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On convergence and stability conditions of homotopy perturbation method for an inverse heat conduction problem

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Abstract

In this paper, we investigate the application of the Homotopy Perturbation Method (HPM) for solving a one-dimensional nonlinear inverse heat conduction problem. In this problem the thermal conductivity term is a linear function with respect to unknown heat temperature in bounded interval. Furthermore, the temperature histories are unknown at the end point of the interval. This problem is ill-posed. So, using the finite difference scheme and discretizing the time interval, the partial differential equation is reduced into a System of Nonlinear Ordinary Differential Equations (SNODE's). Then, using HPM, the approximated solution of the obtained Ordinary Differential Equation (ODE) system is determined. In the sequel, the stability and convergence conditions of the proposed method are investigated. Finally, an upper bound of the error is provided.

Keywords: Homotopy perturbation method; Diffusion equation; Disceretizing method; Inverse problem.

1 Introduction

Inverse heat conduction problems are used to describe many important phenomena in physics, chemistry, mechanics, etc. There has been a great amount of investigation to solving inverse heat conduction problems in one and multi dimensional spaces. Many effective methods have been provided. However, lots of inverse heat transfer problems, which arise in natural phenomena, such as radiational heat transfer, modelling of case hardening, gravimetry,

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and etc., have nonlinear forms and so they are not solvable with analytical methods, but unfortunately, most of presented methods are useful just for solving linear forms.

Usually these problems are ill-posed in the sense of Hadamard. Therefore, the regularization method is a successive technique for solving ill-posed problems and it may be applied to entire class of problems which arise from physical observations.

Beck et al. have investigated an inverse problem in one-dimensional space with two general procedures, function specification and regularization methods, and a method of combining these, trial function method, and have implemented all of these methods in a sequential manner [3]. Lesnic et al. in [15] and [19] have considered a special case of distributed (identification) parameter problems in one-dimensional spaces. They have shown that for a one-dimensional quasi-heterogeneous material with square-root harmonic conductivity, a single measurement of the conductivity and the flux on the boundary is sufficient to determine uniquely the unknown physical parameters and the solution function. Alivanov considered the solution of inverse problems by analytical approaches [2]. Qu and Dou [20], Lewandowski [18], and Jia et al. [16], have studied the nonlinear diffusion equation and provided some numerical techniques. Shidfar and Zakeri in [21] - [23] have investigated the existence and uniqueness of a solution for a two-dimensional nonlinear inverse diffusion problem. Also, Zakeri et al. have begun their research by a Cauchy inverse problem and found a solution by HMP method [26] and in continuation they have gone on by an inverse heat conduction problem and solved it by the HPM again [27]. Also, they have applied an approach which contained a difference method and the HPM together, and solved the problem with a reliable accuracy [28].

In continuation of above researches our intend is investigation of sufficiently condition for HPM for solving inverse heat conduction problems.

In next section, the HPM is introduced shortly and in Section 3, an approximated solution for the inverse heat conduction problem is obtained via HPM . Then, the stability and convergence of the above mentioned method are studied in Section 3 and Section 4, respectively . Some numerical results are illustrated by some tables and figures in Section 6. Finally conclusions and some suggestions for more research are given in last section.

2 Basic concepts of HPM

In this section we introduce the basic concepts of the HPM , according to [26], in brief.

Consider the following nonlinear equation

$$A(u) - f(r) = 0, \quad r \in \Omega, \tag{1}$$

with the boundary conditions

$$B(u, \partial u/\partial n) = 0, \quad r \in \Gamma,$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytic function and Γ is the boundary of the domain Ω. The operator A can be generally divided into two parts L and N, where they are the linear and nonlinear parts of A, respectively. So, Eq. (1) converts into the following form

$$L(u) + N(u) - f(r) = 0. \tag{2}$$

In [10], He constructed a homotopy $H : \Omega \times [0, 1] \longrightarrow \mathbb{R}$ which satisfies:

$$H(v, p) = (1 - p)[L(v) - L(v_0)] + p[A(v) - f(r)] = 0, \tag{3}$$

or

$$H(v, p) = L(v) - L(v_0) + pL(v_0) + p[N(v) - f(r)] = 0, \tag{4}$$

where $r \in \Omega$, and $p \in [0, 1]$ is called the homotopy parameter, and v_0, is an initial approximation for the solution of Eq. (1) which satisfies the boundary conditions. Consequently

$$\begin{cases} H(v, 0) = L(v) - L(v_0) = 0, \\ H(v, 1) = A(v) - f(r) = 0. \end{cases}$$

Now, when p varies from 0 to 1, the homotopy $H(v, p)$, changes from $L(v) - L(v_0)$ to $A(v) - f(r)$.

Applying the perturbation technique due to the fact that $0 \leq \ p \ \leq 1$ is considered as a small parameter, we can assume that the solution of Eq. (3) or Eq.(4) can be expressed as a series in the form

$$v = v_0 + pv_1 + p^2 v_2 + \dots \ .$$

When $p \longrightarrow 1$, Eq.(3) or Eq.(4) corresponds to Eq.(2) and so v becomes the approximate solution of Eq. (2) i.e;

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \dots \ . \tag{5}$$

The series (7) is convergent for most cases and the rate of convergence depends on $A(v)$ [11, 12].

In the next section, a nonlinear inverse heat conduction problem is considered. We discretize the time interval by means of backward finite difference method and apply the HPM. Then, the approximate solution of the problem is yield.

3 Solution of nonlinear inverse heat conduction problem by HPM

Let $T > 0$, and consider the nonlinear parabolic partial differential equation

$$u_t - (D(u)u_x)_x = \Phi(x,t),\ (x,t) \in \Omega = (0,1) \times (0,T), \tag{6}$$

with initial condition

$$u(x,0) = s(x), \quad x \in [0,1], \tag{7}$$

and boundary conditions

$$u(0,t) = f(t), \quad t \in [0,T], \tag{8}$$

$$u(1,t) = h(t), \quad t \in [0,T], \tag{9}$$

where $D(u) = a(t)u + b(t) > 0$, and $a,\ b,\ s$ and f are known functions such that, $b(t)$ is far from zero in $[0,T]$.

If h is given, then the problem (6)-(9) is a direct problem which is solvable by means of common numerical and approximation approaches for solving PDEs, such as finite difference method [7], finite element method [7], radial basis functions [4], homotopy perturbation method [13], Adomian decomposition method [1] and so on.

Now, suppose that h is unknown. Then the problem (6)-(9) becomes an inverse problem. Consequently, an overspecified condition, such as

$$u_x(0,t) = g(t), \quad t \in [0,T], \tag{10}$$

where g is a known function, is used.

For positive integer n, let $\triangle t = \frac{1}{k} = \frac{T}{n},\ t_j = j \triangle t, j \in J_n = \{1, 2, ..., n\}$. Put $u_0(x) = u(x,0) = s(x),\ a_j = a(t_j),\ b_j = b(t_j),\ \Phi_j(x) = \Phi(x,t_j)$, for any $j \in J_n$, such that they are given fixed nodes. Similarly, we consider the $u_j(x)$, as the approximated value of $u(x,t_j),\ j \in J_n$.

Using the backward finite difference scheme for the term u_t in the form

$$u_t(x) \simeq k(\ u_j(x) - u_{j-1}(x)\ ), \qquad j \in J_n,$$

and substituting in Eq. (6), a system of second order ordinary differential equations with respect to x is obtained. We have

$$k(u_j(x) - u_{j-1}(x)) - \frac{d}{dx}\{(a(t_j)u_j(x) + b(t_j))\frac{d}{dx}u_j(x)\} = \Phi_j(x),\ 1 \le j \le n,$$

or

$$\frac{d^2}{dx^2}u_j(x) \ - \{\frac{k}{b(t_j)}(u_j(x) - u_{j-1}(x)) - \frac{a(t_j)}{b(t_j)}\ (\frac{d}{dx}\ (u_j(x)\frac{d}{dx}u_j(x)\ )\ )\ \}$$

$$= \frac{-1}{b(t_j)} \Phi_j(x). \tag{11}$$

For simplicity, define $\mathbf{u}(x) = (u_1(x), u_2(x), \ldots, u_n(x))^T$. Then we can write Eq. (6) as follows

$$A\mathbf{u} \;=\; L_x\mathbf{u} - N\mathbf{u} = \Psi(x,t),$$

where $\Psi(x,t) = (\frac{-\Phi_1(x)}{b(t_1)}, \ldots, \frac{-\Phi_n(x)}{b(t_n)})^T$. Moreover, L_x and N are the linear and nonlinear parts of the operator A, respectively, and are as follows:

$$L_x = \frac{d^2}{dx^2},$$

$$N\mathbf{u} = -\mathbf{M_2}\frac{d}{dx}\ (D(\mathbf{u},\mathbf{u})) + \mathbf{M_1}\mathbf{u} + \mathbf{m},$$

where

$$\mathbf{m} = (\tfrac{-k}{b(t_1)}\ s(x), 0, \ldots, 0)^T{}_{n\times 1}, \qquad \mathbf{M_1} = \ k \begin{pmatrix} \frac{1}{b(t_1)} & & & 0 \\ \frac{-1}{b(t_2)} & \frac{1}{b(t_2)} & & \\ & \ddots & \ddots & \\ 0 & & \frac{-1}{b(t_n)} & \frac{1}{b(t_n)} \end{pmatrix},$$

$$\mathbf{M_2} = \ diag\ (\ \tfrac{a(t_1)}{b(t_1)}, \tfrac{a(t_2)}{b(t_2)},\ \ldots\ ,\ \tfrac{a(t_n)}{b(t_n)}\ ),$$

and

$$D(\mathbf{u}(x),\mathbf{v}(x)) = (u_1(x)\tfrac{d}{dx}v_1(x), u_2(x)\tfrac{d}{dx}v_2(x), \ldots, u_n(x)\tfrac{d}{dx}v_n(x))^T.$$

After twice integration of Eq. (6) with respect to x, and applying the conditions (7)- (9), we obtain

$$\mathbf{u}(x) - x\mathbf{g} - \mathbf{f} - \int_0^x\int_0^x\ N\mathbf{u}\ (x)\ dxdx = \int_0^x\ \int_0^x \Psi(x)dxdx,$$

where $\mathbf{g} = (\ g(t_1),\ \ldots\ ,\ g(t_n)\ )^T$, and $\mathbf{f} = (\ f(t_1),\ \ldots\ ,\ f(t_n)\ )^T$.

Now, using HPM and [8, 26], we choose a convex homotopy such that

$$\mathbf{H}\left(\mathbf{v}(x), p\right) = \mathbf{v}(x) - \mathbf{h}(x) - p\int_0^x\int_0^x\ N\mathbf{v}(x)\ dxdx = 0, \tag{12}$$

and

$$\mathbf{F}(\mathbf{u}(x)) = \mathbf{u}(x) - \mathbf{h}(x) = 0,$$

where

$$\mathbf{h}(x) = x\mathbf{g} + \mathbf{f} + \int_0^x \int_0^x \quad \Psi(x)\ dxdx,$$

and $\mathbf{v}(x) = (\ v(x,t_1),\ \ldots\ v(x,t_n)\ )^T$. Furthermore, Eq. (12) gives

$$\mathbf{v}(x) = \mathbf{h}(x) + p\int_0^x \int_0^x N\mathbf{v}(x)\ dxdx. \tag{13}$$

Combining Eq.s (11) and (16), we obtain the following results

$$\mathbf{v}(x) = x\mathbf{g} + \mathbf{f} + \int_0^x \int_0^x \ \Psi(x)\ dxdx + $$
$$p\int_0^x \int_0^x \{\mathbf{M}_1\ (\ \mathbf{v}(x) - \mathbf{u}_s(x)\ ) - \mathbf{M}_2\ \frac{d}{dx}\ D(\mathbf{u}(x), \mathbf{u}(x))\}dxdx, \tag{14}$$

where $\mathbf{u}_s(x) = (s(x), u_1(x), \ldots, u_{n-1}(x))^T$. Thus, it is concluded that

$$\mathbf{v}_0(x) = \mathbf{h}(x) = x\mathbf{g} + \mathbf{f} + \int_0^x \int_0^x \Psi(x)dxdx = (v_0(x,t_1), \cdots, v_0(x,t_n))^T, \tag{15}$$

and

$$\begin{aligned}\mathbf{v}_1(x) &= \int_0^x \int_0^x \left\{\mathbf{M}_1\ (\mathbf{v}_0(x) - \mathbf{u}_s(x)) - \mathbf{M}_2\ \frac{d}{dx}\ D(\mathbf{v}_0(x), \mathbf{v}_0(x))\ \right\}\ dxdx \\ &= (v_1(x,t_1),\ \cdots\ ,\ v_1(x,t_n))^T.\end{aligned} \tag{16}$$

The above relations are obtained by equating the terms with identical powers of p in Eq. (19). The approximate solution is

$$\mathbf{u}(x) \simeq \mathbf{v}_0(x) + \mathbf{v}_1(x) = \mathbf{v}(x). \tag{17}$$

4 Convergence and stability analysis

In this section, we use continuity of $u(x,t)$ on the compact domain Ω, and prove that $\mathbf{v}(x) \simeq \mathbf{v}_0(x) + \mathbf{v}_1(x)$ depends continuously on the data. Therefore, adding a perturbation term to $a,\ b,\ f,\ g$ and Φ, an upper bound for errors of their solutions are found. In each case, we show that as the perturbation term tends to zero, the solutions errors vanish.

Lemma 1. *Let $M(t) = \int_0^1 |\Phi(x,t)|dx > 0$ is a bounded function such that $M(t) \leq M$ for any $0 \leq t \leq T$, and $\hat{\boldsymbol{v}}_0(x)$ correspond to $\boldsymbol{v}_0(x)$, where $b(t)$ is perturbed by $\delta b(t)$ in Equation (6). Then we have*

$$|\hat{v}_0(x,t_j) - v_0(x,t_j)| = \frac{|\delta b(t_j)|}{|b(t_j)(b(t_j) + \delta b(t_j))|}M, \quad 0 \leq j \leq n, \quad x \in [0,1],$$

consequently, if $|\delta b(t)| \to 0$, *then* $|\hat{v}_0(x,t) - v_0(x,t)| \to 0$, *for any* $t = t_j$, $j = 1, \cdots, n$, *and* $0 \le x \le 1$.

Proof. Using the Equation (15), we have

$$v_0(x,t_j) = f(t_j) + xg(t_j) - \frac{1}{b(t_j)} \int_0^x \int_0^x \Phi_j(x) dx dx. \tag{18}$$

If $b(t_j)$ is perturbed by $\delta b(t_j)$, then

$$\hat{v}_0(x,t_j) = f(t_j) + xg(t_j) - \frac{1}{b(t_j) + \delta b(t_j)} \int_0^x \int_0^x \Phi_j(x) dx dx. \tag{19}$$

Now from Equations (18) and (19), we obtain

$$v_0(x,t_j) - \hat{v}_0(x,t_j) = \frac{-\delta b(t_j)}{b(t_j)(b(t_j) + \delta b(t_j))} \int_0^x \int_0^x \Phi_j(x) dx dx,$$

consequently

$$|v_0(x,t_j) - \hat{v}_0(x,t_j)| \le \frac{|\delta b(t_j)|}{|b(t_j)(b(t_j) + \delta b(t_j))|} M, \quad \textit{for any } 0 \le j \le n,$$

and this completes the proof. □

Lemma 2. *Let* $M(t)$ *and* M *are as defined in lemma 1, and* $\hat{v}_1(x,t_j)$ *is the value of* $v_1(x,t_j)$ *for any* $t = t_j$, $j = 1, \cdots, n$, *when* $b(t_j)$ *is perturbed by* $b(t_j) + \delta b(t_j)$ *as in problem Equation (6), such that* $\delta b(t_0) = 0$. *If* $|\delta b(t_j)| \to 0$, *then* $|\hat{v}_1(x,t_j) - v_1(x,t_j)| \to 0$, *for any* $0 \le x \le 1$, *and* $1 \le j \le n$.

Proof. Similar to detailed proof presented for lemma 1, assume

$$\begin{aligned} v_1(x,t_j) = \int_0^x \int_0^x \{ & \frac{k}{b(t_j)} (v_0(x,t_j) - u_{j-1}(x)) \\ & - \frac{a(t_j)}{b(t_j)} (\frac{d}{dx} \{ v_0(x,t_j) \frac{d}{dx} v_0(x,t_j) \}) \} dx dx. \end{aligned}$$

Suppose that $b(t_j)$ is replaced by $b(t_j) + \delta b(t_j)$, for any $j = 1, \cdots, n$. Then we have

$$\begin{aligned} \hat{v}_1(x,t_j) = \int_0^x \int_0^x \{ & k \frac{\hat{v}_0(x,t_j) - \hat{u}_{j-1}(x)}{b(t_j) + \delta b(t_j)} \\ & - \frac{a(t_j)}{b(t_j) + \delta b(t_j)} (\frac{d}{dx} \{ \hat{v}_0(x,t_j) \frac{d}{dx} \hat{v}_0(x,t_j) \}) \} dx dx, \end{aligned}$$

or

$$
\begin{aligned}
|\hat{v}_1(x,t_j)-v_1(x,t_j)| \leq k\int_0^1\int_0^1 \{ & \frac{|\hat{v}_0(x,t_j)-v_0(x,t_j)|}{|b(t_j)+\delta b(t_j)|}+\frac{|\hat{u}_{j-1}(x)-u_{j-1}(x)|}{|b(t_j)+\delta b(t_j)|} \\
&+\frac{|\delta b(t_j)|C}{|b(t_j)(b(t_j)+\delta b(t_j))|}\}dxdx \\
&+\frac{|a(t_j)||\delta b(t_j)|}{|b(t_j)(b(t_j)+\delta b(t_j))|}\ (\frac{M}{|b(t_j)(b(t_j)+\delta b(t_j))|} \\
&\times \max_{\substack{x\in[0,l],\\ j=1,\cdots,\ n.}} \{|\hat{v}_0(x,t_j)|,|v_0(x,t_j)|\} \\
&+\frac{|v_0^2(x,t_j)|+|f(t_j)^2|+|f(t_j)g(t_j)|}{2|b(t_j)|}),
\end{aligned}
$$

when $C\in\mathbb{R}$ is an upper bound for $|v_0(x,t_j)-u_{j-1}(x)|$. Therefore, for fixed k, from lemma 1, it is derived,

$$\lim_{|\delta b(t_j)|\longrightarrow 0}|\hat{v}_1(x,t_j)-v_1(x,t_j)|\longrightarrow 0,$$

for $\quad 0\leq x\leq 1,\;$ and $\;1\leq j\leq n.$ □

Remark 1. By an induction, it is shown that, when $|\delta b(t_j)|\longrightarrow 0$, the second term of the integral in the above inequality vanishes.

Theorem 1. *Suppose that* $v(x,t_j)$, $v_0(x,t_j)$, $\hat{v}_0(x,t_j)$, $v_1(x,t_j)$, $\hat{v}_1(x,t_j)$, M, $M(t)$, $b(t_j)$ *and* $\delta b(t_j)$, *are the same as defined in lemmas 1 and 2. Then* $v(x,t_j)=v_0(x,t_j)+v_1(x,t_j)$ *depends continuously on the data.*

Proof. Obviously, by considering lemmas 1 and 2, the statement of Theorem 1 is proved. □

Theorem 2. *Suppose that* $\delta a(t_j)$ *is the perturbation term that perturbs* $a(t_j)$ *to* $a(t_j)+\delta a(t_j)$, *and* v_0, v_1, $\hat{v}_0$, $\hat{v}_1$, M, $M(t)$, $b(t_j)$ *and* $\delta b(t_j)$ *satisfy assumptions of Theorem 1. Then* $v(x,t_j)$ *depends continuously on the data.*

Proof. The first part of $v(x,t)$ is independent of $a(t)$. Then, using Theorem 1, there is nothing to prove for $v_0(x,t_j)$, $\;1\leq j\leq n$. Now we just prove that $v_1(x,t_j)$ depends continuously on the data. We have

$$\hat{v}_1(x,t_j)=\int_0^x\int_0^x\{k\frac{\hat{v}_0(x,t_j)-\hat{u}_{j-1}(x)}{b(t_j)+\delta b(t_j)}-\frac{a(t_j)+\delta a(t_j)}{b(t_j)+\delta b(t_j)}(\frac{d}{dx}\{\hat{v}_0(x,t_j)\frac{d}{dx}\hat{v}_0(x,t_j)\})\}dxdx,$$

then, it is concluded that

$$
\begin{aligned}
|\hat{v}_1(x,t_j)-v_1(x,t_j)| \leq k\int_0^1\int_0^1 \{ & \frac{|\hat{v}_0(x,t_j)-v_0(x,t_j)|}{|b(t_j)+\delta b(t_j)|}+\frac{|\hat{u}_{j-1}(x)-u_{j-1}(x)|}{|b(t_j)+\delta b(t_j)|} \\
&+\frac{|\delta b(t_j)|}{|b(t_j)(b(t_j)+\delta b(t_j))|}\}dxdx \\
&+\frac{|a(t_j)||\delta b(t_j)|}{|b(t_j)(b(t_j)+\delta b(t_j))|}\ (\frac{M}{|b(t_j)(b(t_j)+\delta b(t_j))|}
\end{aligned}
$$

$$\times \max_{\substack{x\in[0,1],\\ j=1,\cdots,n.}} \{|\hat{v}_0(x,t_j)|, |v_0(x,t_j)|\}$$
$$+ \frac{|v_0^2(x,t_j)| + |f(t_j)^2| + |f(t_j)g(t_j)|}{2|b(t_j)|})$$
$$+ |\delta a(t_j)| \frac{|\hat{v}_0^2(x,t_j)|}{2}.$$

□

Similarly, it is shown that, the approximate solution $\mathbf{v}(x)$ in (17), depends continuously on the data, when $f(t)$, $g(t)$ and $\Phi(x,t)$ are perturbed by small perturbation terms in their domains. So, we give the following theorem.

Theorem 3. *Let $f(t)$, $g(t)$ and $\Phi(x,t)$ be the same as defined in Equation (6). If*

$$f(t) \longmapsto f(t) + \delta f(t),$$
$$g(t) \longmapsto g(t) + \delta g(t),$$
$$\Phi(x,t) \longmapsto \Phi(x,t) + \delta\Phi(x,t),$$

and $v \longmapsto v + \delta v$, then $|\delta v(x,t)| \longmapsto 0$, when

$$\max_{\substack{0\le x\le l\\ 0\le t\le T}} \{|\delta f(t)| + |\delta g(t)| + |\delta\Phi(x,t)|\} \longrightarrow 0.$$

Furthermore, we have

$$|\hat{v}(x,t_j) - v(x,t_j)| \le |\delta f(t_j)| + |\delta g(t_j)| + \frac{\|\delta\Phi(x,t_j)\|}{2|b(t_j)|}$$
$$+ \frac{|k|}{|b(t_j)|}\int_0^1\int_0^1 \{|\hat{v}_0(x,t_j) - v_0(x,t_j)| + |\hat{u}_{j-1} - u_{j-1}|\}dxdx$$
$$+ \frac{|a(t_j)|}{2|b(t_j)|}|\hat{v}_0^2(x,t_j) - v_0^2(x,t_j)|.$$

Proof. We have

$$\hat{v}_0(x,t_j) = f(t_j) + \delta f(t_j) + xg(t_j) + x\delta g(t_j) - \frac{1}{b(t_j)}\int_0^x\int_0^x (\Phi(x,t_j) + \delta\Phi(x,t_j))dxdx,$$

and so

$$|\hat{v}_0(x,t_j) - v_0(x,t_j)| \le |\delta f(t_j)| + |\delta g(t_j)| + \frac{\|\delta\Phi(x,t_j)\|}{2|b(t_j)|}. \tag{20}$$

So, $|\hat{v}_0(x,t) - v_0(x,t)| \longrightarrow 0$, when

$$\max_{\substack{0\le x\le 1\\ 0\le t\le T}} \{|\delta f(t)| + |\delta g(t)| + |\delta\Phi(x,t)|\} \longrightarrow 0.$$

Similarly, putting

$$\hat{v}_1(x,t_j)=\int_0^x\int_0^x\left\{\frac{k}{b(t_j)}(\hat{v}_0(x,t_j)-\hat{u}_{j-1}(x))-\frac{a(t_j)}{b(t_j)}\frac{d}{dx}(\hat{v}_0(x,t_j)\frac{d}{dx}\hat{v}_0(x,t_j))\right\}dxdx;$$

which via lemma 2 and Equation (20), simplifies to the form

$$|\hat{v}_1(x,t_j)-v_1(x,t_j)|\le\frac{|k|}{|b(t_j)|}\int_0^1\int_0^1\{|\hat{v}_0(x,t_j)-v_0(x,t_j)|+|\hat{u}_{j-1}-u_{j-1}|\}\,dxdx + \frac{|a(t_j)|}{2|b(t_j)|}|\hat{v}_0^2(x,t_j)-v_0^2(x,t_j)|. \quad (21)$$

Now, $|\hat{v}_1(x,t)-v_1(x,t)|\longrightarrow 0$, whenever

$$\max_{\substack{0\le x\le l\\0\le t\le T}}\{|\delta f(t)|+|\delta g(t)|+|\delta\Phi(x,t)|\}\longrightarrow 0.$$

By adding the two sides of Equations (20) and (21), we obtain

$$|\hat{v}(x,t_j)-v(x,t_j)|\le|\delta f(t_j)|+|\delta g(t_j)|+\frac{\|\delta\Phi(x,t_j)\|}{2|b(t_j)|} + \frac{|k|}{|b(t_j)|}\int_0^1\int_0^1\{|\hat{v}_0(x,t_j)-v_0(x,t_j)|+|\hat{u}_{j-1}-u_{j-1}|\}dxdx + \frac{|a(t_j)|}{2|b(t_j)|}|\hat{v}_0^2(x,t_j)-v_0^2(x,t_j)|.$$

Finally,$|\hat{v}(x,t)-v(x,t)|\longrightarrow 0$, whenever

$$\max_{\substack{0\le x\le 1\\0\le t\le T}}\{|\delta f(t)|+|\delta g(t)|+|\delta\Phi(x,t)|\}\longrightarrow 0.$$

□

In the next section, a necessary condition for convergence of the approximate solution, when the step size Δt, tends to zero is obtainsd.

5 Convergence conditions for the problem (6)-(9)

In this section, we apply the error term of finite difference method in the relations (15), (16) and (17) and then convergence condition of the solution will be investigated. So, first, we give the following theorem.

Theorem 4. *Let* $|\Delta t\ b(t)|>1$ *for any* $0\le t\le T$. *If* $|\Delta t|\longrightarrow 0$, *then* $u_j(x)\longrightarrow u(x,t)$, *for any* $0\le j\le n,\ \ 0\le x\le 1$.

Proof. Using Taylor's series expansion, we have

$$v_j(x) \simeq v(x,t_j) + \delta\Phi,$$

where $\delta\Phi = \Delta t \frac{\partial u(x,\theta_j)}{\partial t}$, and $t_j < \theta_j < t_{j+1}$.
So, from (18) we obtain

$$v_0(x,t_j) = f(t_j) + xg(t_j) - \frac{1}{b(t_j)} \int_0^x \int_0^x \Phi_j(x) dx dx,$$

or

$$v_{0j} = f(t_j) + xg(t_j) - \frac{1}{b(t_j)} \int_0^x \int_0^x \Phi_j(x) dx dx + \delta\Phi_j.$$

Clearly, if $\delta\Phi_j \longrightarrow 0$, then $v_{0j} \longrightarrow v_0(x,t_j)$.

Again, by (2) we have

$$v_1(x,t_j) = \int_0^x \int_0^x \frac{k}{b(t_j)} (v_0(x,t_j) - u_{j-1}(x)) - \frac{a(t_j)}{b(t_j)} (\frac{d}{dx}\{v_0(x,t_j) \frac{d}{dx} v_0(x,t_j)\}) dx dx.$$

thus

$$\begin{aligned} v_{1j} = &\int_0^x \int_0^x \frac{k}{b(t_j)} \{(v_{0j} - \delta\Phi_j) - (u_{j-1}(x) - \delta\Phi_{j-1})\} \\ &- \frac{a(t_j)}{b(t_j)} (\frac{d}{dx}\{(v_{0j} - \delta\Phi_j) \frac{d}{dx}(v_{0j} - \delta\Phi_j)\}) dx dx \\ &+ \delta\Phi_j, \end{aligned}$$

and

$$\begin{aligned} v_{1j} = v_1(x,t_j) &- \frac{k}{b(t_j)} \int_0^x \int_0^x (\delta\Phi_j - \delta\Phi_{j-1}) dx dx \\ &+ \frac{a(t_j)}{b(t_j)} \{\frac{\delta\Phi_j^2}{2} - \int_0^x v_{0j} \frac{d}{dx} \delta\Phi_j dx - \int_0^x \delta\Phi_j \frac{d}{dx} v_{0j} dx\}. \end{aligned}$$

That means if $\delta\Phi_j \longrightarrow 0$, then $v_{1j} \longrightarrow v_1(x,t_j)$, and $v_j \longrightarrow v(x,t_j)$.

□

6 Numerical results

In this section, we give a numerical example.
Let

$$u_t - \frac{\partial}{\partial x}\{(\frac{1}{6}\ e^{-t} u + (t+5)\ e^{-t}) \frac{\partial u}{\partial x}\} = -\frac{7}{3} t - 9, \quad (x,t) \in [0,1] \times [0,1],$$

$$u(x,0) = x^2, \quad 0 \le x \le 1,$$

$$u(0,t) = t, \qquad 0 \le t \le 1,$$
$$u_x(0,t) = 0, \qquad 0 \le t \le 1. \tag{22}$$

Obviously
$\Phi(x,t) = -\frac{7}{3}t - 9, \qquad a(t) = \frac{1}{6}\ e^{-t}, \qquad b(t) = (t+5)\ e^{-t}.$
The exact solution is $u(x,t) = x^2 e^t + t$. We obtain the approximate solution by applying equations (15), (16) and (17), at $x = 0.1, 0.2, \ldots, 1$, where $t = 0.25, 0.5, 0.75, 1$, and we assume that $\triangle t = 0.25$. Consequently the solution will be constructed in the form

$$\mathbf{v}_{0_j}(x) = h(x,t_j) = t_j - \frac{1}{(t_j+5)e^{-t_j}}\ (-\tfrac{7}{3}t - 9)\frac{x^2}{2},$$
$$\mathbf{v}_{1_j}(x) = \int_0^x \int_0^x \left\{ \frac{4e^{t_j}}{(t_j+5)}(\mathbf{v}_{0_j}(x) - u(x,t_{j-1})) - \frac{\frac{1}{6}}{(t_j+5)}\frac{d}{dx}\left(\ \mathbf{v}_{0_j}(x)\frac{d}{dx}\mathbf{v}_{0_j}(x)\right)\right\}\ dxdx,$$

for $j = 1, 2, 3, 4$.
The exact solution, approximate solution and relative error for the above problem are given in Tables $2-5$ and $6-9$ at $t = t_j = j\triangle t$, $j = 1, 2, 3, 4$. To illustrate stability, according to Table 1, we enter some noise terms into data functions in Eq. (22).

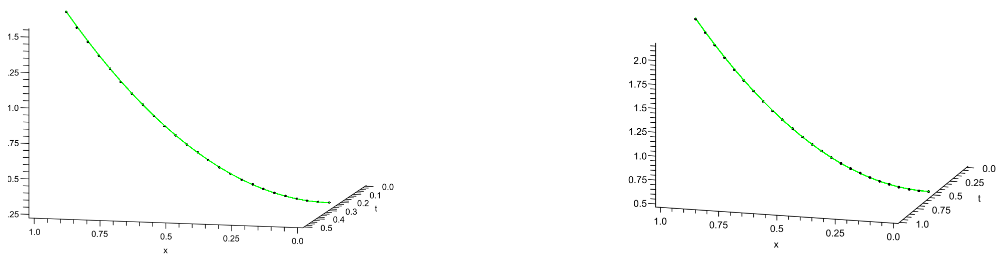

Figure 1: Approximate $(\cdots)$ and exact solution of $u(x,t_j)$ in $t = 0.25$ and $t = 0.5$.

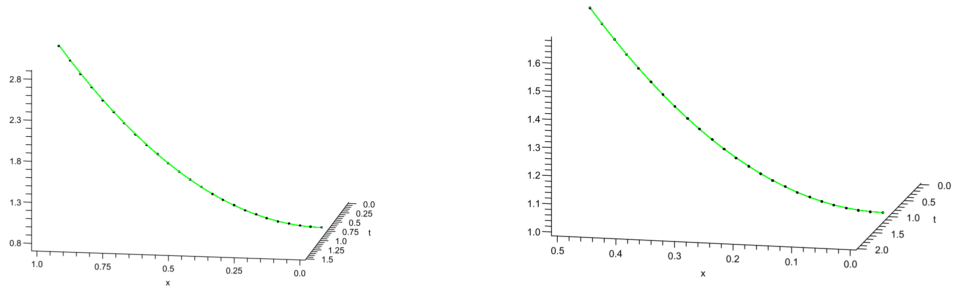

Figure 2: Approximate $(\cdots)$ and exact solution of $u(x,t_j)$ in $t=0.75$ and $t=1$.

Table 1: Perturbation terms in problem (22)

j	a_j	b_j	f_j	g_j	Φ_j
1	-0.006486161986	-0.001002685512	0.001219370604	-0.001002685512	0.001219370604
2	0.01848017099	-0.0008600955762	-0.0002264198307	-0.0008600955762	-0.0002264198307
3	-0.005506853028	0.002919491298	-0.0004964518833	0.002919491298	-0.0004964518833
4	-0.006487155975	-0.001056710212	-0.0004964988898	-0.001056710212	-0.0004964988898

Table 2: Exact and approximate solution of $u_j(x)$ at $t_j=0.25$ with and without perturbation terms in a and b

x	exact solution	approximate solution	relative error	perturbed solution	relative error
0.1	0.2628402542	0.2628483725	3.08×10^{-5}	0.2628562616	6.09×10^{-5}
0.2	0.3013610167	0.3013841491	7.67×10^{-5}	0.3014170267	1.85×10^{-4}
0.3	0.3655622875	0.3655793088	4.65×10^{-5}	0.3656582382	2.62×10^{-4}
0.4	0.4554440667	0.4553871491	1.24×10^{-4}	0.4555397988	2.10×10^{-4}
0.5	0.5710063542	0.5707422861	4.62×10^{-4}	0.5710055740	1.36×10^{-6}
0.6	0.7122491501	0.7115606556	9.66×10^{-4}	0.7119833904	3.37×10^{-4}
0.7	0.8791724543	0.8777395127	1.62×10^{-3}	0.8783850345	8.95×10^{-4}
0.8	1.071776267	1.069157431	2.44×10^{-3}	1.070106259	1.55×10^{-3}
0.9	1.290060588	1.285674303	3.40×10^{-3}	1.287026770	2.35×10^{-3}
1	1.534025417	1.527131339	4.49×10^{-3}	1.529010242	3.26×10^{-3}

Table 3: Exact and approximate solution of $u_j(x)$ at $t_j = 0.5$ with and without perturbation terms in a and b

x	exact solution	approximate solution	relative error	perturbed solution	relative error
0.1	0.5164872127	0.5165050073	3.44×10^{-5}	0.5164663253	4.04×10^{-5}
0.2	0.5659488508	0.5660064728	1.01×10^{-4}	0.5658439475	1.85×10^{-4}
0.3	0.6483849144	0.6484638498	1.21×10^{-4}	0.6480689075	4.87×10^{-4}
0.4	0.7637954034	0.7638099282	1.90×10^{-5}	0.7630349271	9.95×10^{-4}
0.5	0.9121803178	0.9119513987	2.50×10^{-4}	0.9105938906	1.73×10^{-3}
0.6	1.093539658	1.092769640	7.04×10^{-4}	1.090556514	2.72×10^{-3}
0.7	1.307873423	1.306121720	1.33×10^{-3}	1.302693219	3.96×10^{-3}
0.8	1.555181613	1.551841629	1.09×10^{-3}	1.546735182	5.43×10^{-3}
0.9	1.835464230	1.829741743	3.11×10^{-3}	1.822375591	7.13×10^{-3}
1	2.148721271	2.139614496	4.23×10^{-3}	2.129271087	9.05×10^{-3}

Table 4: Exact and approximate solution of $u_j(x)$ at $t_j = 0.75$ with and without perturbation terms in a and b

x	exact solution	approximate solution	relative error	perturbed solution	relative error
0.1	0.7711700002	0.7711983689	3.36×10^{-5}	0.7712065247	4.73×10^{-5}
0.2	0.8346800007	0.8347737522	1.12×10^{-4}	0.8348115589	1.57×10^{-4}
0.3	0.9405300015	0.9406671824	1.45×10^{-4}	0.9407717933	2.57×10^{-4}
0.4	1.088720003	1.088781041	5.60×10^{-5}	1.089016095	2.71×10^{-4}
0.5	1.279250004	1.278980053	2.11×10^{-4}	1.279447063	1.54×10^{-4}
0.6	1.512120006	1.511092655	6.79×10^{-4}	1.511943216	1.16×10^{-4}
0.7	1.787330008	1.784912741	1.35×10^{-3}	1.786361766	5.41×10^{-4}
0.8	2.104880011	2.100201772	2.22×10^{-3}	2.102541999	1.11×10^{-3}
0.9	2.464770014	2.456691211	3.27×10^{-3}	2.460309231	1.80×10^{-3}
1	2.867000017	2.854085288	4.50×10^{-3}	2.859479316	2.62×10^{-3}

Table 5: Exact and approximate solution of $u_j(x)$ at $t_j = 1$ with and without perturbation terms in a and b

x	exact solution	approximate solution	relative error	perturbed solution	relative error
0.1	1.027182818	1.027222369	3.85×10^{-5}	1.027311557	1.25×10^{-4}
0.2	1.108731273	1.108860738	1.16×10^{-4}	1.109229081	4.48×10^{-4}
0.3	1.244645364	1.244829262	1.47×10^{-4}	1.245701389	8.48×10^{-4}
0.4	1.434925092	1.434986046	2.42×10^{-5}	1.436644053	1.19×10^{-3}
0.5	1.679570457	1.679134916	2.59×10^{-4}	1.681940678	1.41×10^{-3}
0.6	1.978581458	1.977027872	7.85×10^{-4}	1.981444677	1.44×10^{-3}
0.7	2.331958096	2.328368191	1.53×10^{-3}	2.334981455	1.29×10^{-3}
0.8	2.739700370	2.732814161	2.51×10^{-3}	2.742351004	9.67×10^{-4}
0.9	3.201808281	3.189983398	3.69×10^{-3}	3.203330804	4.75×10^{-4}
1	3.718281828	3.699457703	5.06×10^{-3}	3.717678990	1.62×10^{-4}

Table 6: Exact and approximate solution of $u_j(x)$ at $t_j = 0.25$ with and without perturbation terms in f, g and Φ

x	exact solution	approximate solution	relative error	perturbed solution	relative error
0.1	0.2628402542	0.2628483725	3.08×10^{-5}	0.2630645568	8.53×10^{-4}
0.2	0.3013610167	0.3013841491	7.67×10^{-5}	0.3015988237	7.89×10^{-4}
0.3	0.3655622875	0.3655793088	4.65×10^{-5}	0.3657914404	6.26×10^{-4}
0.4	0.4554440667	0.4553871491	1.24×10^{-4}	0.4555956649	3.32×10^{-4}
0.5	0.5710063542	0.5707422861	4.62×10^{-4}	0.5709460580	1.05×10^{-4}
0.6	0.7122491501	0.7115606556	9.66×10^{-4}	0.7117584843	6.88×10^{-4}
0.7	0.8791724543	0.8777395127	1.62×10^{-3}	0.8779301112	1.41×10^{-3}
0.8	1.071776267	1.069157431	2.44×10^{-3}	1.069339409	2.27×10^{-3}
0.9	1.290060588	1.285674303	3.40×10^{-3}	1.285846152	3.26×10^{-3}
1	1.534025417	1.527131339	4.49×10^{-3}	1.527291415	4.38×10^{-3}

Table 7: Exact and approximate solution of $u_j(x)$ at $t_j = 0.5$ with and without perturbation terms in f, g and Φ

x	exact solution	approximate solution	relative error	perturbed solution	relative error
0.1	0.5164872127	0.5165050073	3.44×10^{-5}	0.5154115152	2.08×10^{-3}
0.2	0.5659488508	0.5660064728	1.01×10^{-4}	0.5648920604	1.86×10^{-3}
0.3	0.6483849144	0.6484638498	1.21×10^{-4}	0.6473145948	1.65×10^{-3}
0.4	0.7637954034	0.7638099282	1.90×10^{-5}	0.7626119527	1.54×10^{-3}
0.5	0.9121803178	0.9119513987	2.50×10^{-4}	0.9106908844	1.63×10^{-3}
0.6	1.093539658	1.092769640	7.04×10^{-4}	1.091432842	1.92×10^{-3}
0.7	1.307873423	1.306121720	1.33×10^{-3}	1.304694990	1.30×10^{-3}
0.8	1.555181613	1.551841629	1.09×10^{-3}	1.550311436	3.13×10^{-3}
0.9	1.835464230	1.829741743	3.11×10^{-3}	1.828094690	4.01×10^{-3}
1	2.148721271	2.139614496	4.23×10^{-3}	2.137837344	5.06×10^{-3}

Table 8: Exact and approximate solution of $u_j(x)$ at $t_j = 0.75$ with and without perturbation terms in f, g and Φ

x	exact solution	approximate solution	relative error	perturbed solution	relative error
0.1	0.7711700002	0.7711983689	3.36×10^{-5}	0.7736467637	3.21×10^{-3}
0.2	0.8346800007	0.8347737522	1.12×10^{-4}	0.8372983250	3.13×10^{-3}
0.3	0.9405300015	0.9406671824	1.45×10^{-4}	0.9433190828	2.96×10^{-3}
0.4	1.088720003	1.088781041	5.60×10^{-5}	1.091611970	2.65×10^{-3}
0.5	1.279250004	1.278980053	2.11×10^{-4}	1.282042479	2.18×10^{-3}
0.6	1.512120006	1.511092655	6.79×10^{-4}	1.514440029	1.53×10^{-3}
0.7	1.787330008	1.784912741	1.35×10^{-3}	1.788599719	7.10×10^{-4}
0.8	2.104880011	2.100201772	2.22×10^{-3}	2.104284426	2.82×10^{-4}
0.9	2.464770014	2.456691211	3.27×10^{-3}	2.461227246	1.43×10^{-3}
1	2.867000017	2.854085288	4.50×10^{-3}	2.859134249	2.74×10^{-3}

Table 9: Exact and approximate solution of $u_j(x)$ at $t_j = 1$ with and without perturbation terms in f, g and Φ

x	exact solution	approximate solution	relative error	perturbed solution	relative error
0.1	1.027182818	1.027222369	3.85×10^{-5}	1.025635295	1.50×10^{-3}
0.2	1.108731273	1.108860738	1.16×10^{-4}	1.107171622	1.40×10^{-3}
0.3	1.244645364	1.244829262	1.47×10^{-4}	1.242968576	1.34×10^{-3}
0.4	1.434925092	1.434986046	2.42×10^{-5}	1.432882007	1.42×10^{-3}
0.5	1.679570457	1.679134916	2.59×10^{-4}	1.676712572	1.70×10^{-3}
0.6	1.978581458	1.977027872	7.85×10^{-4}	1.974208173	2.21×10^{-3}
0.7	2.331958096	2.328368191	1.53×10^{-3}	2.325067038	2.95×10^{-3}
0.8	2.739700370	2.732814161	2.51×10^{-3}	2.728941436	3.92×10^{-3}
0.9	3.201808281	3.189983398	3.69×10^{-3}	3.185441974	5.11×10^{-3}
1	3.718281828	3.699457703	5.06×10^{-3}	3.694142422	6.49×10^{-3}

7 Conclusions

The HPM for the one-dimensional inverse problems has been presented. The method described is mathematically simple and computationally effective. As we see in Tables $2-9$, small errors in the data make small errors in the solution, so that, the solution depends continuously on the data. In this paper, the noise terms that are shown in Table 1, are made randomized and have standard normal distributions. *Maple 16 packages* have been used to compute the solution before and after adding noise terms. Rapidity , accuracy and stability are advantages of this formulation.

References

1. Adomian, G. *A new approach to the heat equation-An application of the decomposition method*, J. Math. Anal. Appl. 113 (1986) 202-209.

2. Alifonov, O.M. *Inverse heat transfer problems*, Springer-Verlag, 1994.

3. Beck, J.V., Blackwell, B. and St. Clair. Ch. J. *Inverse heat conduction, ill-posed problems*, John Willey and Sons, Inc, 1985.

4. Buhmann Martin, D. *Radial Basis Functions: Theory and Implementations*, Cambridge University Press, 2003.

5. Cannon, J.R. *The one dimensional heat equation*, Addison-Welsey , New york, 1984.

6. Dehghan, M. and Shakeri, F. *Solution of a partial differential equation subject to temperature overspecification by He's homotopy perturbation method*, Phys. Scr., 75 (2007) 778-787.

7. Douglas, J.J. and Russell, T.F. *Numerical Methods for Convection-Dominated Diffusion Problems Based on Combining the Method of Characteristics with Finite Element or Finite Difference Procedures*, SIAM J. Numer. Anal., 19 (5) (2006) 871-885

8. Ghasemi, M., Tavassoli Khanjani, M. and Davari, A. *Numerical solution of two -dimensional nonlinear differential equation by homotopy perturbation method*, Appl. Math. Comput. 189 (2007) 341-345.

9. He, J. H. *Some asymptotic methods for strongly nonlinear equations*, International Journal of Modern Physics B, 20 (2006) 1141-1199.

10. He, J. H. *An elementary introduction to recently developed asymptotic methods and nanomechanics in textile engineering*, International Journal of Modern Physics B, 22 (2008) 3487-3578.

11. He, J. H. *Homotopy perturbation technique*, Comput. Methods. Appl. Mech. Engrg., 178 (1999) 257-262.

12. He, J. H. *A coupling method of a homotopy technique and a perturbation technique for non-linear problems*, Int. J. Non-Linear Mechanics, 35 (2000) 37-43.

13. He, J. H. *The homotopy perturbation method for nonlinear oscilators with discontinuities*, Appl. Math. Comput., 5 (2006) 287-292.

14. He, J. H. *Homotopy perturbation method for solving boundary value problems*, Phys. lett. A., 35 (2006) 87-88.

15. Johansson, B.T. and Lesnic. D. *A method of fundamental solutions for transient heat conduction*, Engineering Analysis with Boundary Elements, 32 (2008) 697-703.

16. Jia, H., Xu, W., Zhao, X. and Li, Z. *Separation of variables and exact solutions to nonlinear diffusion equations with $x-$dependent convection and absorption*, J. Math. Anal. Appl., 339 (2008) 982-995.

17. Johansson, B.T. and Lesnic, D. *Determination of spacewise depedent heat suorce*, Journal of computational and Applied Mathematics, 209 (2007) 66-88.

18. Lewandowski, J.L.V. *Marker method for the solution of nonlinear diffusion equations*, JCAM., 196 (2006) 523-539.

19. Onyango, T.T.M., Ingham, D.B., Lesnic, D. and Soldicka, M. *Determination of a time-dependent heat transfer cofficient from non-standard boundary measurements*, Mathematics and Computers in Simulation, 79 (2009) 1577-1548

20. Qu, C., Ji, L. and Dou, J. *Exact solution and generalized conditional symmetries to* $(n+1)-$*dimensional nonlinear diffusion equations with source term*, Physics Letters A, 343 (2005) 139-147.

21. Shidfar, A. and Zakeri, A. *A numerical method for backward inverse heat conduction problem with two unknown functions*, International Journal of Engineering science, 16 (2008) 71-74.

22. Shidfar, A. and Zakeri, A. *Asymptotic solution for an inverse parabolic problem*, Mathematica Balkanica, 18 (2004) 475-483.

23. Shidfar, A. and Zakeri, A. *A two-dimensional inverse heat conduction problem for estimating heat source*, International Journal of Applied Mathematics and Mathematical Science, 10 (2005) 1633-1641.

24. Siddiqui, A.M., Mahmood, R. and Ghori, Q.K. *Homotopy perturbation method for thin film of a fourth grade fluid down a vertical cylinder*, Phys. lett. A., 352 (2006) 404-410.

25. Vogel, C.R. *Computational methods for inverse problems*, SIAM, 2002.

26. Zakeri, A., Aminataei, A. and Jannati, Q. *Application of He's homotopy perturbation method for Cauchy problem of ill-posed nonlinear diffusion equation*, Discrete Dynamics in Nature and Society, Vol. 2010, ID 780207.

27. Zakeri, A. and Jannati, Q. *An inverse problem for parabolic partial differential equations with nonlinear conductivity term in one dimensional space*, Scholarly Research Exchange, 2009, ID 468570.

28. Zakeri, A. and Jannati, Q. *Stability of homotopy perturbtion technique for an inverse diffusion problem*, The proceeding of fourty fourth Annual Iranian Mathematics Conference, pp. 682-685, Rafsanjan, Aug . 2014.

بررسی شرایط همگرایی و پایداری یک جواب تقریبی مسأله معکوس هدایت گرما توسط روش اختلال هموتوپی

قدسیه جنتی و علی ذاکری

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چکیده : در این مقاله کاربرد روش اختلال هموتوپی برای حل یک مسأله معکوس هدایت گرمایی غیرخطی در فضای یک بعدی مورد بررسی قرار می‌گیرد. در این مسأله رسانش به صورت تابع خطی از دمای ناشناخته در یک دامنه کران‌دار است. علاوه بر این مقدار دما در یک کران دامنه تعریف مجهول است. مسأله موردنظر یک مسأله بدخیم است. با به کار بستن روش تفاضلات متناهی و گسسته‌سازی متغیر زمانی، مسأله به یک دستگاه معادلات دیفرانسیل غیرخطی تبدیل می‌شود که جواب تقریبی آن با استفاده از روش اختلال هموتوپی تعیین می‌گردد. در ادامه شرایط همگرایی و پایداری روش پیشنهادی مورد بررسی قرار می‌گیرد. در آخر یک کران بالای خطا ارائه می‌گردد.

کلمات کلیدی : روش اختلال هموتوپی ؛ معادلات نفوذ؛ روش‌های گسسته‌سازی؛ مسائل معکوس.