multiple inputs to produce multiple outputs. This approach allocates one efficiency score to each DMU so it is possible to compare the DMUs with each other. For the first time Charnes et al (1978) laid the foundation of DEA through introducing the CCR model. Then Banker et al (1984) introduced BCC model by removing the condition of constant returns to scale. Up to this time, since DEA has been utilized in various problems, many studies have been conducted on this issue for example the presented papers by Charnes et al (1985,1989), Seiford (1996) and etc. A review of DEA can be found in (Cook et al, 2009).

One of the important questions in linear programming is that to find out what is the influence of data modification on optimal solution. Sensitivity analysis of DEA models which is based on the linear programming are both theoretically and practically important. The first DEA sensitivity analysis paper is presented by Charnes et al (1985). Afterwards Many studies have been conducted on this issue by Seiford et al (1998), Zhu (2001), Cooper et al (2001), Jahanshahloo et al (2004, 2005a, 2005b, 2012) and etc. Heretofore the existent studies have considered the data modification of inputs and outputs. In this paper the number of DMUs is increased and a stability region is obtained in T_v by applying defining hyperplanes in which if the added DMU (only one DMU) is in this region, then all of the extreme efficient DMUs will be remained on frontier. Thereafter we claim that the obtained region is the largest stability region.

The present study is organized as follows: First some basic DEA models and related concepts are reviewed. Then the largest stability region is obtained by defining hyperplanes. Thereafter the mentioned stability region for a number of DMUs is obtained. Finally the results are synthesized and concluded.

Preliminary

Suppose *n DMUs* are evaluated, each of them consumes m inputs to produce s outputs. Suppose $X_j = (x_{1j}, x_{2j}, \dots, x_{mj})^t$ and $Y_j = (y_{1j}, y_{2j}, \dots, y_{sj})^t$ are the corresponding inputs and outputs of DMU_j for $j = 1, \dots, n$. Banker defined production possibility set under the variable returns to scale condition as follows:

$$T_v = \left\{ (X,Y) \mid X \geq \sum_{j=1}^{n} \lambda_j X_j , Y \leq \sum_{j=1}^{n} \lambda_j Y_j , \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \geq 0; j = 1, \dots, n \right\}$$

Envelopment forms of BCC model in input and output oriented have been presented respectively as follows:

$$\begin{aligned}
&Min \;\; \theta - \varepsilon\left(\textstyle\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+\right) \\
&S.t. \quad \textstyle\sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = \theta x_{io} \qquad i = 1, \dots, m \qquad (1)\\
&\qquad\;\; \textstyle\sum_{j=1}^{n} \lambda_j y_{rj} - s_r^- = y_{ro} \qquad r = 1, \dots, s \\
&\qquad\;\; \textstyle\sum_{j=1}^{n} \lambda_j = 1 \\
&\qquad\;\; \lambda_j \geq 0 \qquad\qquad j = 1, \dots, n \\
&\qquad\;\; s_i^- \geq 0 \qquad\qquad i = 1, \dots, m \\
&\qquad\;\; s_r^+ \geq 0 \qquad\qquad i = 1, \dots, m
\end{aligned}$$

$$\begin{array}{lll} Max & \varphi + \varepsilon(\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+) & \\ S.t. & \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = x_{io} & i = 1, \dots, m \qquad (2) \\ & \sum_{j=1}^{n} \lambda_j y_{rj} - s_r^- = \varphi y_{ro} & r = 1, \dots, s \\ & \sum_{j=1}^{n} \lambda_j = 1 & \\ & \lambda_j \geq 0 & j = 1, \dots, n \\ & s_i^- \geq 0 & i = 1, \dots, m \\ & s_r^+ \geq 0 & i = 1, \dots, m \end{array}$$

And multiplier forms of BCC model in input and output oriented have been presented respectively as follows:

$$\begin{array}{lll} Max & \sum_{r=1}^{s} u_r y_{ro} + u_o & \\ S.t. & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + u_o \leq 0 & \qquad (3) \\ & & j = 1, \dots, n \\ & \sum_{i=1}^{m} v_i x_{io} = 1 & \\ & v_i \geq \varepsilon & i = 1, \dots, m \\ & u_r \geq \varepsilon & r = 1, \dots, s \end{array}$$

$$\begin{array}{lll} Min & \sum_{i=1}^{m} v_i x_{io} + v_o & \\ S.t. & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} - v_o \leq 0 & \qquad (4) \\ & & j = 1, \dots, n \\ & \sum_{r=1}^{s} u_r y_{ro} = 1 & \\ & v_i \geq \varepsilon & i = 1, \dots, m \\ & u_r \geq \varepsilon & r = 1, \dots, s \end{array}$$

Where $o \in \{1, \dots, n\}$ is the index of evaluated DMU and $\varepsilon > 0$ is a nonArchimedean element.

Definition1. DMU_o is called efficient if the optimal value of objective function of model (3) (model (4)) equals one, otherwise DMU_o is called inefficient.

Definition2. The set

$$EF=\left\{(X,Y) \in T_v \mid \forall (X', Y') \in R^{m+s} \left((-X', Y') \gneqq (-X, Y) \Rightarrow (X', Y') \notin T_v\right)\right\}$$

is called efficient frontier.

Definition3. H is called a defining hyperplane of T_v if the following conditions are satisfied:

1) H is supporting.
2) T_v is enlarged by removing H.

Theorem1.

Suppose that $\bar{U} = (\bar{u}_1, \dots, \bar{u}_s) \in R^s$, $\bar{V} = (\bar{v}_1, \dots, \bar{v}_m) \in R^m$ and $\bar{w} \in R$. $H = \{(X,Y) \in R^{m+s} | \bar{U}Y - \bar{V}X + \bar{w} = 0\}$ is a defining hyperplane of T_v if and only if $(\bar{U}, \bar{V}, \bar{w})$ is an extreme direction of

$$Q = \{(U, V, w) |\ UY_j - VX_j + w \leq 0; \quad j = 1, \dots, n, \ U \geq 0, V \geq 0\}$$

Proof.

Refer to (Wei et al, 2007).

Suppose (U^l, V^l, w^l) for $l = 1, \dots, k$ are all of the extreme directions of Q. From the theorem1, it can be concluded that

$H_l = \{(X,Y) \in R^{m+s} | U^l Y - V^l X + w^l = 0\}$ for $l = 1, \dots, k$ are all of the defining hyperplanes of T_v. Corresponding with each defining hyperplane H_l, halfspace H_l^- is defined as follows:

$$H_l^- = \{(X,Y) \in R^{m+s} | U^l Y - V^l X + w^l \leq 0\} \quad l = 1, \dots, k$$

Theorem2.

If (U^l, V^l, w^l) for $l = 1, \dots, k$ are all of the extreme directions of Q then $T_v = \{(X,Y) \in R^{m+s} | U^l Y - V^l X + w^l \leq 0; \quad l = 1, \dots, k, X \geq 0,\ Y \geq 0\}$

Proof.

Refer to (Wei et al, 2007).

Definition4. The set

$$F = T_v \cap \left(\bigcup_{l=1}^{k} H_l \right)$$

is called frontier.

Theorem3. Suppose $(X,Y) \in T_v$ is evaluated by models (1) and (2). $(X,Y) \in F$ if and only if

$\theta^* = 1$ or $\varphi^* = 1$.

The proof is evident.

Definition5. A DMU is called extreme efficient if it lies on some $m+s$ linearly independent defining hyperplanes of T_v.

Obtaining the stability region by defining hyperplanes of T_v

In this section a stability region is obtained by using defining hyperplanes in which if the added DMU (only one DMU) is in this region then all of the extreme efficient DMUs will be remained on frontier. It is clear that with deleting each DMU all of the extreme efficient DMUs will be remained on frontier. Because the optimal value of objective function of models (1) and (2) will not be better through this deletion. Therefore we focus on adding DMUs. For obtaining the stability region the following method is suggested:

Step1:

All of the defining hyperplanes are obtained. Observe (Wei et al, 2007) and (Jahanshahloo et al, 2010).

Step2:

All of the extreme efficient DMUs are determined. With considering this point that some $m+s$ linearly independent defining hyperplanes are binding at each extreme efficient DMU, determining them (extreme efficient DMUs) is possible easily. Without lost of generality suppose $DMU_1, \dots, DMU_t$ are extreme efficient DMUs .

Step3:

The stability regions $S_1, \dots, S_t$ corresponding with $DMU_1, \dots, DMU_t$ are separately obtained. Suppose the binding defining hyperplanes at DMU_j are $H_{1j}, \dots, H_{(m+s)j}$ for $j = 1, \dots, t$. For obtaining S_j each time one of these binding defining hyperplanes is preserved and the others are deleted. So $m+s$ stability regions $S_{1j}, \dots, S_{(m+s)j}$ corresponding to DMU_j are obtained as $S_{pj} = H_{pj}^-$ for $p = 1, \dots, m+s$. Then $S_j = \bigcup_{p=1}^{m+s} S_{pj}$ as the stability region of DMU_j is concluded.

Step4:

The stability region for all of the extreme efficient DMUs as $S = \bigcap_{j=1}^{t} S_j$ is obtained.

Theorem4. Suppose that DMU_j is an extreme efficient DMU.

a) DMU_j is still remained on frontier through adding DMU in S_j.

b) S_j is the largest stability region for DMU_j.

Proof.

a) Suppose $(\bar{X}, \bar{Y}) \in S_j$ is the added DMU. Through definition of S_j, $(\bar{X}, \bar{Y}) \in S_{pj}$ for at least one $p \in \{1, \dots, m+s\}$. By noticing the definition of S_{pj} it is evident that at least one of the binding defining hyperplanes at DMU_j $(H_{1j}, \dots, H_{(m+s)j})$ is still supporting. Therefore DMU_j is still remained on frontier through adding $(\bar{X}, \bar{Y})$.

b) Suppose S_j is not the largest stability region for DMU_j. It means there is $(\hat{X},\hat{Y}) \notin S_j$ which with adding it, DMU_j is still remained on frontier. Since $(\hat{X},\hat{Y}) \notin S_j$, based on the definition of S_j $(\hat{X},\hat{Y}) \notin S_{pj}$ for $p = 1,\dots,m+s$. It means that none of the defining hyperplanes $H_{1j},\dots,H_{(m+s)j}$ which have already passed through DMU_j, are not defining and also supporting with adding $(\hat{X},\hat{Y})$.

Now it should be proved that there is no new defining hyperplane which pass through DMU_j. To this end suppose that before adding $(\hat{X},\hat{Y})$ the following linearly independent defining hyperplanes have been passed through DMU_j:

$$H_{pj} = \{(X,Y) \in R^{m+s} | U_j^p Y - V_j^p X + w_j^p = 0\} \quad p = 1,\dots,m+s \tag{5}$$

On the other hand since $(\hat{X},\hat{Y}) \notin S_j$ We have:

$$U_j^p \hat{Y} - V_j^p \hat{X} + w_j^p > 0 \quad p = 1,\dots,m+s \tag{6}$$

Furthermore based on the contrary assumption with adding $(\hat{X},\hat{Y})$, DMU_j is still remained on frontier. Therefore at least one binding defining hyperplane at DMU_j exists. Suppose this hyperplane is $\bar{H} = \{(X,Y) \in R^{m+s} | \bar{U}Y - VX+w=0$. So

$$\bar{U}Y_j - \bar{V}X_j + \bar{w} = 0 \tag{7}$$

Since this hyperplane is supporting we have:

$$\bar{U}\hat{Y} - \bar{V}\hat{X} + \bar{w} \le 0 \tag{8}$$

With regarding to this fact that $\left\{\left(U_j^p, V_j^p\right) \text{ for } p = 1,\dots,m+s\right\}$ is linearly independent, there exist scalars $\alpha_1,\dots,\alpha_{m+s}$ that

$$(\bar{U},\bar{V}) = \alpha_1\left(U_j^1, V_j^1\right) + \cdots + \alpha_{m+s}\left(U_j^{m+s}, V_j^{m+s}\right) \tag{9}$$

By noticing to (5), (7), (9) we have

$$\bar{w} = -\bar{U}Y_j + \bar{V}X_j = \sum_{p=1}^{m+s} \alpha_p\left(-U_j^p Y_j + V_j^p X_j\right) = \sum_{p=1}^{m+s} \alpha_p w_j^p \tag{10}$$

Therefore

$$(\bar{U},\bar{V},\bar{w}) = \alpha_1\left(U_j^1, V_j^1, w_j^1\right) + \cdots + \alpha_{m+s}\left(U_j^{m+s}, V_j^{m+s}, w_j^{m+s}\right) \tag{11}$$

If $\alpha_p \ge 0$ for each $p = 1,\dots,m+s$ then with multiplying the (p)th inequality of (6) in α_p and summing them together, (11) implies that $\bar{U}\hat{Y} - \bar{V}\hat{X} + \bar{w} > 0$ that is in contradiction with (8).

Now suppose there is at least one $q \in \{1,\dots,m+s\}$ that $\alpha_q < 0$. Consider the production possibility set before adding $(\hat{X},\hat{Y})$ and also the $m+s$ binding defining hyperplanes at DMU_j. Suppose that $(\tilde{X},\tilde{Y})$ is a nonextreme point of the edge that is obtained by $\bigcap_{p\neq q} H_{pj}$. Therefore

$$\begin{cases} U_j^p \tilde{Y} - V_j^p \tilde{X} + w_j^p = 0 \\ U_j^q \tilde{Y} - V_j^q \tilde{X} + w_j^q < 0 \end{cases} \tag{12}$$

$$p = 1,\dots,m+s;\ p \neq q$$

With multiplying the (p)th equality or inequality of (12) in α_p for $p = 1,\dots,m+s$ and summing them together, (11) implies that

$$\bar{U}\tilde{Y} - \bar{V}\tilde{X} + \bar{w} > 0 \tag{13}$$

On the other hand since $(\tilde{X},\tilde{Y})$ belongs to T_v so it will belong to the new T_v (after adding $(\tilde{X},\tilde{Y})$). Thus

$$\bar{U}\tilde{Y} - \bar{V}\tilde{X} + \bar{w} \le 0$$

Which it is in contradiction with (13).

Theorem 5.

a) All of the extreme efficient DMUs are remained on frontier through adding DMU in S.

b) S is the largest stability region.

Proof.

a) By considering the definition of S and theorem1 part a, the proof is evident

b) Suppose that S is not the largest stability region. So there exists $(\hat{X},\hat{Y}) \notin S$ which with adding it, $DMU_1,\dots,DMU_t$ are still remained on frontier. With regarding to theorem1 part b, $(\hat{X},\hat{Y}) \in S_j$ for $j=1,\dots,t$. So it can be concluded $(\hat{X},\hat{Y}) \in S$ by definition of S which it is in contradiction with the contrary assumption.

Example:

In this example the stability region for a set of DMUs is obtained. Consider four DMUs with one input and one output as they are defined in Table1. Now the presented method for obtaining the stability region is used for this set of $DMUs$.

Table1: data of numerical example

	DMU_1	DMU_2	DMU_3	DMU_4
Input	1	2	4	3
output	1	2	3	1.5

Step1:

All of the hyperplanes of T_v are obtained.

$H_1=\{(x,y)|y-3=0\}$

$H_2=\{(x,y)|y-x=0\}$

$H_3=\{(x,y)|2y-x-2=0\}$

$H_4=\{(x,y)|-x+1=0\}$

Step2:

All of the extreme efficient DMUs are obtained. They are DMU_1, DMU_2 and DMU_3 in which the binding defining hyperplanes are $\{H_2,H_4\},\{H_2,H_3\}$ and $\{H_1,H_3\}$ respectively.

Step3:

The stability regions S_1,S_2 and S_3 corresponding with DMU_1, DMU_2 and DMU_3 are separately obtained.

$S_{11}=\{(x,y)|y-x\le 0\}$,

$S_{21}=\{(x,y)|-x+1\le 0\}$

$S_1=S_{11}\cup S_{21}=\{(x,y)|\ y-x\le 0 \vee -x+1\le 0\}$

$S_{12}=\{(x,y)|y-x\le 0\}$,

$S_{22}=\{(x,y)|2y-x-2\le 0\}$

$S_2=S_{12}\cup S_{22}=\{(x,y)|\ y-x\le 0 \vee 2y-x-2\le 0\}$

$S_{13}=\{(x,y)|y-3\le 0\}$,

$S_{23}=\{(x,y)|2y-x-2\le 0\}$

$S_3=S_{13}\cup S_{23}=\{(x,y)|\ y-3\le 0 \vee 2y-x-2\le 0\}$

Step4:

The stability region S is obtained.

$$S=S_1\cap S_2\cap S_3=\{(x,y)|\ (y-x\le 0\ \&\ y-3\le 0) \vee (y-x\le 0\ \&\ 2y-x-2\le 0) \vee (-x+1\le 0\ \&\ 2y-x-2\le 0)\}$$

S is the largest region which DMU_1, DMU_2 and DMU_3 will be remained on frontier if the added DMU belongs to it.

The stability region S is shown by shadow area in figure1.

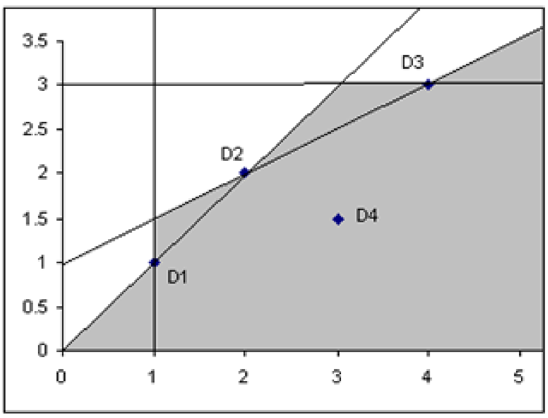

Figure1: The stability region for data of table1

Conclusion

One of the important issues in data envelopment analysis (DEA) is sensitivity analysis. In this paper a stability region in T_v by utilizing the defining hyperplanes has been obtained and it has been proved that if a DMU is added in this region then all of the extreme efficient DMUs are still remained on frontier. It has been proved that the obtained region is the largest stability region too. Finally the presented algorithm for finding stability region has been used for a set of DMUs through one example and the results have been reported.

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Sensitivity Analysis and Finding the Stability Region with Adding DMUs in DEA

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تحلیل حساسیت و یافتن ناحیه پایداری با اضافه کردن واحدهای تصمیم گیرنده در تحلیل پوششی داده ها

چکیده: یکی از موضوعات مهم در تحلیل پوششی داده ها تحلیل حساسیت می‌باشد. مطالعات موجود در این زمینه تاکنون تغییر در مقادیر ورودی و خروجی های یک یا چندین واحد تصمیم گیرنده را مورد بررسی قرار داده اند. در این مقاله تعداد واحدها افزایش یافته و با استفاده از ابرصفحه های تعریف کننده در مجموعه امکان تولید با بازده به مقیاس متغیر ناحیه پایداری ای به دست می‌آید که با اضافه شدن تنها یک واحد تصمیم گیرنده در آن ناحیه، تمامی واحدهای کارای راسی روی مرز باقی خواهند ماند. سپس ثابت می‌شود ناحیه پایداری به دست آمده بزرگترین ناحیه پایداری با این خاصیت می باشد. در پایان، ناحیه پایداری مذکور برای یک مجموعه مفروض از واحدهای تصمیم گیرنده به دست آمده و نتایج مربوطه ارایه خواهد شد.

کلمات کلیدی: : تحلیل پوششی داده ها، تحلیل حساسیت، کارایی، ابرصفحه، مرز.