



## **Estimating the Efficient Portfolio in Non-Radial DEA and DEA-R Models**

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Received 23 November 2015, Revised 2 February 2016, Accepted 19 February 2016

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### **Abstract**

The portfolio is a perfect combination of stock or assets, which an investor buys them. The objective of the portfolio is to divide the investment risk among several shares. Using non-parametric DEA and DEA-R methods can be of great significance in estimating portfolio. In the present paper, the efficient portfolio is estimated by using non-radial DEA and DEA-R models. By proposing non-radial models in DEA-R when there is ratio data the efficient portfolio is determined. At the end of the study, an applicatory example based on article [2] with non-radial DEA and DEA-R models has been conducted and results are presented.

**Keywords:** DEA, DEA-R, Efficient Portfolio.

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## 1. Introduction

DEA is a technique to calculate the relative efficiency of a set of decision-making units conducted by the mathematical programming. DEA divides decision-making units into two categories of efficient and inefficient and compares decision-making units by determining the amount of efficiency. For the first time Farrell [1] proposed using non-parametric methods to determine the efficiency. His suggestion has been extended by Charnes and Cooper in Rhodes' PhD thesis [2] was called DEA. In his study, Rhodes analyzed student achievement in American schools and presented the CCR model in cooperation with Charnes-Cooper model. Then, in 1984, DEA method was developed by Banker, Charnes and Cooper under a model named BCC [3]. Furthermore, in addition to CCR and BCC models other models were also presented later in DEA.

Shareholders usually form a collection of financial assets known as portfolio in order to reduce the investment risk. The monetary or cash value of the portfolio of any legal or natural person is the value of the portfolio. The portfolio is the most important factor in valuation of investment companies listed on the Stock Exchange. In order to reduce risk, The portfolio is selected in a way that in normal circumstances the possibility of reduction in the efficiency of all assets (i.e. purchased shares) is close to zero. In this case, like other countries the portfolio plays an

important role in stock market in Iran. Along with the dramatic growth of DEA and the focus on the input and output data, the subject of ratio data was introduced. With the integration of DEA and Ratio analysis, Despic et al. proposed ratio-based analysis of DEA (DEA-R) [4]. Wei et al. indicated false in 21 medical centers inefficiency in Taiwan by utilizing DEA-R models [5]. Later, Wei et al. studied problems of CCR model in DEA and advantages of DEA-R [6]. In addition, Wei et al. measured the efficiency and super efficiency by developing input oriented DEA-R and constant returns to scale models [7]. With the introduction of DEA-R for ratio data Liu et al. proposed DEA models with hidden input from a different viewpoints and studied 15 research institutes with ratio data in China [8]. Following this path, Mozaffari et al. studied the relationship between DEA and DEA-R models [9]. Mozaffari et al. compared efficiency of cost and revenue in DEA and DEA-R [10].

Overall efficient portfolio is estimated by non-parametric methods due to large data and multiple inputs and outputs. Even if there are ratio input and output, the portfolio is estimated as optimized. Another approach was considered in the study published by Joro and Na [11], and Vorst [12]. By considering diversification, Briec et al. also developed a quadratic constraint nonlinear DEA model in the mean-variance framework for a single period [15]. Liu et al. [13] applied the DEA

model for Estimation of portfolio efficiency. Juo et al. [14] discussed profit-oriented productivity change. Brier and Kerstens extended the multi-horizon mean-variance portfolio analysis. [16,17]. In the present study, in addition to the radial and non-radial models ratio-based models are also utilized in order to estimate efficient portfolio. In specific, in this study a proper algorithm is recommended for efficient portfolio using non-parametric methods such as the DEA and DEA-R. This article is structured as follows: in the second section the basic concepts of statistics, DEA and DEA-R are presented. In the third and fourth sections, non-radial models are recommended for determining efficient portfolio in DEA and DEA-R. In the last section, a case study will be provided.

**2. Basic Concepts**

In this section the basic concepts of statistics, DEA and DEA-R are briefly stated.

**2-1.Review of Co-Variance and Correlation Coefficient**

If X is a one-dimensional random variable in discrete space, the expected value and variance of X are obtained from the following equation, respectively:

$$V(X) = E(X^2) - (E(X))^2 \text{ \& } E(X) = \sum_{i=1}^n x_i P(x_i)$$

Where P (x<sub>i</sub>) is the probability density function.

**Definition 1:** in variance if the number of variables is more than one, there is new concept called Covariance calculated from the following equation:

$$\delta_{ij} = \text{Cov}(X_i, Y_j) = E \left( (X_i - E(X_i))(Y_j - E(Y_j)) \right) = E(X_i Y_j) - E(X_i)E(Y_j)$$

**Definition 2:** In the case of more than one variable in addition to covariance another concept called correlation coefficient that is a double operator defined as

$$\rho(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{V(X)V(Y)}} \text{ such that } |\rho_{X, Y}| \leq 1. \text{ Thus,}$$

we have a correlation coefficient matrix which is the positive-definite matrix and its main diagonal elements are all one.

**Note 1.** If X is a random vector and ‘a’ is a vector that is the transpose of the row matrix, a<sup>T</sup>, the variance and standard deviation are defined as follows:

$$\sum_{i=1}^n \sum_{j=1}^n a_i a_{ij} a_j = V(a^T X) = \sigma^2_{a^T X}$$

$$\sigma_{a^T X} = \sqrt{\sum_{i=1}^n \sum_{j=1}^n a_i a_{ij} a_j}$$

**2.2 Overview of the concepts of DEA**

The production function is a function that provides maximum output for any combination of input. In most cases, obtaining a production function is not an easy task, due to the complexity of the production process, change in the production technology and the multi-validness of the production function. This means that in most cases, a vector of inputs such as X<sub>j</sub>=(x<sub>1</sub>,...,x<sub>m</sub>) produces an output vector such as Y<sub>j</sub>=(y<sub>1</sub>,...,y<sub>s</sub>).

The purpose of DEA is to calculate the efficiency of a decision-making unit according to the production function. Meanwhile, calculating the production function is not an

easy task and in some cases, it is impossible. Therefore, a set named the production possibility set is made and its boundary is considered as an approximate of the production function. The production function obtained from the production possibility set is the proximate boundary which is intended according to the production technologies. The production possibility set shown by T is defined as follows:

$$T = \{(X, Y) \mid X \text{ vector produces } Y \text{ vector}\}.$$

Consider the production possibility of  $(\theta X_o, Y_o)$  where  $Y \geq Y_o$  and  $0 \leq \theta X_o \leq X_o$ . If we want put this possibility on the frontier, then  $\theta$  is the lowest amount which  $(\theta X_o, Y_o)$  will be on the frontier of Tc. If  $\theta < 1$ , then the production possibility in which the input is  $\theta X_o$  and its output is at least  $Y_o$ , i.e. the  $\theta X_o \leq X_o$  and  $Y \geq Y_o$  will dominate DMU<sub>o</sub>.

Therefore, in order to reduce inputs and to evaluate DMU<sub>o</sub>, Charnes et al. introduced CCR model as follows:

$$\begin{aligned} \text{Min } & \theta \\ \text{S.T } & \sum_{j=1}^n \lambda_j x_{ij} \leq \theta x_{io} \quad i=1, \dots, m, \\ & \sum_{j=1}^n \lambda_j y_{rj} \geq y_{ro} \quad r=1, \dots, s, \\ & \lambda_j \geq 0 \quad j=1, \dots, n. \end{aligned} \quad (1)$$

### 3. Determining the portfolio in non-radial model of DEA

Assume that there are m portfolios under analysis such that  $a^j = (a_1^j, a_2^j, \dots, a_n^j)$  and  $\sum_{i=1}^n a^j r_i = r^{(j)}$  and  $\sigma^j = \sqrt{\sum_{i=1}^n \sum_{l=1}^n a_i^j \sigma_{il} a_l^j}$  for  $j = 1, 2, \dots, m$  are the weight of the portfolio, the expected return and standard

deviation (risk) of the  $j^{\text{th}}$  portfolio, respectively. First consider the subject as risk-free assets. In Figure (1) the horizontal axis represents the standard deviation and the vertical axis represents the expected return and  $B_1CB_2$  curve represents the efficient frontier of portfolio. Suppose that A  $(r, \sigma)$  is one share and  $B_1(r^{(1)}, \sigma^{(1)})$ ,  $B_2(r^{(2)}, \sigma^{(2)})$  and  $C(r^{(3)}, \sigma^{(3)})$  are the optimized portfolio on the boundary. Therefore, using different distances different PE (efficient portfolios) is defined as follows:

$$PE_r^R = \frac{r}{r^{(1)}} \quad (2)$$

$$PE_\sigma^R = \frac{\sigma^{(2)}}{\sigma} \quad (3)$$

$$PE_{r\sigma}^S = \frac{1 - \frac{\sigma - \sigma^{(3)}}{\sigma}}{1 + \frac{r^{(3)} - r}{r}} \quad (4)$$

$$PE_{r\sigma}^q = \frac{1}{2} \left( \left( 1 - \frac{\sigma - \sigma^{(3)}}{\sigma} \right) + \left( 1 + \frac{r^{(3)} - r}{r} \right) \right) \quad (5)$$

In this section, by using additive and enhanced Russell models and taking into account the expected return and risk of portfolio decision making units are determined.

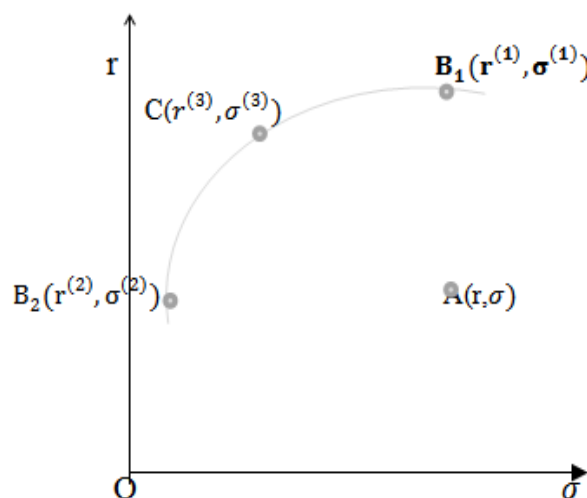


Figure (1). The efficiency of risk free portfolio

Considering  $n$  decision-making units with inputs of  $\sigma^{(j)}$  and outputs of  $r^{(j)}$  for  $DMU_j$  the additive model is proposed as follows:

$$\begin{aligned} \gamma^* = & \text{Max } \sum_{i=1}^m s_i + \sum_{f=1}^s s_f \\ \text{S.T } & \sum_{j=1}^n \lambda_j \sigma^{(j)} + s_i = \sigma^o \quad i=1, \dots, m, \\ & \sum_{j=1}^n \lambda_j r^{(j)} - s_f = r^o \quad f=1, \dots, s, \quad (6) \\ & s_i \geq 0, \quad s_f \geq 0, \quad i = 1, \dots, m, \quad f = 1, \dots, s, \\ & \lambda_j \geq 0, \quad j = 1, \dots, n. \end{aligned}$$

Model (6) which is a non-radial model evaluates  $DMU_o$  with the input and output of  $(\sigma^o, r^o)$ .

**Axiom 1:** model (6) is always feasible.

**Proof:** Considering  $\lambda_j = e_o$  and  $s_f = 0$  and  $s_i = 0$  this model is always feasible. The model shows the inefficiency scale for  $DMU_o = (\sigma^o, r^o)$  and  $1-\gamma^*$  indicates its scale inefficiency. ■

Furthermore, in order to assess the  $DMU_o$ , enhanced Russell model is presented as follows:

$$\begin{aligned} T^* = & \text{Min } \frac{\frac{1}{m} \sum_{i=1}^m \theta_i}{\frac{1}{s} \sum_{f=1}^s \varphi_f} \\ \text{S.T. } & \sum_{j=1}^n \lambda_j \sigma^{(j)} \leq \sigma_i^o \theta_i \quad i = 1, \dots, m, \\ & \sum_{j=1}^n \lambda_j r^{(j)} \geq r_f^o \varphi_f \quad f = 1, \dots, s, \quad (7) \\ & \theta_i \leq 1, \quad \varphi_f \geq 1 \quad i = 1, \dots, m, \quad f = 1, \dots, s, \\ & \lambda_j \geq 0 \quad j = 1, \dots, n. \end{aligned}$$

In model (7), since the fractional objective function, it is not linear and the objective is to decrease all inputs and increase all outputs.

#### 4-Determination of the Portfolio in Non-Radial DEA-R Models

The DEA based on the analysis of fraction (DEA-R) makes use of linear programming models in order to evaluate DMUs. In case of

ratio data in DEA-R by defining efficiency as a weighted sum of the ratio of input to output or vice versa scale efficiency is calculated. The production possibility set in DEA - R is defined as follows:

$$\begin{aligned} T_R = & \left\{ \left( \frac{x}{y} \right) \left| \sum_{j=1}^n \lambda_j \left( \frac{x_{ij}}{y_{rj}} \right) \leq \frac{x}{y} \& \sum_{j=1}^n \lambda_j = 1, \right. \right. \\ & \left. \left. \lambda_j \geq 0, \quad j = 1, \dots, n \right\} \quad (8) \end{aligned}$$

If there is ratio data and a ratio of inputs to outputs (or vice versa) is defined, for instance  $\frac{x_{ij}}{y_{rj}}$ , the input-oriented DEA - R models in constant returns to scale technology are as follows:

$$\begin{aligned} \text{Min } & \theta \\ \text{S.T } & \theta \left( \frac{x_{io}}{y_{ro}} \right) \in T_R \quad (9) \end{aligned}$$

Output-oriented Model (9) can be obtained as follows:

$$\begin{aligned} \text{Max } & \theta \\ \text{S.T } & \sum_{j=1}^n \lambda_j \left( \frac{y_{rj}}{x_{ij}} \right) \geq \theta \left( \frac{y_{ro}}{x_{io}} \right) \quad i = 1, \dots, m, \\ & \quad \quad \quad r = 1, \dots, s \\ & \sum_{j=1}^n \lambda_j = 1 \quad (10) \\ & \lambda_j \geq 0, \quad j = 1, \dots, n \end{aligned}$$

Model (10) is a linear programming problem that has been introduced in constant returns to scale technology in DEA-R. (See [9])

In DEA-R Models some axioms are hold.

$T_R$  is a closed and bounded set. (See [8])

The inclusion principle of observations related to ratios of  $\frac{x_{ij}}{y_{rj}}$  is hold.

The convexity principle in DEA-R is hold.

(See [8]).

In DEA models, the efficiency is equal to the weighted sum of the output divided into the weighted sum of the input and the relative efficiency is defined as the absolute efficiency divided into the maximum defined absolute efficiency. In this regard, the following problems exist:

First, what is the reason for defining efficiency? Secondly, the use of non-Archimedean number ( $\epsilon$ ) that prevents zero weights, such that neither nominator nor denominator becomes zero. Thirdly, in DEA the aforementioned reasons may lead to the false inefficiency. In these regards, the DEA-R model is helpful and it creates no problem.

In addition, the scale efficiency in input-oriented models of DEA-R is equal to or less than the scale efficiency in DEA. The scale efficiency in DEA and DEA - R models when there is one output and multiple inputs is exactly equivalent to each other, and it can easily be proved (Wei et al, 2011).

The following additive model in non-radial DEA-R models is recommended for recognizing the efficiency and inefficiency among units in determining the portfolio:

$$\begin{aligned} \text{Max } \alpha &= \sum_{i=1}^m \sum_{f=1}^s s_{if} \\ \text{S.T } \sum_{j=1}^n \lambda_j \left( \frac{\sigma^{(j)}}{r^{(j)}} \right) + s_{if} &= \frac{\sigma^o}{r^o} \\ i=1, \dots, m \quad , \quad f=1, \dots, s, \\ \sum_{j=1}^n \lambda_j &= 1 \quad (11) \\ s_{if} &\geq 0 \quad i = 1, \dots, m \quad , \quad f = 1, \dots, s, \\ \lambda_j &\geq 0 \quad j=1, \dots, n. \end{aligned}$$

$DMU_o$  in model (11) is efficient if and only if

$$\alpha^* = 0.$$

Also, Russell's additive model is presented for evaluating DMUo in non-radial DEA-R models as follows:

$$\begin{aligned} \text{Min } \beta &= \frac{1}{ms} \sum_{i=1}^m \sum_{f=1}^s \theta_{if} \\ \text{S.T } \sum_{j=1}^n \lambda_j \left( \frac{\sigma^{(j)}}{r^{(j)}} \right) &\leq \frac{\sigma^o}{r^o} \theta_{if} \\ i=1, \dots, m \quad , \quad f=1, \dots, s, \\ \sum_{j=1}^n \lambda_j &= 1 \quad (12) \\ \theta_{if} &\geq 0 \quad i = 1, \dots, m \quad , \quad f = 1, \dots, s, \\ \lambda_j &\geq 0 \quad j=1, \dots, n. \end{aligned}$$

In each optimal solution of model (12), all input and output constraints are enforced. In model (12),  $DMU_o$  is efficient if and only if  $\beta^* = 1$ .

### 5-Numerical Example

In this section, using the data in the article [13], the expected returns and covariance matrix, which its statistical details are presented in Table (1) and (2). The Random sample weights of  $m = 10, 50$  and  $100$  are created by the EXEL software. In addition, EXEL and GAMS software are utilized to produce efficient portfolio. By selecting the samples of  $m = 10, 50$  and  $100$ , the obtained solutions from model (4), (5), (6) and (7) are shown in Table (3), (4) and (5). The objective function and constraints in model (6) using GAMS program are as follows:

Equations

Objective,    Const1(i),    Const2(f);

Objective1..  $z=e=\text{sum}(i, s(i))+ \text{sum}(f, t(f));$

Const 1(i)..

$\text{Sum}(j, \text{SIGMA}(i,j)*\text{Lambda}(j))+s(i)=e-\text{SIGMA}o(i);$

Const 2(f)..

$\text{Sum}(j, \text{EXR}(f,j)*\text{Lambda}(j))-t(f)=e-\text{EXRo}(f) ;$

Also, the objective function and constraints in model (7) using GAMS program is as follows:

Equations
Objective , Const1(i) , Const2(f), Const3(f) , Const4(i) ,Const5(i);
Objective..
$z=e=\text{Sum}(i,\text{Teta}(i).m).\text{Sum}(f,\text{Phi}(f).s);$
Const1(i)..
$\text{Sum}(j,\text{sigma}(i,j)*\text{Lambda}(j))=l=\text{Teta}(i)*\text{sigmao}(i);$
Const2(f)..
$\text{Sum}(j,\text{exr}(f,j)*\text{Lambda}(j))=g=\text{Phi}(f)*\text{yo}(f);$
const3(f).. $\text{Phi}(f)=g=1;$
const4(i).. $\text{Teta}(i)=l=1;$
const5(i).. $\text{Teta}(i)=g=0;$

Finally, in table (5), the coefficient correlation between each pair of solutions obtained from models (5) and (6) and solutions obtained from models (4) and (7) are presented as following: By selecting a sample with  $m = 10$ , in Table 3 from the left side to right, the first column shows the obtained solutions from model (6), second column shows obtained solutions from model (5), third column shows obtained solutions from model (7) and the fourth column shows obtained solutions from model (4).

Table 1.Expected return(%)

	1	2	3	4	5
Returns	57.2	36.1	83.1	49,1	85.0

Table 2. Covariance Matrix

Covariance	5	4	3	2	1
1	28.91	11.74	50.75	14.97	40.174
2	02.133	97.143	87.12	52.280	14.97
3	04.72	28.107	62.256	87.125	50.75
4	83.88	83.172	28.107	97.143	11.74
5	25.168	83.88	04.72	02.133	28.91

Table 3. The obtained solutions from model and definition with 10 samples

Model (6)	Definition (5)	Model (7)	Model (4)
42.80	0.71	0.85	0.41
90.57	0.77	0.77	0.36
90.20	2.12	0.53	0.25
77.26	0.88	0.78	0.36
71.20	0.94	0.78	0.36
119.06	0.93	0.67	0.31
77.15	0.84	0.79	0.36
2.15	1.05	0.99	0.46
0.00	1.10	1.00	0.46
114.45	1.36	0.59	0.27
77.53	0.69	0.82	0.38
136.72	0.69	0.71	0.33
88.42	1.03	0.72	0.33
31.97	0.85	0.90	0.42
52.49	1.08	0.81	0.38
95.41	1.02	0.70	0.33
118.76	0.95	0.67	0.31
109.58	1.05	0.67	0.31
118.60	1.04	0.65	0.30
123.36	1.58	0.53	0.24
92.77	1.17	0.68	0.31
136.28	1.08	0.60	0.28
74.23	0.70	0.82	0.38
48.91	2.05	0.70	0.32
32.01	0.91	0.89	0.41
91.58	0.88	0.75	0.35
63.19	0.84	0.82	0.38
77.66	0.75	0.80	0.37
63.56	0.73	0.84	0.39
48.24	0.89	0.85	0.39
119.15	1.06	0.64	0.30
69.88	1.42	0.70	0.32
64.20	0.87	0.81	0.38
46.49	0.81	0.87	0.40
98.33	1.05	0.69	0.32
148.99	1.14	0.56	0.26
14.86	0.95	0.95	0.44
117.03	0.67	0.75	0.35
76.18	0.81	0.80	0.37
66.45	1.04	0.78	0.36
81.93	0.89	0.76	0.35
55.68	0.93	0.82	0.38
41.58	1.19	0.83	0.38
31.49	1.79	0.81	0.38
118.55	1.02	0.65	0.30
55.88	0.96	0.82	0.38
15.24	0.82	0.95	0.44
74.26	1.29	0.71	0.33
74.77	0.93	0.77	0.36
160.67	1.02	0.58	0.27

By selecting a sample with  $m = 50$ , in Table 4 from the left side to right, the first column shows the obtained solutions from model (6), the second column shows obtained solutions from model (5), the third column shows solutions obtained from the model (7) and the fourth column shows solutions obtained from the model (4).

Selecting a model with  $m = 100$ , we go through the same procedures like samples of 10 and 50.

In Table 5, the first row shows the sample size, the second shows the coefficient correlation of obtained solutions from model (5) and (6) and the third row shows the coefficient correlation of obtained solutions from the model (4) and (7).

**6. Conclusion**

In this study, we define the portfolio by DEA frontier. In other words, it is possible to utilize a frontier from DEA in order to approximate the real frontier and to estimate the efficient portfolio. We also consider various limitations that exist for investments in the market. Given that DEA is a linear programming model and, it needs to a large extent simple calculations, with selecting several different sample, using covariance matrix and non-radial DEA and DEA-R models we show that by increasing the sample size, the frontier of the efficient portfolio gradually becomes close to the frontier of DEA and this is the practical application of DEA models.

Table 4. Obtained solutions from the model and definition with 50 samples

Model (6), (11)	Definition (5)	Model (7), (12)	Model (4)
42.80	0.71	0.85	0.41
90.57	0.77	0.77	0.36
90.20	2.12	0.53	0.25
77.26	0.88	0.78	0.36
71.20	0.94	0.78	0.36
119.06	0.93	0.67	0.31
77.15	0.84	0.79	0.36
2.15	1.05	0.99	0.46
0.00	1.10	1.00	0.46
114.45	1.36	0.59	0.27
77.53	0.69	0.82	0.38
136.72	0.69	0.71	0.33
88.42	1.03	0.72	0.33
31.97	0.85	0.90	0.42
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74.77	0.93	0.77	0.36
160.67	1.02	0.58	0.27



Table 5. Correlation coefficients of efficiency ranks with different sample sizes

Sample Size	10	50100
the coefficient correlation of obtained solutions from model (5) and (6 or 11),	-0.28991	0.007682 -0.01786
the coefficient correlation of obtained from the model (4) and (7 or 12)	0.996806	0.997481 0.996867

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## Estimating the Efficient Portfolio in Non-Radial DEA and DEA-R Models

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### تخمین سبد اوراق بهادار کارا در مدل‌های غیر شعاعی DEA و DEA-R

چکیده: سبد اوراق بهادار ترکیبی مناسب از سهام یا دارایی‌ها است، که یک سرمایه‌گذار آنها را خریداری کرده است. هدف از تشکیل سبد سهام، تقسیم کردن ریسک سرمایه‌گذاری بین چند سهم است. استفاده از روش‌های غیر پارامتری DEA و DEA-R می‌تواند در تخمین سبد اوراق بهادار کارا بسیار حائز اهمیت باشد. در این مقاله با استفاده از مدل‌های غیر شعاعی DEA و DEA-R سبد اوراق بهادار کارا تخمین زده می‌شود. با پیشنهاد مدل‌های غیر شعاعی در DEA-R در حالتی که داده‌های نسبی وجود دارد سبد اوراق بهادار کارا تعیین می‌شود. در خاتمه مطالعه کاربردی بر گرفته از مقاله [۲] با مدل‌های غیر شعاعی DEA و DEA-R انجام شده و نتایج ارائه شده است.

کلمات کلیدی: سبد اوراق بهادار کارا، DEA، DEA-R