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A nonstandard finite difference scheme for solving three-species food chain with fractional-order Lotka-Volterra model

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Abstract

In this paper, we introduce fractional-order for a model of tritrophic food chain Lotka-Volterra. Moreover, we discuss the stability analysis of fractional system. The nonstandard finite difference (NSFD) scheme is implemented to study the dynamic behaviors in the fractional-order Lotka-Volterra system. Numerical results show that the NSFD approach is easy to implement and accurate when applied to fractional-order Lotka-Volterra system.

Keywords: Fractional differential equations; Lotka-Volterra model; prey-predator system; Nonstandard finite difference scheme; Stability.

1 Introduction

Biological systems have been studied for many years. In these systems, it is common that state variables represent nonnegative quantities, such as concentrations, physical properties, the size of populations and the amount of chemical compounds [15]. These biological models are commonly based on the systems of ordinary differential equations (ODEs). Exact solutions of these systems are rarely in access and usually complicated; hence good approximations are required. Numerical methods are often the method of choice. They should describe the dynamic behavior of the systems, produce the nonnegative solutions, and reproduce the real dynamics of the biological systems. The interspecies interaction is among the most intensively explored fields of biology. The existance of many mathematical models in that area

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help us understand the population dynamics of analyzed biological systems. Mathematical models of predator-prey systems, characterized by decreasing growth rate of one of the interacting populations and increasing growth rate of the other, consist of systems of ODEs. In most of the modeled interactions, all rates of change are assumed to be time independent, which makes the corresponding systems autonomous. It is not always possible to find the exact solutions of the nonlinear models that have at least two ODEs. It is sometimes more useful to find numerical solutions of these types of systems in order to programme easily and visualize the results. By applying a numerical method on a continuous differential equation system, it becomes a difference equation system, i.e., discrete time system. While applying these numerical methods, it is necessary that the new difference equation system provide the positivity conditions and exhibit the same quantitative behaviours of continuous systems such as stability, bifurcation and chaos. It is well known that some traditional and explicit schemes such as forward Euler and Runge-Kutta are unsuccessful at generating oscillation, bifurcations, chaos and false steady states, despite using adaptative step size [13,17,18]. For forward Euler method, if the step size h is chosen small enough and the positivity conditions are satisfied, the local asymptotic stability for a fixed point is saved while in some special cases Hopf bifurcation cannot be seen. Instead of classical methods, NSFD schemes can alternatively be used to obtain more qualitative results and remove numerical instabilities. These schemes are developed for compensating the weaknesses, such as numerical instabilities that may be caused by standard finite difference methods. Also, the dynamic consistency can be represented by NSFD schemes [10]. The most important advantage of this scheme is that by choosing a convenient denominator function instead of the step size h, better results can be obtained. If the step size h is chosen small enough, the obtained results do not change significantly, but if the step size h gets larger this advantage comes into focus.

As it is well known, in the field of mathematical biology, the traditional Lotka-Volterra systems are very important mathematical models which describe multispecies population dynamics in a nonautonomous environment. Many important and interesting results of the dynamic behaviors for the Lotka-Volterra systems have been found in [3, 19, 20], such as the existence and uniqueness of solutions, the permanence, extinction, global asymptotic behavior and bifurcation. Because of the good memory and hereditary properties of fractional derivatives, it is often necessary to study the corresponding fractional systems. Therefore, the dynamical analysis of the fractional Lotka-Volterra systems has attracted a great deal of attention due to its theoretical and practical significance.

Many important results regarding stability of fractional systems have been obtained. For instance, the stability, existence, uniqueness and numerical solution of the fractional logistic equation are investigated in [7]. The stability and solutions of fractional predator-prey and rabies models are discussed in [1]. In addition, bifurcation properties of fractional systems have been studied in some papers. For example, conditions for the occurrence of Hopf 's bifurcation are explored based on numerical simulations in [29]. The critical values of the fractional order are identified for which Hopf 's bifurcation may occur based on the stability analysis in [29]. Thus, it is significant to study the dynamical behaviors in the fractional population systems.

Analysis of fractional Lotka-Volterra equations which are obtained from the classical Lotka-Volterra equations in mathematical modeling by the replacing first order derivatives by fractional derivative of order α $(0 < \alpha \leq 1)$ have been the focus of recent research in this field. Lots of universal phenomena can be modeled to a greater degree of accuracy by using the property of these evolution equations. The fractional differential equations have gained much attention recently due to the fact that fractional order system response ultimately converges to the integer order system response.

The current technological advance has made it possible for humans to disturb the environmental balance in nature that may cause immense damages, such as species extinction or starvation. Therefore, understanding the behaviour of the interaction between the species may help biologists and other related parties to prevent those events from happening. The real interaction of prey-predator in nature is complex and comprises both interspecies and external environmental factors. Therefore, several simplifications are usually assumed so that a basic model can be constructed and then developed or modified to approach the real system.

The Lotka-Volterra equations are a system of ODEs in the following form:

$$x' = ax - bxy,$$

$$y' = -cy + dxy,$$

$$x(0) = x_0, \qquad y(0) = y_0,$$

where x and y are prey and predactor, respectively. Here a is the prey growth rate in the absence of the predators, b is the capture rate of prey per predator, d is the rate at which each predator converts captured prey into predator births and c is the constant rate at which death occurs in the absence of prey. They show that ditrophic food chains (i.e. prey-predator systems) permanently oscillate for any initial conditions if the prey growth rate is constant and the predator functional response is linear.

The classical food chain models with only two trophic levels are shown to be insufficient to produce realistic dynamics [5]. Therefore, in this paper, by modifying the classical Lotka-Volterra model, we analyse and simulate the dynamics of a three-species food chain interaction. With non-dimensionalisation, the system of three-species food chain can be written as

$$
\begin{aligned}
&x' = ax - bxy,\\
&y' = dxy - cy - eyz,\\
&z' = gzy - fz,\\
&x(0) = x_0, \qquad y(0) = y_0, \qquad z(0) = z_0,
\end{aligned} \tag{1}
$$

where x, y and z denote the non-dimensional population density of the prey, predator and top predator, respectively. The predator y preys on x and the predator z preys on y. Furthermore a, b, c, d, e, f and g are the intrinsic growth rate of the prey, the death rate of the predator, the death rate of the top predator, predation rate of the predator, the conversion rate, predation rate of the top predator and the conversion rate, respectively.

This paper is organized as follows: In the next section, we give some basic definitions and properties of the Grünwald-Letnikov (GL) approximation and provide a brief overview of the important feature of the procedures for constructing NSFD schemes for ODEs. In Section 3, we introduce fractional order into the model that describes Lotka-Volterra system and also stability theorem and fractional Routh-Hurwitz stability conditions are given for the local asymptotic stability of the fractional systems. In Section 4, we will discuss the stability analysis of fractional system. In Section 5, we present the idea of NSFD scheme for solving the fractional order Lotka-Volterra model. Finally in the last section, numerical results show that the NSFD approach is easy to be implemented and accurated when applied to fractional-order Lotka-Volterra system.

2 Preliminaries and notations

In this section, some basic definitions and properties of the fractional calculus theory and nonstandard discretization are discussed.

2.1 Fundamentals of fractional-order

Fractional differential equations (FDEs) have gained considerable importance due to their application in various sciences, such as physics, mechanics, chemistry and engineering [16]. In the recent years, the dynamic behaviors of fractional-order differential systems have received increasing attention. Although the concept of the fractional calculus was discussed in the same time interval of integer-order calculus, the complexity and the lack of applications postponed its progress till a few decades ago. Recently, most of the dynamical systems based on the integer-order calculus have been modified into the fractional order domain due to the extra degrees of freedom and the flexibility which can be used to precisely fit the experimental data much better than the integer-order modeling. For example, new fundamentals have been investigated in the fractional-order domain for the first time and do not exist in the integer-order systems such as those presented in [9, 16].

2.2 GL approximation

The GL method of approximation for the one-dimensional fractional derivative is as follows [16]:

$$D^{\alpha}x(t) = f(t, x(t)), \qquad x(0) = x_0, \qquad t \in [0, t_f], \tag{2}$$

$$D^{\alpha}x(t) = \lim_{h\to 0} h^{-\alpha} \sum_{j=0}^{[\frac{t_f}{h}]} (-1)^j \binom{\alpha}{j} x(t - jh),$$

where $0 < \alpha < 1$, D^{α} denotes the fractional derivative and h is the step size and $[\frac{t_f}{h}]$ denotes the integer part of $\frac{t_f}{h}$. Therefore, Eq. (2) is discretized as follows:

$$\sum_{j=0}^{n} c_j^{\alpha} x_{n-j} = f(t_n, x_n), \qquad n = 1, 2, 3, ...$$

where $t_n = nh$ and c_j^{α} are the GL coefficients defined as:

$$c_j^{\alpha} = (1 - \frac{1+\alpha}{j})c_{j-1}^{\alpha}, \qquad c_0^{\alpha} = h^{-\alpha}, \qquad j = 1, 2, 3, ...$$

2.3 NSFD discretization

The initial foundation of NSFD schemes came from the exact finite difference schemes. These schemes are well developed by Mickens [13, 14] in the past decades. These schemes are developed for compensating the weaknesses such as numerical instabilities that may be caused by standard finite difference methods. Regarding the positivity, boundedness and monotonicity of solutions, NSFD schemes have a better performance over the standard finite difference schemes, due to flexibility to construct a NSFD scheme that can preserve certain properties and structures, which are obeyed by the original equations.

The advantages of NSFD schemes have been shown in many numerical applications. Gonzalez-Parra et al. [4] developed NSFD schemes to solve population and biological models. Jordan [8] constructed NSFD schemes for heat transfer problems.

We now give an outline of the critical points which will allow the construction of NSFD discretizations for ODEs.

Consider the autonomous ODE given by

$$x' = f(x), \qquad x(0) = x_0, \qquad t \in [0, t_f],$$

where $f(x)$ is, in general, a nonlinear function of x. For a discrete-time grid with step size, $\triangle t = h$, we replace the independent variable t by

$$t \approx t_n = nh, \qquad n = 0, 1, 2, \dots, N$$

where $h = \frac{t_f}{N}$. The dependent variable $x(t)$ is replaced by

$$x(t) \approx x_n,$$

where x_n is the approximation of $x(t_n)$.

The first NSFD requirement is that the dependent functions should be modeled on the discrete-time computational grid. Particular examples of this include the following functions [13, 14].

$$\begin{cases} xy \approx 2x_{n+1}y_n - x_{n+1}y_{n+1}, \\ x^2 \approx x_{n+1}x_n, \\ x^3 \approx (\dfrac{x_{n+1} + x_{n-1}}{2})x_n^2. \end{cases}$$

A standard way for representing a discrete first-derivative is given by

$$x' \cong \frac{x_{n+1} - x_n}{h}.$$

However, the NSFD scheme requires that x' has a more general representation

$$x' \cong \frac{x_{n+1} - x_n}{\phi},$$

where the denominator function, i.e. ϕ has the following properties:

(i) $\phi(h) = h + O(h^2)$,

(ii) $\phi(h)$ is an increasing function of h,

(iii) $\phi(h)$ may depend on the parameters appearing in the differential equations.

The paper by Mickens [14] gives a general procedure for determining $\phi(h)$ for systems of ODEs. An example of the NSFD discretization process is its application to the decay equation

$$x' = -\lambda x,$$

where λ is a constant. The discretization scheme is as follows [14]

$$\frac{x_{n+1} - x_n}{\phi} = -\lambda x_n, \qquad \phi(h, \lambda) = \frac{1 - e^{-\lambda h}}{\lambda}.$$

Another example is given by

$$x' = \lambda_1 x - \lambda_2 x^2,$$

where the NSFD scheme is

$$\frac{x_{n+1} - x_n}{\phi} = \lambda_1 x_n - \lambda_2 x_{n+1} x_n, \qquad \phi(h, \lambda_1) = \frac{e^{\lambda_1 h} - 1}{\lambda_1}.$$

It should be noted that the NSFD schemes for these two ODEs are exact in the sense that $x_n = x(t_n)$ for all applicable values of $h > 0$. In general, for an ODE with polynomial terms,

$$x' = ax + (NL) \qquad NL \equiv \text{Nonlinear terms},$$

the NSFD discretization for the linear expressions is given by Mickens [14]

$$\frac{x_{n+1} - x_n}{\phi} = ax_n + (NL)_n,$$

where the denominator function is

$$\phi(h, a) = \frac{e^{ah} - 1}{a}.$$

It follows that if x' is a function of x which does not have a linear term, then the denominator function is just h, i.e. $\phi(h) = h$.

By applying this technique and using the GL discretization method, the following relations are yielded:

$$x_{n+1} = \frac{-\sum_{j=1}^{n+1} c_j^\alpha x_{n+1-j} + f(t_{n+1}, x_{n+1})}{c_0^\alpha}, \qquad n = 0, 1, 2, ...$$

where $c_0^\alpha = \phi(h)^{-\alpha}$.

3 Fractional-order Lotka-Volterra model

Now we introduce fractional-order into the model (1) of Lotka-Volterra chaotic system. The new system is described by the following set of fractional ODEs of order $\alpha_1, \alpha_2, \alpha_3 > 0$, in the following form

$$\begin{aligned} &D^{\alpha_1} x(t) = ax - bxy, \\ &D^{\alpha_2} y(t) = dxy - cy - eyz, \\ &D^{\alpha_3} z(t) = gzy - fz, \\ &x(0) = x_0, \qquad y(0) = y_0, \qquad z(0) = z_0, \\ &0 < \alpha_i \leq 1, \qquad i = 1, 2, 3. \end{aligned} \tag{3}$$

Now, stability theorem on fractional-order systems, fractional Routh-Hurwitz stability conditions and their related results are introduced. The first stability theorem has been given for incommensurate fractional-order systems.

Theorem 1. *([12]) Consider the incommensurate fractional-order system*

$$D^{\alpha} x(t) = f(x(t)), \qquad x(0) = x_0, \tag{4}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_i \in (0, 1]$ for $i = 1, 2, \ldots, n$ and $x \in \mathbb{R}^n$. The equilibrium points of (4), are calculated by solving the equations:

$$f(x) = 0.$$

These points are locally asymptotically stable if all eigenvalues λ of the Jacobian matrix $J \equiv \frac{\partial f}{\partial x}$ evaluated at the equilibrium points satisfy:

$$|arg(\lambda)| > \frac{\alpha^{\star}\pi}{2}, \qquad \alpha^{\star} = \max(\alpha_1, \ldots, \alpha_n).$$

Theorem 2. *([11]) Consider the commensurate fractional-order system (4), i.e., $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha^{\star}$. If all eigenvalues of the Jacobian matrix of an equilibrium point satisfy:*

$$|arg(\lambda)| > \frac{\alpha^{\star}\pi}{2},$$

then, the fractional system is locally asymptotically stable at the equilibrium point.

Consider the system of ODEs given by

$$\begin{aligned} X' &= F(X, Y, Z), \\ Y' &= G(X, Y, Z), \\ Z' &= H(X, Y, Z), \end{aligned} \tag{5}$$

where F, G and H are nonlinear functions. Let $\bar{X}$, $\bar{Y}$ and $\bar{Z}$ be the steady-state solution, i.e.,

$$F(\bar{X},\bar{Y},\bar{Z}) = G(\bar{X},\bar{Y},\bar{Z}) = H(\bar{X},\bar{Y},\bar{Z}) = 0.$$

Now consider small perturbations to the steady-state solutions

$$\begin{aligned} X(t) &= \bar{X} + x(t), \\ Y(t) &= \bar{Y} + y(t), \\ Z(t) &= \bar{Z} + z(t). \end{aligned}$$

Frequently these are called perturbations of the steady-state. Substituting, we arrive at

$$\begin{aligned} (\bar{X}+x)' &= F(\bar{X}+x,\bar{Y}+y,\bar{Z}+z), \\ (\bar{Y}+y)' &= G(\bar{X}+x,\bar{Y}+y,\bar{Z}+z), \\ (\bar{Z}+z)' &= H(\bar{X}+x,\bar{Y}+y,\bar{Z}+z). \end{aligned}$$

On the left-hand side we expand the derivatives and that by definition

$$\bar{X}' = \bar{Y}' = \bar{Z}' = 0.$$

On the right-hand side we now expand F, G and H in a Taylor series about the point $(\bar{X},\bar{Y},\bar{Z})$. The result is

$$\begin{aligned} x' &= F(\bar{X},\bar{Y},\bar{Z}) + F_x(\bar{X},\bar{Y},\bar{Z})x + F_y(\bar{X},\bar{Y},\bar{Z})y \\ &+ F_z(\bar{X},\bar{Y},\bar{Z})z + \text{terms of order}\, x^2, y^2, z^2, xy, \\ &yz, xz, \text{and higher}, \end{aligned}$$

$$\begin{aligned} y' &= G(\bar{X},\bar{Y},\bar{Z}) + G_x(\bar{X},\bar{Y},\bar{Z})x + G_y(\bar{X},\bar{Y},\bar{Z})y \\ &+ G_z(\bar{X},\bar{Y},\bar{Z})z + \text{terms of order}\, x^2, y^2, z^2, xy, \\ &yz, xz, \text{and higher}, \end{aligned}$$

$$\begin{aligned} z' &= H(\bar{X},\bar{Y},\bar{Z}) + H_x(\bar{X},\bar{Y},\bar{Z})x + H_y(\bar{X},\bar{Y},\bar{Z})y \\ &+ H_z(\bar{X},\bar{Y},\bar{Z})z + \text{terms of order}\, x^2, y^2, z^2, xy, \\ &yz, xz, \text{and higher}. \end{aligned}$$

Again by definition,

$$F(\bar{X},\bar{Y},\bar{Z}) = G(\bar{X},\bar{Y},\bar{Z}) = H(\bar{X},\bar{Y},\bar{Z}) = 0,$$

so we are left with

$$\begin{aligned} x' &= a_{11}x + a_{12}y + a_{13}z, \\ y' &= a_{21}x + a_{22}y + a_{23}z, \\ z' &= a_{31}x + a_{32}y + a_{33}z, \end{aligned}$$

where the matrix of coefficients

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$
$$= \begin{pmatrix} F_x(\bar{X},\bar{Y},\bar{Z}) & F_y(\bar{X},\bar{Y},\bar{Z}) & F_z(\bar{X},\bar{Y},\bar{Z}) \\ G_x(\bar{X},\bar{Y},\bar{Z}) & G_y(\bar{X},\bar{Y},\bar{Z}) & G_z(\bar{X},\bar{Y},\bar{Z}) \\ H_x(\bar{X},\bar{Y},\bar{Z}) & H_y(\bar{X},\bar{Y},\bar{Z}) & H_z(\bar{X},\bar{Y},\bar{Z}) \end{pmatrix},$$

is the Jacobian of the system (5). Hence, the problem has been reduced to a linear system, i.e., $w' = Aw$ with $w = (x, y, z)^T$, for states that are in proximity to the steady-state $(\bar{X}, \bar{Y}, \bar{Z})$.

The Jacobian matrix J of the system (3) at the equilibrium point $E = (x^*, y^*, z^*)$ is computed as

$$J(E) = \begin{pmatrix} a - by^* & -bx^* & 0 \\ dy^* & -c + dx^* - ez^* & -ey^* \\ 0 & gz^* & -f + gy^* \end{pmatrix}. \quad (6)$$

The existence and local stability conditions of these equilibrium points are as follows:

Let $D(P)$ denotes the discriminant of a polynomial P

$$P(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0, \quad (7)$$

and

$$D(P) = 18a_1a_2a_3 + (a_1a_2)^2 - 4a_3(a_1)^3 - 4(a_2)^3 - 27(a_3)^2,$$

using the results of [2], we have the following Routh-Hurwitz stability conditions for FDEs:

(i) If $D(P) > 0$, then the necessary and sufficient condition for the equilibrium point E to be locally asymptotically stable is $a_1 > 0, a_3 > 0, a_1a_2 - a_3 > 0$.

(ii) If $D(P) < 0$, $a_1 \geq 0, a_2 \geq 0, a_3 > 0$, then the equilibrium point E is locally asymptotically stable for $\alpha < 2/3$. However, if $D(P) < 0, a_1 < 0, a_2 < 0, \alpha > 2/3$, then all roots of polynomial (7) satisfy the condition $|arg(\lambda)| < \frac{\alpha\pi}{2}$.

(iii) If $D(P) < 0$, $a_1 > 0, a_2 > 0, a_1a_2 - a_3 = 0$, then the equilibrium point E is locally asymptotically stable for all $\alpha \in [0, 1)$.

(iv) The necessary condition for the equilibrium point E to be locally asymptotically stable is $a_3 > 0$.

In the next section, we discuss the asymptotic stability of the equilibrium point E of the system (3).

4 Stability analysis of the model

To evaluate the equilibrium points of the system (3), let

$$ax - bxy = 0,$$
$$dxy - cy - eyz = 0,$$
$$gzy - fz = 0,$$

then the equilibrium points are $E_0 = (0, 0, 0)$, $E_1 = (0, \frac{f}{g}, -\frac{c}{e})$ and $E_2 = (\frac{c}{d}, \frac{a}{b}, 0)$. All calculations were performed by MAPLE. The local stability conditions of these equilibrium points are as follows:

(i) The Jacobian matrix (6) at the equilibrium point $E_0 = (0, 0, 0)$ is

$$J(0,0,0) = \begin{pmatrix} a & 0 & 0 \\ 0 & -c & 0 \\ 0 & 0 & -f \end{pmatrix}, \tag{8}$$

with the characteristic equation

$$P(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0,$$

where

$$a_1 = f + c - a, \qquad a_2 = cf - af - ac, \qquad a_3 = -fac,$$

and $D(P)$ in the above equation is

$$D(P) = (c - f)^2(a + f)^2(a + c)^2.$$

Therefore, the eigenvalues of the Jacobian matrix (8) corresponding to the equilibrium point E_0 are $\lambda_1 = a$, $\lambda_2 = -c$ and $\lambda_3 = -f$.

Clearly, if $c \neq f$ then $D(P) > 0$. Now, since $a_3 < 0$; therefore, based on part (i) in Routh-Hurwitz stability conditions, the equilibrium point E_0 is unstable.

(ii) The Jacobian (6) at the equilibrium point $E_1 = (0, \frac{f}{g}, -\frac{c}{e})$ is

$$J(0, \frac{f}{g}, -\frac{c}{e}) = \begin{pmatrix} \dfrac{ag - bf}{g} & 0 & 0 \\ \dfrac{fd}{g} & 0 & -\dfrac{ef}{g} \\ 0 & -\dfrac{gc}{e} & 0 \end{pmatrix},$$

where the characteristic equation is

$$P(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0,$$

with

$$a_1 = \frac{bf - ag}{g}, \qquad a_2 = -cf, \qquad a_3 = \frac{cf(ag - bf)}{g},$$

and again

$$D(P) = \frac{4cf(g^2(a^2 - cf) + bf(bf - 2ag))^2}{g^4}.$$

Here, the corresponding eigenvalues are

$$\lambda_1 = \frac{ag - bf}{g}, \qquad \lambda_2 = \sqrt{cf}, \qquad \lambda_3 = -\sqrt{cf}.$$

Obviously, if $g^2(a^2 - cf) + bf(bf - 2ag) \neq 0$ then $D(P) > 0$. Then as $a_1a_2 - a_3 = 0$; therefore, based on part (i) in Routh-Hurwitz stability conditions, the equilibrium point E_1 is an unstable point.

(iii) The Jacobian (6) at the equilibrium point $E_2 = (\frac{c}{d}, \frac{a}{b}, 0)$ is

$$J(\frac{c}{d}, \frac{a}{b}, 0) = \begin{pmatrix} 0 & -\frac{bc}{d} & 0 \\ \frac{ad}{b} & 0 & -\frac{ea}{b} \\ 0 & 0 & \frac{ag - bf}{b} \end{pmatrix}, \tag{9}$$

In this case, the characteristic equation is also

$$P(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0,$$

where

$$a_1 = -\frac{ag - bf}{b}, \qquad a_2 = ac, \qquad a_3 = -\frac{(ag - bf)ac}{b},$$

and

$$D(P) = -\frac{4ac(bf(bf - 2ag) + g^2a^2 + b^2ca)^2}{b^4}.$$

Therefore, the eigenvalues of the Jacobian matrix (9) corresponding to the equilibrium point E_2 are

$$\lambda_1 = \frac{ag - bf}{b}, \qquad \lambda_2 = i\sqrt{ac}, \qquad \lambda_2 = -i\sqrt{ac}.$$

Clearly, if $bf(bf-2ag)+g^2a^2+b^2ca \neq 0$ then $D(P)<0$. Now if $bf>ag$ then $a_1>0, a_2>0, a_1a_2-a_3=0$ and based on part (iii) in Routh-Hurwitz stability conditions the equilibrium point E_2 is locally asymptotically stable for all $\alpha \in [0,1)$.

5 NSFD for fractional-order Lotka-Volterra model

For system (3) and applying Mickens scheme by replacing the step size h by a function $\phi(h)$ and using the GL discretization method, the following equations are obtained:

$$\begin{aligned}
&\sum_{j=0}^{n+1} c_j^{\alpha_1} x_{n+1-j} = a x_n - b x_{n+1} y_n,\\
&\sum_{j=0}^{n+1} c_j^{\alpha_2} y_{n+1-j} = -c y_{n+1} + d x_{n+1} y_n - e y_{n+1} z_n,\\
&\sum_{j=0}^{n+1} c_j^{\alpha_3} z_{n+1-j} = -f z_{n+1} + g z_n y_{n+1}.
\end{aligned} \tag{10}$$

Comparing equations (10) with system (3), we note the following:
1. The linear and nonlinear terms on the right-hand side of the first equation in system (3) are in the forms

$$x \approx x_n, \qquad -xy \approx -x_{n+1} y_n.$$

2. The linear and nonlinear terms on the right-hand side of the second equation in (3) are

$$-y \approx -y_{n+1}, \qquad xy \approx x_{n+1} y_n, \qquad -yz \approx -y_{n+1} z_n.$$

3. The linear and nonlinear terms on the right-hand side of the third equation in (3) are

$$-z \approx -z_{n+1}, \qquad zy \approx z_n y_{n+1}.$$

Doing some algebraic manipulations to equations (10) yields the following relations:

$$
\begin{aligned}
x_{n+1} &= \frac{-\sum_{j=1}^{n+1} c_j^{\alpha_1} x_{n+1-j} + a x_n}{c_0^{\alpha_1} + b y_n},\\
y_{n+1} &= \frac{-\sum_{j=1}^{n+1} c_j^{\alpha_2} y_{n+1-j} + d x_{n+1} y_n}{c_0^{\alpha_2} + c + e z_n},\\
z_{n+1} &= \frac{-\sum_{j=1}^{n+1} c_j^{\alpha_3} z_{n+1-j} + g z_n y_{n+1}}{c_0^{\alpha_3} + f},
\end{aligned}
\tag{11}
$$

where

$$c_0^{\alpha_1} = \phi_1(h)^{-\alpha_1}, \qquad c_0^{\alpha_2} = \phi_2(h)^{-\alpha_2}, \qquad c_0^{\alpha_3} = \phi_3(h)^{-\alpha_3},$$

with [21]

$$\phi_1(h) = \frac{e^{ah} - 1}{a}, \qquad \phi_2(h) = \frac{e^{ch} - 1}{c}, \qquad \phi_3(h) = \frac{e^{fh} - 1}{f}.$$

Proposition 1. *The numerical solutions obtained from system (11) for case* $0 < \alpha_i \leq 1, \quad i = 1, 2, 3$ *satisfy*

$$
\begin{array}{lcl}
x_n > 0 & & x_{n+1} > 0\\
y_n > 0 & \Rightarrow & y_{n+1} > 0\\
z_n > 0 & & z_{n+1} > 0
\end{array}
\tag{12}
$$

for all the relevant values of n*.*

Proof. Since $c_0^{\alpha_i} > 0$ and by recursive relation

$$c_j^{\alpha_i} = (1 - \frac{1 + \alpha_i}{j}) c_{j-1}^{\alpha_i}, \qquad j = 1, 2, 3, ...$$

we have $c_j^{\alpha_i} < 0, \quad j > 0$. Now system (11) shows that relations (12) is established. For case $\alpha_i = 1, \quad i = 1, 2, 3$ we should consider the following system:

$$
\begin{aligned}
\frac{x_{n+1} - x_n}{\phi_1} &= a x_n - b x_{n+1} y_n,\\
\frac{y_{n+1} - y_n}{\phi_2} &= -c y_{n+1} + d x_{n+1} y_n - e y_{n+1} z_n,\\
\frac{z_{n+1} - z_n}{\phi_3} &= -f z_{n+1} + g z_n y_{n+1}.
\end{aligned}
$$

By solving this system for x_{n+1}, y_{n+1} and z_{n+1} we conclude that relation (12) holds. □

6 Numerical results

Analytical studies always remain incomplete without numerical verification of the results. In this section, we present numerical simulation to illustrate the results obtained in the previous sections. The numerical experiments are designed to show the dynamical behaviour of the system in three main different sets of parameters and initial conditions:

(i) The case where $bf = ag$,

(ii) The case where $bf > ag$,

(iii) The case where $bf < ag$.

To show the dynamics of the system (3), set the parameter $a = b = c = d = e = f = 1$ given as fixed parameters and g as a varied parameter.

(i) The case where $bf = ag$

For the case $bf = ag$ the equilibrium point E_2 has three eigenvalues with zero real part corresponding with stable centre point in xy plane . We consider the case $\alpha_1 = \alpha_2 = \alpha_3 = 1$ which corresponds to the classical Lotka-Volterra system. Figures 1 and 2 represents the phase portrait for solutions where parameter $g = 1$ with the initial conditions $(x(0), y(0), z(0)) = (0.5, 1, 2)$, for simulation time 40s and step size $h = 0.1$ and $h = 0.5$. In this case, prey x, predator y and top predator z persist and have populations that vary periodically over time in a common period.

Once again an equilibrium is achieved within the system, such that each predator population increases as the population of its respective prey increases. Each predator population also peaks and then begins to decrease shortly after its respective prey population peaks and begins to decrease. The plots of populations x and y are essentially the same as they were in the $2D$ system, and the new predator population z behaves similarly with respect to y as y behaves with respect to x. All three populations share a common period.

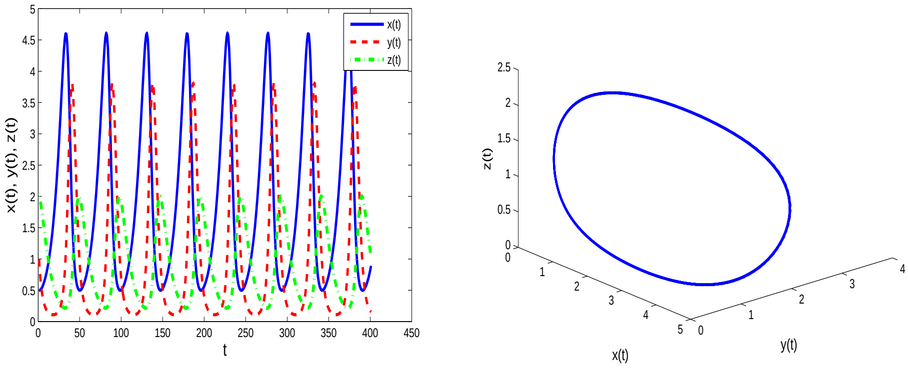

Figure 1: Plot of populations x, y and z over time for the case $bf = ag$ with $\alpha_1 = \alpha_2 = \alpha_3 = 1$ and $h = 0.1$.

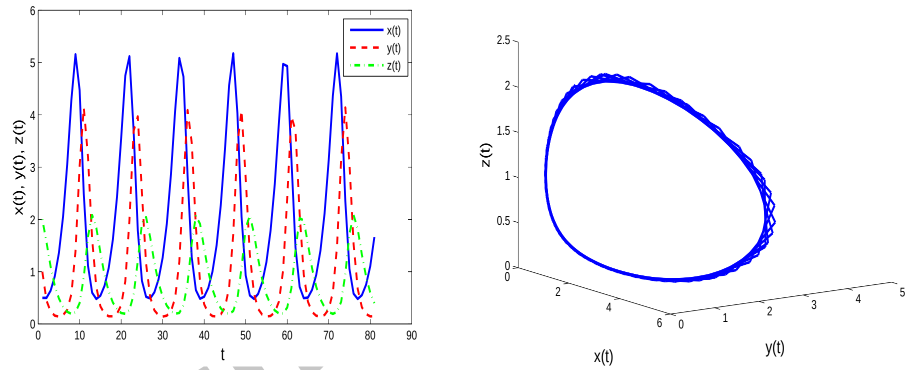

Figure 2: Plot of populations x, y and z over time for the case $bf = ag$ with $\alpha_1 = \alpha_2 = \alpha_3 = 1$ and $h = 0.5$.

Figures 3 and 4 depict the phase trajectory of the fractional-order Lotka-Volterra chaotic system (3) for commensurate order $\alpha_1 = \alpha_2 = \alpha_3 = 0.90$ and parameters $g = 1$ with the initial conditions $(x(0), y(0), z(0)) = (0.5, 1, 2)$, for simulation time 40s and step size $h = 0.1$ and $h = 0.5$.

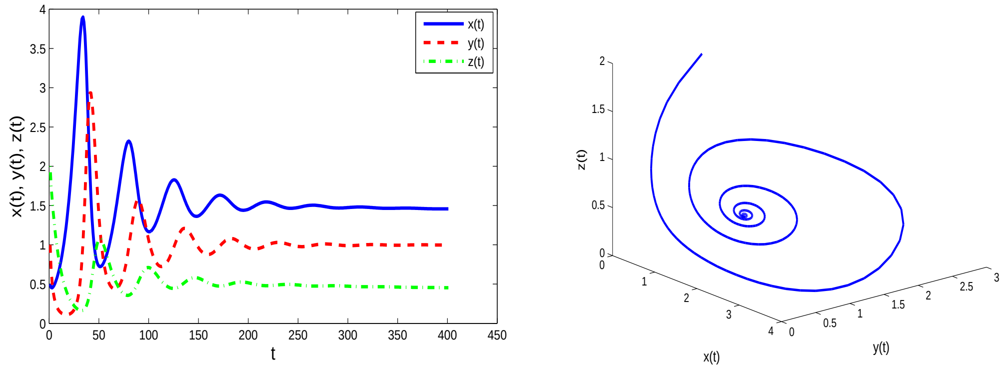

Figure 3: Plot of populations x, y and z over time for the case $bf = ag$ with $\alpha_1 = \alpha_2 = \alpha_3 = 0.90$ and $h = 0.1$.

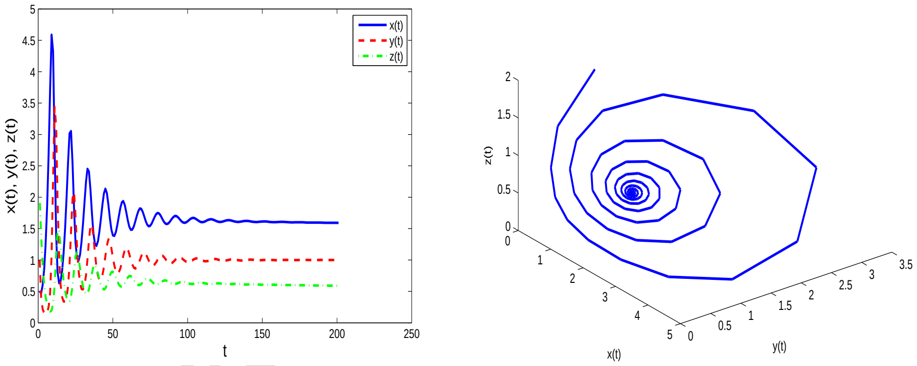

Figure 4: Plot of populations x, y and z over time for the case $bf = ag$ with $\alpha_1 = \alpha_2 = \alpha_3 = 0.90$ and $h = 0.5$.

Figures 5 and 6 depict the phase trajectory of the fractional-order Lotka-Volterra chaotic system for incommensurate order and parameters $g = 1$ with the initial conditions $(x(0), y(0), z(0)) = (0.5, 1, 2)$, for simulation time 40s and step size $h = 0.1$ and $h = 0.5$.

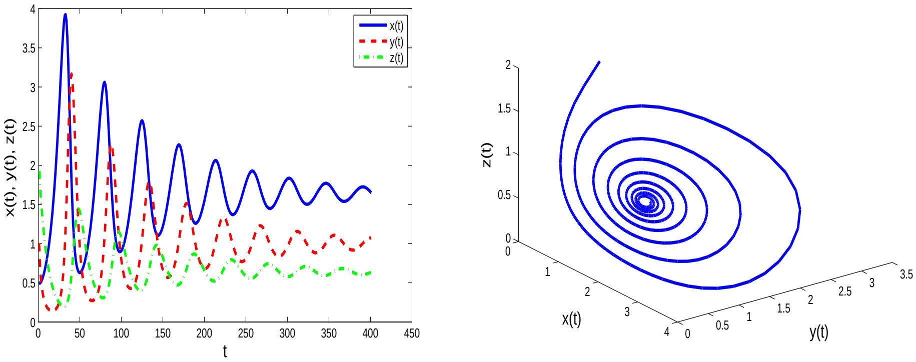

Figure 5: Plot of populations x, y and z over time for the case $bf = ag$ with $\alpha_1 = 0.99, \alpha_2 = 0.95, \alpha_3 = 0.90$ and $h = 0.1$.

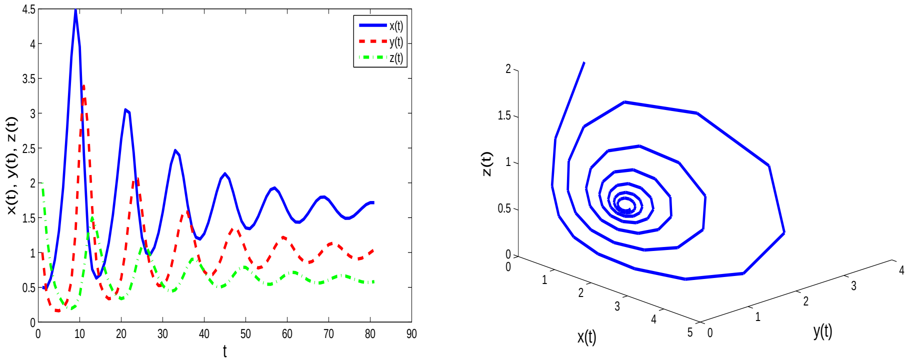

Figure 6: Plot of populations x, y and z over time for the case $bf = ag$ with $\alpha_1 = 0.95, \alpha_2 = 0.90, \alpha_3 = 0.80$ and $h = 0.5$.

In Figure 7, the phase trajectory of the fractional-order Lotka-Volterra chaotic system is depicted for incommensurate order and parameters $a = 1, b = 2, c = 5, d = 4, e = 3, f = 3, g = 6$ with the initial conditions $(x(0), y(0), z(0)) = (0.5, 1, 2)$, for simulation time 40s and step size $h = 0.1$.

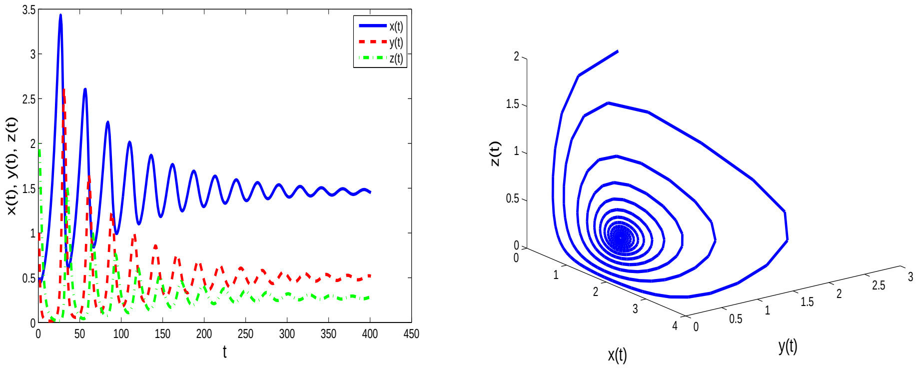

Figure 7: Plot of populations x, y and z over time for the case $bf = ag$ with $\alpha_1 = 0.99, \alpha_2 = 0.95, \alpha_3 = 0.90$ and $h = 0.1$.

(ii) The case where $bf > ag$

For the case where $bf > ag$, two eigenvalues for E_2 are pure imaginary initially-spiral stability corresponding with centre manifold in xy plane and one negative real eigenvalue corresponding with stable one-dimensional invariant curve in z axis. Hence, the equilibrium point E_2 is locally stable spiral sink. On the other hand, prey x and predator y persist and has populations that vary periodically over time with a common period. The solutions are plotted in Figures 8 and 9 for commensurate order $\alpha_1 = \alpha_2 = \alpha_3 = 1$ and parameters $g = 0.88$ with the initial conditions $(x(0), y(0), z(0)) = (0.5, 1, 2)$, for simulation time 100s and step size $h = 0.1$ and $h = 0.5$.

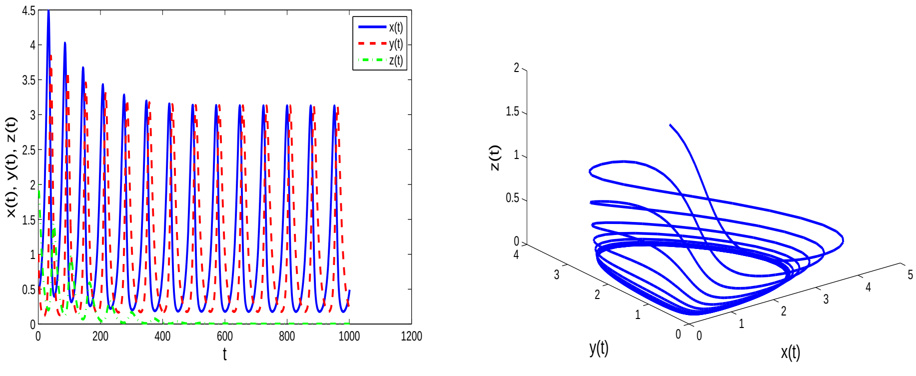

Figure 8: Plot of populations x, y and z over time for the case $bf > ag$ with $\alpha_1 = \alpha_2 = \alpha_3 = 1$ and $h = 0.1$.

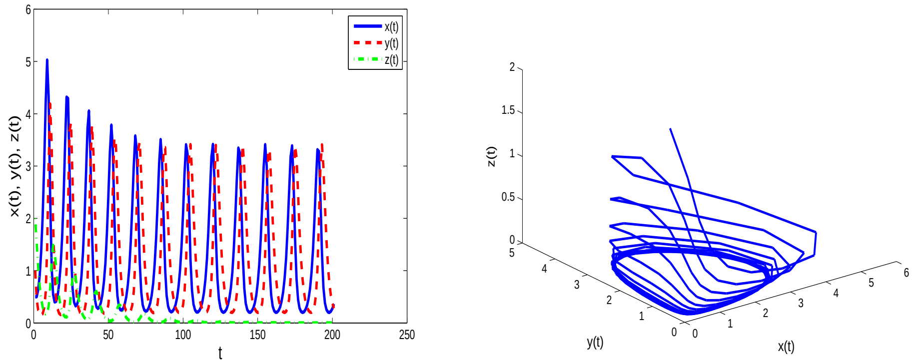

Figure 9: Plot of populations x, y and z over time for the case $bf > ag$ with $\alpha_1 = \alpha_2 = \alpha_3 = 1$ and $h = 0.5$.

Figures 10 and 11 depict the phase trajectory of the fractional-order Lotka-Volterra chaotic system (3) for incommensurate order $\alpha_1 = 0.90$, $\alpha_2 = 0.80$, $\alpha_3 = 0.70$ and parameters $g = 0.88$ with the initial conditions $(x(0), y(0), z(0)) = (0.5, 1, 2)$, for simulation time 100s and step size $h = 0.1$ and $h = 0.5$.

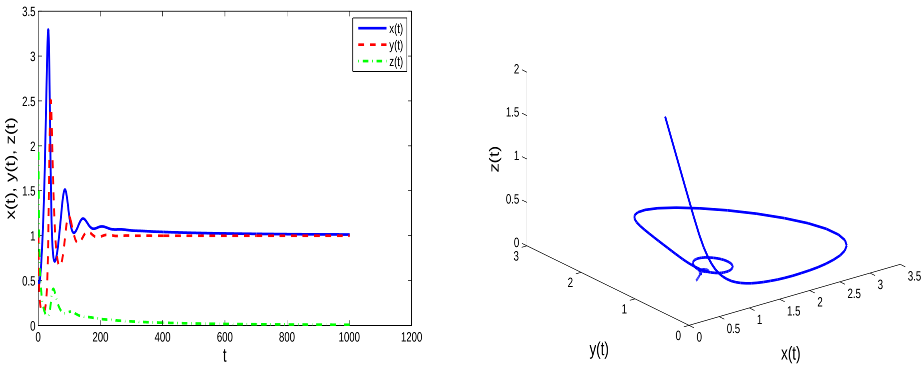

Figure 10: Plot of populations x, y and z over time for the case $bf > ag$ with $\alpha_1 = 0.90, \alpha_2 = 0.80, \alpha_3 = 0.70$ and $h = 0.1$.

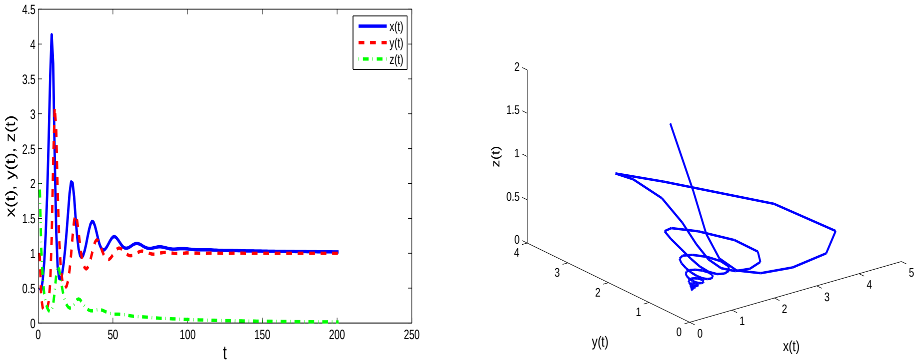

Figure 11: Plot of populations x, y and z over time for the case $bf > ag$ with $\alpha_1 = 0.90, \alpha_2 = 0.80, \alpha_3 = 0.70$ and $h = 0.5$.

(iii) The case where $bf < ag$

For the case where $bf < ag$, two eigenvalues for E_2 is pure imaginary initially-spiral stability corresponding with centre manifold in xy plane and one positive real eigenvalue corresponding to unstable one-dimensional invariant curve in z axes. Hence the equilibrium point E_2 is a locally unstable spiral source. In this case, the prey x and top predator z can survive, growing periodically unstable. On the other hand, predator y persists and has populations that vary periodically stable. The solutions for this case are shown in Figure 12 for commensurate order $\alpha_1 = \alpha_2 = \alpha_3 = 1$ and parameters $g = 1.6$ with the initial conditions $(x(0), y(0), z(0)) = (0.5, 1, 2)$, for simulation time 50s and step size $h = 0.1$.

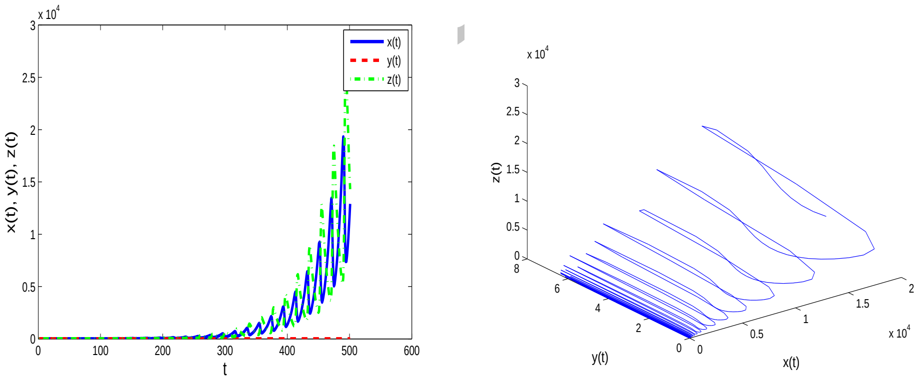

Figure 12: Plot of populations x, y and z over time for the case $bf < ag$ with $\alpha_1 = \alpha_2 = \alpha_3 = 1$ and $h = 0.1$.

In Figure 13 the phase trajectory of the fractional-order Lotka-Volterra chaotic system (3) is depicted for commensurate order $\alpha_1 = \alpha_2 = \alpha_3 =$

0.50 and parameters $g = 1.6$ with the initial conditions $(x(0), y(0), z(0)) = (0.5, 1, 2)$, for simulation time 50s and step size $h = 0.5$.

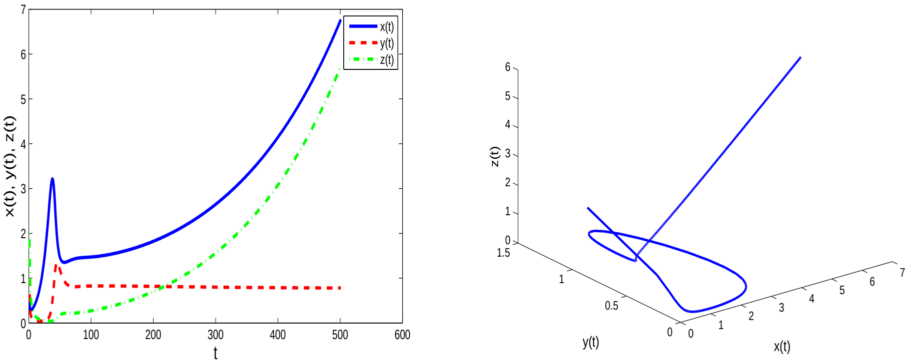

Figure 13: Plot of populations x, y and z over time for the case $bf < ag$ with $\alpha_1 = \alpha_2 = \alpha_3 = 0.50$ and $h = 0.5$.

The solutions for this case are shown in Figure 14 for incommensurate order and parameters $g = 1.6$ with the initial conditions $(x(0), y(0), z(0)) = (0.5, 1, 2)$, for simulation time 50s and step size $h = 0.1$.

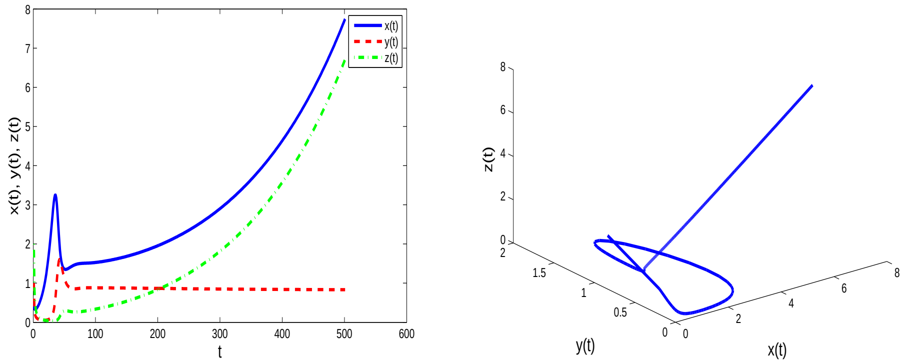

Figure 14: Plot of populations x, y and z over time for the case $bf < ag$ with $\alpha_1 = 0.60, \alpha_2 = 0.50, \alpha_3 = 0.40$ and $h = 0.1$.

In Figure 15 the phase trajectory of the fractional-order Lotka-Volterra chaotic system is depicted for incommensurate order and parameters $g = 1.6$ with the initial conditions $(x(0), y(0), z(0)) = (0.5, 1, 2)$, for simulation time 50s and step size $h = 0.5$.

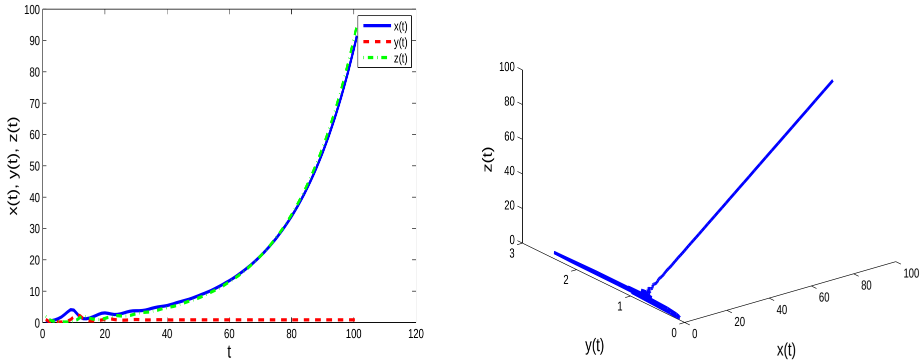

Figure 15: Plot of populations x, y and z over time for the case $bf < ag$ with $\alpha_1 = 0.8, \alpha_2 = 0.6, \alpha_3 = 0.5$ and $h = 0.5$.

In Table 1 for different step size h, the qualitative results, obtained by NSFD scheme, of the fixed point E_2 are respectively compared to classical methods such as forward Euler and 4th order Runge-Kutta. From Table 1, it follows that the CPU time of the method NSFD is less than the CPU time of the forward Euler and Runge-Kutta methods. Also if step size h is chosen small enough, the results of the proposed NSFD scheme are similar with the results of the other two numerical methods. But if the step size h is chosen larger, the efficiency of NSFD scheme is clearly seen.

Table 1: Qualitative results of the equilibrium point E_2 for different time step sizes, t= 0-200 for the case where $bf = ag$

h	Euler	CPU time	Runge-Kutta	CPU time	NSFD	CPU time
0.001	Convergence	0.016342	Convergence	0.032029	Convergence	0.000206
0.01	Convergence	0.014760	Convergence	0.028096	Convergence	0.000205
0.1	Convergence	0.013917	Convergence	0.027959	Convergence	0.000203
0.2	Divergence	–	Convergence	0.025959	Convergence	0.000202
2	Divergence	-	Divergence	–	Convergence	0.000201
10	Divergence	-	Divergence	-	Convergence	0.000201

In Figure 16 the numerical solution of forward Euler and fourth order Runge-Kutta methods are compared with NSFD scheme graphically.

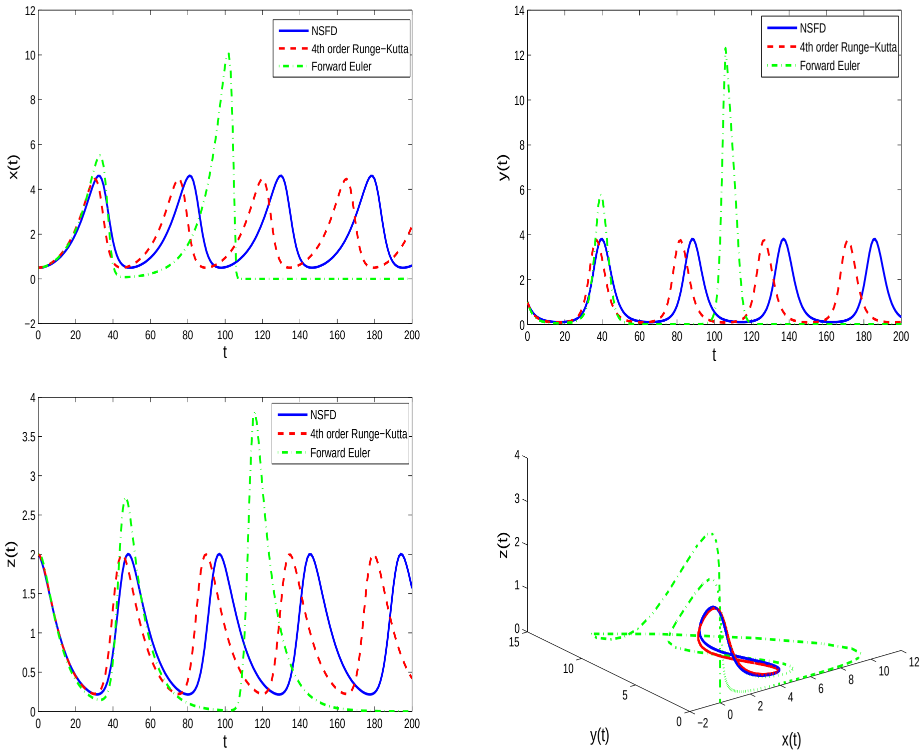

Figure 16: Numerical solutions for forward Euler and fourth order Runge-Kutta and NSFD methods with $h = 0.1$ for the case $bf = ag$ and $\alpha_1 = \alpha_2 = \alpha_3 = 1$.

7 Conclusion

In this paper, we study the fractional-order Lotka-Volterra model. The stability of equilibrium points is studied. Numerical solutions of these models are given. The reason for considering a fractional order system instead of its integer order counterpart is that fractional order differential equations are generalizations of integer order differential equations. Also using fractional order differential equations can help us to reduce the errors arising from the neglected parameters in modelling real life phenomena.

We argue that the fractional order models are at least as good as integer order ones in modeling biological, economic and social systems (generally complex adaptive systems) where memory effects are important.

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یک طرح تفاضلی متناهی غیر استاندارد برای حل شبکه‌ی غذایی سه بعدی با مدل لوتکا-ولترا مرتبه‌ی کسری

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چکیده : در این مقاله یک مدل مرتبه‌ی کسری از شبکه‌ی غذایی سه بعدی لوتکا-ولترا را معرفی می‌کنیم. تحلیل پایداری سیستم کسری را شرح می‌دهیم. طرح تفاضلی متناهی غیراستانداردی بیان می‌شود که رفتار دینامیکی سیستم لوتکا-ولترا مرتبه‌ی کسری را مورد مطالعه قرار می‌دهد. نتایج عددی نشان می‌دهند که تقریبات طرح تفاضلی متناهی غیر استاندارد زمانی که برای سیستم لوتکا-ولترا مرتبه‌ی کسری استفاده می‌شوند، بسیار دقیق هستند.

کلمات کلیدی : معادلات دیفرانسیل کسری؛ مدل لوتکا-ولترا؛ سیستم شکار-شکارچی؛ طرح تفاضلی متناهی غیراستاندارد؛ پایداری.