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# A contractive mapping on fuzzy normed linear spaces

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**Abstract**

In this paper, we use the definition of fuzzy normed spaces given by Bag and Samanta and provide four types of fuzzy versions of contraction. We show that these mappings necessarily have unique fixed points in fuzzy normed linear spaces. Moreover we prove that the presented theorems are indeed fuzzy extensions of their classical counterparts.

**Keywords:** Fuzzy norm; Fuzzy normed linear space; Fixed point; $\alpha-$ semi-norm; Contractive conditions.

## 1 Introduction

Banach contraction mapping principle is one of the fundamental consequences of analysis. This contraction mapping is an important object in metric fixed point theory. Also its emphasis lies on its wide applicability in branches of mathematics. Some contractive conditions have been introduced in [4, 6, 7, 9, 10].

A natural question is whether we can provide contractive conditions which imply existence of fixed point in a fuzzy Banach space. Recently, Shukla and Chauhan [11] defined the concept of cyclic representation and proved some fixed point results for operators on complete fuzzy metric spaces. In [1], AL-Mayahi and Hadi proved that $\alpha$-$\eta$-$\varphi$-contraction functions have a fixed point on fuzzy metric space. Das and Saha [5] considered uniformly locally contractive mappings on a fuzzy metric space and showed that these functions have a unique fixed point. Manro and Tomar [8] focused on the compatibility and non-compatibility of pair of self-maps and established existence of fixed point of the compatible maps on fuzzy metric space.
In this paper, we use the definition of fuzzy normed spaces given in [2] and discuss four types of fuzzy versions of contraction and some corollaries.
We give below some basic preliminaries required for this paper.

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**Definition 1.** [2] Let $X$ be a linear space over $R$ (real number) and $N$ be A fuzzy subset of $X \times R$ such that for all $x, u \in X$ and $c \in R$
$(N1)$ $N(x,t) = 0$ for all $t \leq 0$,
$(N2)$ $x = 0$ if and only if $N(x,t) = 1$ for all $t > 0$,
$(N3)$ If $c \neq 0$ then $N(cx,t) = N(x, t/|c|)$ for all $t \in R$,
$(N4)$ $N(x+u, s+t) \geq min\{N(x,s), N(u,t)\}$ for all $s,t \in R$,
$(N5)$ $N(x,.)$ is a nondecreasing function of $R$ and $\lim_{t\to\infty} N(x,t) = 1$.
Then $N$ is called a fuzzy norm on $X$.

Assume that
$(N6)$ $N(x,t) > 0$ for all $t > 0$ implies $x = 0$,
$(N7)$ For $x \neq 0$, $N(x,.)$ is a continuous function of $R$ and strictly increasing on the subset $\{t : 0 < N(x,t) < 1\}$ of $R$.

**Definition 2.** [3] Let $(X,N)$ be a fuzzy normed linear space.
i) A sequence $\{x_n\} \subseteq X$ is said to converge to $x \in X$ ($\lim\limits_{n\to\infty} x_n = x$), if $\lim\limits_{n\to\infty} N(x_n - x, t) = 1$, for all $t > 0$.
ii) A sequence $\{x_n\} \subseteq X$ is called Cauchy, if $\lim\limits_{n,m\to\infty} N(x_n - x_m, t) = 1$, for all $t > 0$.

**Definition 3.** If $X$ is a vector space over $R$, a seminorm is a function $p : X \longrightarrow [0,\infty)$ having the properties:
$(i)$ $p(c(x)) = |c|p(x)$ for all $c \in R$ and $x \in X$.
$(ii)$ $p(x+y) \leq p(x) + p(y)$ for all $x, y \in X$.

**Theorem 1.** *Let $(X,N)$ be a fuzzy normed linear space. Define*

$$\|x\|_\alpha = \inf\{t > 0 : N(x,t) \geq \alpha\}, \ \alpha \in (0,1).$$

*Then $\{\|.\|_\alpha : \alpha \in (0,1)\}$ is an ascending family of seminorms on $X$ and they are called $\alpha$-seminorms on $X$ corresponding to the fuzzy norm $N$ on $X$.*

*Proof.* $(i)$ Let $x \in X$, $c \in R$ and $\alpha \in (0,1)$, we have

$$\begin{aligned}\|cx\|_\alpha &= \wedge\{t > 0 : N(cx,t) \geq \alpha\} \\ &= \wedge\{t > 0 : N(x, t/|c|) \geq \alpha\} \\ &= \wedge\{|c|t > 0 : N(x,t) \geq \alpha\} \\ &= |c|\|x\|_\alpha.\end{aligned}$$

$(ii)$ Let $x, y \in X$ and $\alpha \in (0,1)$, we obtain that

$$N(x+y, \|x\|_\alpha + \|y\|_\alpha + \epsilon) \geq min\{N(x, \|x\|_\alpha + \epsilon/2), N(y, \|y\|_\alpha + \epsilon/2)\} \geq \alpha,$$

hence $\|x+y\|_\alpha \leq \|x\|_\alpha + \|y\|_\alpha + \epsilon$, as $\epsilon \longrightarrow 0$ then $\|x+y\|_\alpha \leq \|x\|_\alpha + \|y\|_\alpha$. □

## 2 Fixed point theorems

At first we introduce the following notation.
Denote $\Psi$ to be the set of functions $\psi : [0, +\infty) \longrightarrow [0, +\infty)$ satisfying the following hypotheses:
(i) $\psi$ is continuous and nondecreasing,
(ii) $\psi(t) = 0$ if and only if $t = 0$.
We denote by $\Phi$ the set of functions $\phi : [0, +\infty) \longrightarrow [0, +\infty)$ satisfying the following hypotheses:
(i) $\phi$ is continuous and strictly increasing,
(ii) $\phi(t) = 0$ if and only if $t = 0$.

**Theorem 2.** *Let $(X, N)$ be a fuzzy Banach space and $f : X \longrightarrow X$ be a selfmap such that for all $x, y \in X$, $t \in \mathbb{R}$ and $\alpha \in (0, 1]$,*

$$N(x - y, t) \geq \alpha \text{ implies that } N(f(x) - f(y), t - \psi(t)) \geq \alpha,$$

*where $\psi \in \Psi$. Then $f$ has a unique fixed point in $X$.*

*Proof.* Let $x_0 \in X$ and $x_{n+1} = f(x_n)$, for all $n \in \mathbb{N}$. Suppose that $t > 0$, we have

$$N(x - y, t) \leq N(f(x) - f(y), t - \psi(t)), \text{ for all } x, y \in X.$$

Therefore

$$N(x_{n+1} - x_n, t) \leq N(x_{n+2} - x_{n+1}, t - \psi(t)) \leq N(x_{n+2} - x_{n+1}, t),$$

for all $n \in \mathbb{N}$. Hence $\{N(x_{n+1} - x_n, t)\}$ is a bounded and nondecreasing sequence. Thus $\lim_{n\to\infty} N(x_{n+1} - x_n, t)$ exists. Now we have

$$N(x_1 - x_0, t + \psi(t)) \leq N(x_2 - x_1, t + \psi(t) - \psi(t + \psi(t))) \leq N(x_2 - x_1, t),$$

by induction on n, we obtain that

$$N(x_1 - x_0, t + n\psi(t)) \leq N(x_{n+1} - x_n, t), \text{ for all } n \in \mathbb{N}.$$

As $n \longrightarrow \infty$, (N5) implies $\lim_{n\to\infty} N(x_{n+1} - x_n, t) = 1$, for all $t > 0$.
Let $t > 0$, $\epsilon > 0$ and $N \in \mathbb{N}$ such that $1 - \epsilon \leq N(x_{N+1} - x_N, t/2)$ and $1 - \epsilon \leq N(x_{N+1} - x_N, \psi(t/2))$. If $1 - \epsilon \leq N(x - x_N, t/2)$ then

$$\begin{aligned} N(f(x) - x_N, t/2) &\geq \min\{N(f(x) - f(x_N), t/2 - \psi(t/2)), \\ &\quad N(f(x_N) - x_N, \psi(t/2))\} \\ &\geq \min\{N(x - x_N, t/2), N(x_{N+1} - x_N, \psi(t/2))\} \\ &\geq 1 - \epsilon. \end{aligned}$$

Therefore $1-\epsilon \leq N(x_n - x_N, t/2)$, for all $n \geq N$, so

$$N(x_n - x_m, t) \geq \min\{N(x_n - x_N, t/2), N(x_m - x_N, t/2)\} \geq 1-\epsilon,$$

for all $n, m \geq N$. Since $\epsilon$ is arbitrary, $\{x_n\}$ is Cauchy, hence it is convergent. Assume that $\lim_{n\to\infty} x_n = x$. Let $\varepsilon > 0$ and $t > 0$. Then there exists $N_0 > 0$ such that $1-\varepsilon \leq N(x_n - x, t)$ and $1-\varepsilon \leq N(x - x_n, \psi(t))$, for all $n \geq N_0$. Hence

$$\begin{aligned} N(f(x)-x,t) &\geq \min\{N(f(x)-x_{n+1}, t-\psi(t)), N(x_{n+1}-x, \psi(t))\} \\ &\geq \min\{N(x-x_n,t), N(x_{n+1}-x,\psi(t))\} \\ &\geq 1-\epsilon, \text{ for all } n \geq N_0. \end{aligned}$$

Therefore $N(f(x)-x,t) = 1$, for all $t > 0$. Hence $f(x) = x$.
To prove the uniqueness of the fixed point, we let $y$ be any other fixed point of $f$ in $X$. Suppose that $t > 0$. Similarly, we have

$$N(x-y, t+n\psi(t)) \leq N(f(x)-f(y), t) = N(x-y,t), \text{ for all } n \in \mathbb{N}.$$

As $n \longrightarrow \infty$, we obtain that $N(x-y,t) = 1$, for all $t > 0$, hence $x = y$. □

**Corollary 1.** *Let $(X, N)$ be a fuzzy Banach space and $f : X \longrightarrow X$ be a selfmap such that for all $x, y \in X$ and $t \in \mathbb{R}$,*

$$N(f(x)-f(y), t-\psi(t)) \geq N(x-y,t),$$

*where $\psi \in \Psi$. Then $f$ has a unique fixed point in $X$.*
**Example 1.** Let $(X, \|.\|)$ be a Banach space and $f : X \longrightarrow X$ be a function such that

$$\|f(x)-f(y)\| \leq \|x-y\| - \psi(\|x-y\|), \text{ for all } x, y \in X,$$

where $\psi \in \Psi$. Assume that $I - \psi$ is a nondecreasing function and $\psi(\beta t) \leq \beta\psi(t)$, for all $t \in [0, +\infty)$, $\beta \in [0,1]$. Define a fuzzy norm $N$ as follows:

$$N(x,t) = \begin{cases} t/\|x\| & , \quad 0 < t \leq \|x\| \\ 1 & , \quad \|x\| < t \\ 0 & , \quad t \leq 0. \end{cases}$$

Suppose that $x, y \in X$, $t > 0$, $\alpha \in (0,1]$ and $N(x-y,t) \geq \alpha$.
Case1: Let $\|x-y\| < t$. Since $I-\psi$ is nondecreasing,

$$\|f(x)-f(y)\| \leq \|x-y\| - \psi(\|x-y\|) \leq t - \psi(t).$$

So $N(f(x)-f(y), t-\psi(t)) = 1 \geq \alpha$.
Case2: Let $0 < t \leq \|x-y\|$. So $t/\|x-y\| = N(x-y,t) \geq \alpha$. Hence $\alpha\|x-y\| \leq t$. Therefore

$$\alpha\|f(x)-f(y)\| \le \alpha\|x-y\|-\alpha\psi(\|x-y\|) \le \alpha\|x-y\|-\psi(\alpha\|x-y\|) \le t-\psi(t).$$

Thus $N(f(x)-f(y), t-\psi(t)) = (t-\psi(t))/(\|f(x)-f(y)\|) \ge \alpha$. By Theorem 2, $f$ has a unique fixed point in $X$.

**Example 2.** Let $[0,1] = X$ and $\|x-y\| = |x-y|$, for all $x, y \in X$. Also let $f : X \longrightarrow X$ and $\psi : [0,+\infty) \longrightarrow [0,+\infty)$ be defined as

$$f(x) = x - (1/2)x^2, \text{ for all } x \in X,$$

$$\psi(t) = (1/2)t^2, \text{ for all } t \ge 0.$$

It is clear that $I - \psi$ is a nondecreasing function and $\psi(\beta t) \le \beta\psi(t)$, for all $t \in [0,+\infty)$, $\beta \in [0,1]$. Define a fuzzy norm $N$ as follows:

$$N(x,t) = \begin{cases} t/\|x\| & , \quad 0 < t \le \|x\| \\ 1 & , \quad \|x\| < t \\ 0 & , \quad t \le 0. \end{cases}$$

Suppose that $x, y \in X$. Without loss of generality, we assume that $x > y$. Then

$$\begin{aligned} \|f(x)-f(y)\| &= (x-(1/2)x^2)-(y-(1/2)y^2) \\ &= (x-y)-(1/2)(x-y)(x+y) \\ &\le (x-y)-(1/2)(x-y)^2 \\ &\le \|x-y\|-\psi(\|x-y\|). \end{aligned}$$

By Example 1, $f$ has a unique fixed point in $X$.

**Theorem 3.** *Let $(X, N)$ be a fuzzy Banach space such that $N$ satisfies (N7) and $\gamma : (0,+\infty) \longrightarrow [0,1)$ be a decreasing function, also $f : X \longrightarrow X$ be a selfmap such that for all $x, y \in X$, $t > 0$ and $\alpha \in (0,1]$,*

$$N(x-y,t) \ge \alpha \text{ implies that } N(f(x)-f(y), \phi^{-1}(\gamma(t)\phi(t)) \ge \alpha,$$

*where $\phi \in \Phi$. Then $f$ has a unique fixed point in $X$.*



*Proof.* Let $x_0 \in X$ and $x_{n+1} = f(x_n)$, for all $n \in \mathbb{N}$. Suppose that $t > 0$, we have

$$N(x-y,t) \le N(f(x)-f(y), \phi^{-1}(\gamma(t)\phi(t)).$$

Therefore

$$N(x_{n+1}-x_n, t) \le N(x_{n+2}-x_{n+1}, \phi^{-1}(\gamma(t)\phi(t))) \le N(x_{n+2}-x_{n+1}, t),$$

for all $n \in \mathbb{N}$. Hence $\{N(x_{n+1} - x_n, t)\}$ is a bounded and nondecreasing sequence. Thus $\lim_{n\to\infty} N(x_{n+1} - x_n, t)$ exists.

Assume that there exists $t > 0$ such that $\lim_{n\to\infty} N(x_{n+1}-x_n, t) < \beta < 1$. Since $N(x_{n+2} - x_{n+1}, s) \geq N(x_{n+1} - x_n, s)$, for all $s > 0$, it follows that

$$0 < t \leq \|x_{n+2} - x_{n+1}\|_\beta \leq \|x_{n+1} - x_n\|_\beta, \text{ for all } n \in \mathbb{N}.$$

Hence $\lim_{n\to\infty} \|x_{n+1} - x_n\|_\beta$ exists. Let $\lim_{n\to\infty} \|x_{n+1} - x_n\|_\beta = b \geq t > 0$. If $N(x_{n+1} - x_n, s) \geq \beta$ then

$$N(x_{n+2} - x_{n+1}, \phi^{-1}(\gamma(s)\phi(s))) \geq N(x_{n+1} - x_n, s) \geq \beta.$$

Therefore $\|x_{n+2} - x_{n+1}\|_\beta \leq \phi^{-1}(\gamma(s)\phi(s))$. Thus

$$\phi(\|x_{n+2} - x_{n+1}\|_\beta) \leq \gamma(s)\phi(s) \leq \gamma(\|x_{n+1} - x_n\|_\beta)\phi(s) \leq \gamma(b)\phi(s).$$

As $s \longrightarrow \|x_{n+1} - x_n\|_\beta$, we get $\phi(\|x_{n+2} - x_{n+1}\|_\beta) \leq \gamma(b)\phi(\|x_{n+1} - x_n\|_\beta)$. As $n \longrightarrow \infty$, one can obtain that $0 < \phi(t) \leq \phi(b) \leq \gamma(b)\phi(b)$. So $1 \leq \gamma(b)$, which is a contradiction. Hence $\lim_{n\to\infty} N(x_{n+1} - x_n, t) = 1$, for all $t > 0$. Let $t > 0$, $\epsilon > 0$ and $N \in \mathbb{N}$ such that

$$1 - \epsilon \leq N(x_{N+1} - x_N, t/2 - \phi^{-1}(\gamma(t/2)\phi(t/2)).$$

If $1 - \epsilon \leq N(x - x_N, t/2)$ then

$$\begin{aligned} N(f(x) - x_N, t/2) &\geq \min\{N(f(x) - x_{N+1}, \phi^{-1}(\gamma(t/2)\phi(t/2))), \\ &\qquad N(x_{N+1} - x_N, t/2 - \phi^{-1}(\gamma(t/2)\phi(t/2)))\} \\ &\geq \min\{N(x - x_N, t/2), \\ &\qquad N(x_{N+1} - x_N, t/2 - \phi^{-1}(\gamma(t/2)\phi(t/2)))\} \\ &\geq 1 - \epsilon \end{aligned}$$

Therefore $1 - \epsilon \leq N(x_n - x_N, t/2)$, for all $n \geq N$, so

$$N(x_n - x_m, t) \geq \min\{N(x_n - x_N, t/2), N(x_m - x_N, t/2)\} \geq 1 - \epsilon,$$

for all $n, m \geq N$. Since $\epsilon$ is arbitrary, $\{x_n\}$ is Cauchy, hence it is convergent. Assume that $\lim_{n\to\infty} x_n = x$. Let $\varepsilon > 0$ and $t > 0$, then there exists $N_0 > 0$ such that $1 - \varepsilon \leq N(x_n - x, t - \phi^{-1}(\gamma(t)\phi(t)))$, for all $n \geq N_0$. Hence

$$\begin{aligned} N(f(x) - x, t) &\geq \min\{N(f(x) - x_{N+1}, \phi^{-1}(\gamma(t)\phi(t))), \\ &\qquad N(x_{N+1} - x, t - \phi^{-1}(\gamma(t)\phi(t)))\} \\ &\geq \min\{N(x - x_N, t), N(x_{N+1} - x, t - \phi^{-1}(\gamma(t)\phi(t)))\} \\ &\geq 1 - \epsilon, \text{ for all } n \geq N_0. \end{aligned}$$

Therefore $N(f(x) - x, t) = 1$, for all $t > 0$. Hence $f(x) = x$.
To prove the uniqueness of the fixed point, we let $y$ be any other fixed point

of $f$ in $X$. If there exists $t>0$ such that $0<N(x-y,t)<1$, then

$$\begin{aligned} N(x-y,t) &\le N(f(x)-f(y),\phi^{-1}(\gamma(t)\phi(t))) \\ &\le N(f(x)-f(y),t) \\ &= N(x-y,t), \end{aligned}$$

therefore $N(x-y,\phi^{-1}(\gamma(t)\phi(t)))=N(x-y,t)$. By $(N7)$, $t=\phi^{-1}(\gamma(t)\phi(t))$, we get $\phi(t)=\gamma(t)\phi(t)$. Hence $\gamma(t)=1$, which is a contradiction. Thus $N(x-y,t)=1$, for all $t>0$, so $x=y$. $\square$

**Corollary 2.** *Let $(X,N)$ be a fuzzy Banach space such that $N$ satisfying (N7) and $\gamma:(0,+\infty)\longrightarrow[0,1)$ be a decreasing function, also $f:X\longrightarrow X$ be a selfmap such that for all $x,y\in X$ and $t>0$,*

$$N(f(x)-f(y),\phi^{-1}(\gamma(t)\phi(t))\ge N(x-y,t),$$

*where $\phi\in\Phi$. Then $f$ has a unique fixed point in $X$.*

**Example 3.** Let $(X,\|.\|)$ be a Banach space and $\gamma:(0,+\infty)\longrightarrow[0,1)$ be a decreasing function and $f:X\longrightarrow X$ be a function such that

$$\phi(\|f(x)-f(y)\|)\le\gamma(\|x-y\|)\phi(\|x-y\|),\ \text{for all } x,y\in X,$$

where $\phi\in\Phi$.
Assume that $\gamma\phi$ is a nondecreasing function and

$$\beta(\phi^{-1}(\gamma(t)\phi(t)))\le\phi^{-1}(\gamma(\beta t)\phi(\beta t)),\ \text{for all } t\in[0,+\infty),\ \beta\in[0,1].$$

Define a fuzzy norm $N$ as follows:

$$N(x,t)=\begin{cases} t/\|x\| & , \quad 0<t\le\|x\| \\ 1 & , \quad \|x\|<t \\ 0 & , \quad t\le 0. \end{cases}$$

Suppose that $x,y\in X$, $t>0$, $\alpha\in(0,1]$ and $N(x-y,t)\ge\alpha$.
Case1: Let $\|x-y\|<t$, since $\gamma\phi$ is nondecreasing,

$$\phi(\|f(x)-f(y)\|)\le\gamma(\|x-y\|)\phi(\|x-y\|)\le\gamma(t)\phi(t).$$

So $\|f(x)-f(y)\|\le\phi^{-1}(\gamma(t)\phi(t))$. Hence

$$N(f(x)-f(y),\phi^{-1}(\gamma(t)\phi(t)))=1\ge\alpha.$$

Case2: Let $0<t\le\|x-y\|$. So $t/\|x-y\|=N(x-y,t)\ge\alpha$. This implies that $\alpha\|x-y\|\le t$. Therefore

$$\begin{aligned}\alpha\|f(x)-f(y)\| &\leq \alpha(\phi^{-1}(\gamma(\|x-y\|)\psi(\|x-y\|)))\\ &\leq \phi^{-1}(\gamma(\alpha\|x-y\|)\phi(\alpha\|x-y\|))\\ &\leq \phi^{-1}(\gamma(t)\phi(t)).\end{aligned}$$

Thus $N(f(x)-f(y),\phi^{-1}(\gamma(t)\phi(t))) = (\phi^{-1}(\gamma(t)\phi(t)))/(\|f(x)-f(y)\|) \geq \alpha$. By Theorem 3, $f$ has a unique fixed point in $X$.

**Example 4.** Let $[0,1] = X$ and $\|x-y\| = |x-y|$, for all $x,y \in X$. Also let $f : X \longrightarrow X$, $\gamma : (0,+\infty) \longrightarrow [0,1)$ and $\phi : [0,+\infty) \longrightarrow [0,+\infty)$ be defined as

$$f(x) = x^3, \text{ for all } x \in X,$$

$$\phi(t) = (1/2)t^2, \text{ for all } t \geq 0,$$

$$\gamma(t) = 1/t \text{ for all } t > 0.$$

It is clear that $\gamma\phi$ is a nondecreasing function and $\beta(\phi^{-1}(\gamma(t)\phi(t))) \leq \phi^{-1}(\gamma(\beta t)\phi(\beta t))$, for all $t \in (0,+\infty)$, $\beta \in [0,1]$. Define a fuzzy norm $N$ as follows:

$$N(x,t) = \begin{cases} t/\|x\| & , \quad 0 < t \leq \|x\| \\ 1 & , \quad \|x\| < t \\ 0 & , \quad t \leq 0. \end{cases}$$

Suppose that $x,y \in X$. Without loss of generality, we assume that $x > y$. Then

$$\begin{aligned}\|f(x)-f(y)\| &= x^3 - y^3\\ &= (x-y)(x^2+xy+y^2)\\ &\leq (x-y)\\ &\leq \sqrt{x-y}\\ &= \phi^{-1}(\gamma(\|x-y\|)\phi(\|x-y\|)).\end{aligned}$$

By Example 3, $f$ has a unique fixed point in $X$.

**Theorem 4.** *Let $(X,N)$ be fuzzy Banach space such that $N$ satisfying (N7) and $f : X \longrightarrow X$ be a selfmap such that for all $x,y \in X$, $t > 0$ and $\alpha \in (0,1]$,*

$$N(x-y,t) \geq \alpha \text{ implies that } N(f(x)-f(y),\varphi^{-1}(\varphi(t)-\phi(t))) \geq \alpha,$$

*where $\phi,\varphi \in \Phi$ and $\varphi(t) \geq \phi(t)$, for all $t > 0$. Then $f$ has a unique fixed point in $X$.*

*Proof.* Let $x_0 \in X$ and $x_{n+1} = f(x_n)$, for all $n \in \mathbb{N}$. Suppose that $t > 0$, we have

$$N(x-y,t) \leq N(f(x)-f(y),\varphi^{-1}(\varphi(t)-\phi(t))).$$

Therefore

$$N(x_{n+1}-x_n,t) \le N(x_{n+2}-x_{n+1},\varphi^{-1}(\varphi(t)-\phi(t))) \le N(x_{n+2}-x_{n+1},t),$$

for all $n \in \mathbb{N}$. Hence $\{N(x_{n+1}-x_n,t)\}$ is a bounded and nondecreasing sequence, and $\lim_{n\to\infty} N(x_{n+1}-x_n,t)$ exists.
Assume that there exists $t>0$ such that $\lim_{n\to\infty} N(x_{n+1}-x_n,t) < \beta < 1$. Since $N(x_{n+2}-x_{n+1},s) \ge N(x_{n+1}-x_n,s)$, for all $s>0$, then

$$0 < t \le \|x_{n+2}-x_{n+1}\|_\beta \le \|x_{n+1}-x_n\|_\beta, \text{ for all } n \in \mathbb{N}.$$

Therefore $\lim_{n\to\infty}\|x_{n+1}-x_n\|_\beta$ exists. Let $\lim_{n\to\infty}\|x_{n+1}-x_n\|_\beta = b \ge t > 0$. If $N(x_{n+1}-x_n,s) \ge \beta$ then

$$N(x_{n+2}-x_{n+1},\varphi^{-1}(\varphi(s)-\phi(s))) \ge N(x_{n+1}-x_n,s) \ge \beta,$$

and $\|x_{n+2}-x_{n+1}\|_\beta \le \varphi^{-1}(\varphi(s)-\phi(s))$. Thus

$$\varphi(\|x_{n+2}-x_{n+1}\|_\beta) \le \varphi(s)-\phi(s).$$

As $s \longrightarrow \|x_{n+1}-x_n\|_\beta$, one can get

$$\varphi(\|x_{n+2}-x_{n+1}\|_\beta) \le \varphi(\|x_{n+1}-x_n\|_\beta)-\phi(\|x_{n+1}-x_n\|_\beta).$$

As $n \longrightarrow \infty$, we obtain that $0 < \varphi(t) \le \varphi(b) \le \varphi(b)-\phi(b) < \varphi(b)$, which is a contradiction. Hence $\lim_{n\to\infty} N(x_{n+1}-x_n,t) = 1$, for all $t>0$.
Let $t>0$, $\epsilon>0$ and $N \in \mathbb{N}$ such that

$$1-\epsilon \le N(x_{N+1}-x_N,t/2-\varphi^{-1}(\varphi(t/2)-\phi(t/2))).$$

If $1-\epsilon \le N(x-x_N,t/2)$, then

$$\begin{aligned}N(f(x)-x_N,t/2) &\ge \min\{N(f(x)-x_{N+1},\varphi^{-1}(\varphi(t/2)-\phi(t/2))),\\ &\qquad N(x_{N+1}-x_N,t/2-\varphi^{-1}(\varphi(t/2)-\phi(t/2)))\}\\ &\ge \min\{N(x-x_N,t/2),\\ &\qquad N(x_{N+1}-x_N,t/2-\varphi^{-1}(\varphi(t/2)-\phi(t/2)))\}\\ &\ge 1-\epsilon.\end{aligned}$$

Therefore $1-\epsilon \le N(x_n-x_N,t/2)$, for all $n \ge N$, so

$$N(x_n-x_m,t) \ge \min\{N(x_n-x_N,t/2),N(x_m-x_N,t/2)\} \ge 1-\epsilon,$$

for all $n,m \ge N$. Since $\epsilon$ is arbitrary, $\{x_n\}$ is Cauchy, hence it is convergent. Assume that $\lim_{n\to\infty} x_n = x$. Let $\varepsilon>0$ and $t>0$, then there exists $N_0>0$ such that $1-\varepsilon \le N(x_n-x,t-\varphi^{-1}(\varphi(t)-\phi(t)))$, for all $n \ge N_0$. Hence

$$\begin{aligned}N(f(x)-x,t) &\geq \min\{N(f(x)-x_{N+1},\varphi^{-1}(\varphi(t)-\phi(t))),\\ &\qquad N(x_{N+1}-x,t-\varphi^{-1}(\varphi(t)-\phi(t)))\}\\ &\geq \min\{N(x-x_N,t),N(x_{N+1}-x,t-\phi^{-1}(\gamma(t)\phi(t)))\}\\ &\geq 1-\epsilon,\ \text{for all } n\geq N_0.\end{aligned}$$

Therefore $N(f(x)-x,t)=1$, for all $t>0$, so $f(x)=x$.
To prove the uniqueness of the fixed point, we let $y$ be any other fixed point of $f$ in $X$. If there exists $t>0$ such that $0<N(x-y,t)<1$ then

$$\begin{aligned}N(x-y,t) &\leq N(f(x)-f(y),\varphi^{-1}(\varphi(t)-\phi(t)))\\ &\leq N(f(x)-f(y),t)\\ &= N(x-y,t),\end{aligned}$$

therefore $N(x-y,\phi^{-1}(\gamma(t)\phi(t)))=N(x-y,t)$. By $(N7)$, we obtain that $t=\varphi^{-1}(\varphi(t)-\phi(t))$, then $\varphi(t)=\varphi(t)-\phi(t)$. Hence $\phi(t)=0$, and $t=0$, which is a contradiction. Thus $N(x-y,t)=1$, for all $t>0$, so $x=y$. □

**Corollary 3.** *Let $(X,N)$ be fuzzy Banach space such that $N$ satisfying $(N7)$ and $f:X\longrightarrow X$ be a selfmap such that for all $x,y\in X$ and $t>0$,*

$$N(f(x)-f(y),\varphi^{-1}(\varphi(t)-\phi(t)))\geq N(x-y,t),$$

*where $\phi,\varphi\in\Phi$ and $\varphi(t)\geq\phi(t)$, for all $t>0$. Then $f$ has a unique fixed point in $X$.*

**Example 5.** Let $(X,\|.\|)$ be a Banach space and $f:X\longrightarrow X$ be a function such that

$$\varphi(\|f(x)-f(y)\|)\leq\varphi(\|x-y\|)-\phi(\|x-y\|),\ \text{for all } x,y\in X,$$

where $\varphi,\phi\in\Phi$. Assume that $\varphi-\phi$ is a nondecreasing function and

$$\beta(\varphi^{-1}(\varphi(t)-\phi(t)))\leq\varphi^{-1}(\varphi(\beta t)-\phi(\beta t)),\ \text{for all } t\in[0,+\infty),\ \beta\in[0,1].$$

Define a fuzzy norm $N$ as follows:

$$N(x,t)=\begin{cases} t/\|x\| & , \quad 0<t\leq\|x\|\\ 1 & , \quad \|x\|<t\\ 0 & , \quad t\leq 0.\end{cases}$$

Suppose that $x,y\in X$, $t>0$, $\alpha\in(0,1]$ and $N(x-y,t)\geq\alpha$.
Case1: Let $\|x-y\|<t$. Since $\varphi-\phi$ is nondecreasing,

$$\varphi(\|f(x)-f(y)\|)\leq\varphi(\|x-y\|)-\phi(\|x-y\|)\leq\varphi(t)-\phi(t).$$

So $\|f(x)-f(y)\|\leq\varphi^{-1}(\varphi(t)-\phi(t))$. Hence

$$N(f(x)-f(y), \varphi^{-1}(\varphi(t)-\phi(t))) = 1 \geq \alpha.$$

Case2: Let $0 < t \leq \|x-y\|$. So $t/\|x-y\| = N(x-y,t) \geq \alpha$. Hence $\alpha\|x-y\| \leq t$, and

$$\begin{aligned}\alpha\|f(x)-f(y)\| &\leq \alpha(\varphi^{-1}(\varphi(\|x-y\|)-\psi(\|x-y\|)))\\ &\leq \varphi^{-1}(\varphi(\alpha\|x-y\|)-\phi(\alpha\|x-y\|))\\ &\leq \varphi^{-1}(\varphi(t)-\phi(t))\end{aligned}$$

Thus

$$N(f(x)-f(y), \varphi^{-1}(\varphi(t)-\phi(t))) = (\varphi^{-1}(\varphi(t)-\phi(t)))/(\|f(x)-f(y)\|) \geq \alpha$$

By Theorem 4, $f$ has a unique fixed point in $X$.

**Theorem 5.** *Let $(X,N)$ be fuzzy Banach space such that $N$ satisfying $(N7)$ and $f : X \longrightarrow X$ be a selfmap such that for all $x,y \in X$, $s,t > 0$ and $\alpha \in (0,1]$,*

$$N(x-f(y),t) \geq \alpha \text{ and } N(f(x)-y,s) \geq \alpha \text{ implies that}$$

$$N(f(x)-f(y), 1/2(t+s)-\theta(t,s)) \geq \alpha,$$

*where $\theta : [0,+\infty)^2 \longrightarrow [0,+\infty)$ is a continuous mapping such that $\theta(x,y)=0$ if and only if $x=y=0$. Then $f$ has a unique fixed point in $X$.*

*Proof.* Let $x_0 \in X$ and $x_{n+1} = f(x_n)$, for all $n \in \mathbb{N}$. Suppose that $t > 0$, so

$$N(f(x)-f(y), 1/2(t+s)-\theta(t,s)) \geq \min\{N(x-f(y),t), N(f(x)-y,s)\},$$

for all $s > 0$. Therefore

$$\begin{aligned}N(x_{n+1}-x_n, 1/2(2t+s)-\theta(2t,s)) &\geq \min\{N(x_n-x_n,s),\\ &\qquad N(x_{n+1}-x_{n-1},2t)\}\\ &= N(x_{n+1}-x_{n-1},2t),\end{aligned}$$

for all $s > 0$. As $s \longrightarrow 0$, we obtain that

$$\begin{aligned}N(x_{n+1}-x_n, t-\theta(2t,0)) &\geq N(x_{n+1}-x_{n-1},2t)\\ &\geq \min\{N(x_{n+1}-x_n,t), N(x_n-x_{n-1},t)\},\end{aligned}$$

for all $n \in \mathbb{N}$ and all $t > 0$. Let there exists $t_0 > 0$ and $n_0 \in \mathbb{N}$ such that $N(x_{n_0+1}-x_{n_0},t_0) < N(x_{n_0}-x_{n_0-1},t_0)$. By $(N7)$ and $(N5)$, there is $t_1 > 0$ such that

$$0 \leq N(x_{n_0+1}-x_{n_0},t_0) < N(x_{n_0+1}-x_{n_0},t_0+t_1) < N(x_{n_0}-x_{n_0-1},t_0) \leq 1$$

Hence

$$N(x_{n+1}-x_n,(t_0+t_1)) \geq N(x_{n+1}-x_n,(t_0+t_1)-\theta(2(t_0+t_1),0)) \\ \geq N(x_{n+1}-x_n,(t_0+t_1)).$$

Thus $N(x_{n+1}-x_n,(t_0+t_1)) = N(x_{n+1}-x_n,(t_0+t_1)-\theta(2(t_0+t_1),0))$. By $(N7)$, $(t_0+t_1)-\theta(2(t_0+t_1),0) = t_0+t_1$. So $\theta(2(t_0+t_1),0)=0$ which is a contradiction. In addition $N(x_{n+1}-x_n,t) \geq N(x_n-x_{n-1},t)$, for all $t>0$ and all $n\in\mathbb{N}$. Therefore $\lim_{n\to\infty} N(x_{n+1}-x_n,t)$ exists.
Assume that there exists $t>0$ such that $\lim_{n\to\infty} N(x_{n+1}-x_n,t) < \beta < 1$. Since $N(x_{n+2}-x_{n+1},s) \geq N(x_{n+1}-x_n,s)$, for all $s>0$, it follows that

$$0 < t \leq \|x_{n+2}-x_{n+1}\|_\beta \leq \|x_{n+1}-x_n\|_\beta, \text{ for all } n\in\mathbb{N}.$$

Thus $\lim_{n\to\infty}\|x_{n+1}-x_n\|_\beta$ exists. Let $\lim_{n\to\infty}\|x_{n+1}-x_n\|_\beta = b \geq t > 0$. If $N(x_{n+1}-x_n,s)\geq\beta$, then $N(x_{n+2}-x_{n+1},s-\theta(2s,0)) \geq N(x_{n+1}-x_n,s)\geq\beta$. So $\|x_{n+2}-x_{n+1}\|_\beta \leq s-\theta(2s,0)$. As $s\longrightarrow\|x_{n+1}-x_n\|_\beta$, we obtain that $\|x_{n+2}-x_{n+1}\|_\beta \leq \|x_{n+1}-x_n\|_\beta - \theta(2\|x_{n+1}-x_n\|_\beta,0)$. As $n\longrightarrow\infty$, we get $0<t\leq b\leq b-\theta(2b,0)\leq b$ and $\theta(2b,0)=0$, which is a contradiction. Hence $\lim_{n\to\infty} N(x_{n+1}-x_n,t)=1$, for all $t>0$.
Next we show that $\{x_n\}$ is a cauchy sequence. If otherwise, then there exist $t_0>0$, $\beta\in(0,1)$ and increasing sequences of integers $\{m_k\}$ and $\{n_k\}$ such that $N(x_{n_k}-x_{m_k},t_0)<\beta$ and $N(x_{n_k-1}-x_{m_k},t_0)\geq\beta$, for all $k\in\mathbb{N}$. Since $\lim_{n\to\infty} N(x_{n+1}-x_n,t)=1$, for all $t>0$, it follows that

$$\lim_{n\to\infty}\|x_{n_k}-x_{n_k-1}\|_\beta = 0 = \lim_{n\to\infty}\|x_{m_k}-x_{m_k-1}\|_\beta.$$

Moreover

$$\begin{aligned} 0 < t_0 &\leq \|x_{n_k}-x_{m_k}\|_\beta \\ &\leq \|x_{n_k-1}-x_{n_k}\|_\beta + \|x_{m_k}-x_{n_k-1}\|_\beta \\ &\leq t_0 + \|x_{n_k}-x_{n_k-1}\|_\beta, \end{aligned}$$

for all $k\in\mathbb{N}$. As $k\longrightarrow\infty$, which leads to

$$\lim_{n\to\infty}\|x_{n_k}-x_{m_k}\|_\beta = t_0 = \lim_{n\to\infty}\|x_{m_k}-x_{n_k-1}\|_\beta.$$

Now we have

$$\|x_{m_k}-x_{n_k}\|_\beta \leq \|x_{m_k-1}-x_{m_k}\|_\beta + \|x_{m_k-1}-x_{n_k}\|_\beta + \|x_{n_k}-x_{n_k-1}\|_\beta$$

and

$$\|x_{m_k-1}-x_{n_k}\|_\beta \leq \|x_{m_k-1}-x_{m_k}\|_\beta + \|x_{m_k}-x_{n_k}\|_\beta,$$

for all $k \in \mathbb{N}$. As $k \longrightarrow \infty$, we obtain that $\lim_{n\to\infty} \|x_{n_k} - x_{m_k-1}\|_\beta = t_0$. If $N(x_{n_k-1} - x_{m_k}, t) \geq \beta$ and $N(x_{n_k} - x_{m_k-1}, s) \geq \beta$ then

$$N(x_{n_k} - x_{m_k}, 1/2(t+s) - \theta(t,s)) \geq \beta.$$

Hence

$$\|x_{m_k} - x_{n_k}\|_\beta \leq 1/2(t+s) - \theta(t,s).$$

As $t \longrightarrow \|x_{m_k} - x_{n_k-1}\|_\beta$ and $s \longrightarrow \|x_{m_k-1} - x_{n_k}\|_\beta$, we get

$$\|x_{m_k} - x_{n_k}\|_\beta \leq$$

$$1/2(\|x_{m_k} - x_{n_k-1}\|_\beta + \|x_{m_k-1} - x_{n_k}\|_\beta) - \theta(\|x_{m_k} - x_{n_k-1}\|_\beta, \|x_{m_k-1} - x_{n_k}\|_\beta),$$

for all $k \in \mathbb{N}$. As $k \longrightarrow \infty$, we have $t_0 \leq 1/2(t_0 + t_0) - \theta(t_0, t_0)$. Therefore $\theta(t_0, t_0) = 0$ which is a contradiction. Thus $\{x_n\}$ is Cauchy, hence convergent. Assume that $\lim_{n\to\infty} x_n = x$.

Next we show that $\lim_{n\to\infty} N(f(x) - x_n, t) = 1$, for all $t > 0$. If otherwise, then there exist $t_0 > 0$, $\beta \in (0,1)$ and increasing sequences of integers $\{n_k\}$ such that $N(f(x) - x_{n_k}, t_0) < \beta$, for all $k \in \mathbb{N}$. Since $\lim_{n\to\infty} N(x - x_n, t) = 1$, for all $t > 0$, it follows that $\lim_{n\to\infty} \|x_n - x\|_\beta = 0$. We have

$$\|f(x) - x_n\|_\beta \leq \|x - x_n\|_\beta + \|f(x) - x\|_\beta$$

and

$$\|f(x) - x\|_\beta \leq \|f(x) - x_n\|_\beta + \|x_n - x\|_\beta,$$

for all $n \in \mathbb{N}$. As $n \longrightarrow \infty$, we obtain that $\lim_{n\to\infty} \|f(x) - x_n\|_\beta = \|f(x) - x\|_\beta$. If $N(x_{n_k-1} - f(x), t) \geq \beta$ and $N(x_{n_k} - x, s) \geq \beta$, then

$$\begin{aligned} N(f(x) - x_{n_k}, 1/2(t+s) - \theta(t,s)) &\geq \min\{N(x - x_{n_k}, t), N(f(x) - x_{n_k-1}, s)\} \\ &\geq \beta. \end{aligned}$$

Therefore $\|f(x) - x_{n_k}\|_\beta \leq 1/2(t+s) - \theta(t,s)$. As $t \longrightarrow \|f(x) - x_{n_k-1}\|_\beta$ and $s \longrightarrow \|x - x_{n_k}\|_\beta$, we get

$$0 < t_0 \leq \|f(x) - x_{n_k}\|_\beta \leq$$

$$1/2(\|f(x) - x_{n_k-1}\|_\beta + \|x - x_{n_k}\|_\beta) - \theta(\|f(x) - x_{n_k-1}\|_\beta, \|x - x_{n_k}\|_\beta),$$

for all $k \in \mathbb{N}$. As $k \longrightarrow \infty$, one can obtain

$$0 < t_0 \leq \|f(x) - x\|_\beta \leq 1/2\|f(x) - x\|_\beta - \theta(\|f(x) - x\|_\beta, 0).$$

So $\theta(\|f(x)-x\|_\beta, 0) = 0$, which is a contradiction. This implies that $\lim_{n\to\infty} N(f(x)-x_n, t) = 1$, for all $t > 0$. By $(N4)$, we have

$$N(f(x)-x,t) \geq \min\{N(f(x)-x_n, t/2), N(x-x_n, t/2)\}, \text{ for all } t > 0.$$

As $n \longrightarrow \infty$, we obtain that $N(f(x)-x,t) = 1$, for all $t > 0$. Hence $f(x) = x$. To prove the uniqueness of the fixed point, let $y$ be any other fixed point of $f$ in $X$. If there exists $t > 0$ such that $0 < N(x-y,t) < 1$, then

$$\begin{aligned} N(x-y,t) &= N(f(x)-f(y),t) \\ &\geq N(f(x)-f(y), t-\theta(t,t)) \\ &\geq \min\{N(x-f(y),t), N(f(x)-y,t)\} \\ &= \min\{N(x-y,t), N(x-y,t)\} \\ &= N(x-y,t). \end{aligned}$$

Therefore $N(x-y, t-\theta(t,t)) = N(x-y,t)$. By $(N7)$, $\theta(t,t) = 0$ which is a contradiction. Thus $N(x-y,t) = 1$, for all $t > 0$. So $x = y$. □

**Corollary 4.** *Let $(X,N)$ be fuzzy Banach space such that $N$ satisfying $(N7)$. Also $f : X \longrightarrow X$ be a selfmap such that for all $x, y \in X$ and $s, t > 0$,*

$$N(f(x)-f(y), 1/2(t+s)-\theta(t,s)) \geq \min\{N(x-f(y),t), N(f(x)-y,s)\},$$

*where $\theta : [0,+\infty)^2 \longrightarrow [0,+\infty)$ is a continuous mapping such that $\theta(x,y) = 0$ if and only if $x = y = 0$. Then $f$ has a unique fixed point in $X$.*

**Example 6.** Let $(X, \|.\|)$ be a Banach space and $f : X \longrightarrow X$ be a function such that

$$\|f(x)-f(y)\| \leq 1/2(\|x-f(y)\| + \|f(x)-y\|) - \theta(\|x-f(y)\|, \|f(x)-y\|),$$

for all $x, y \in X$, where $\theta : [0,+\infty)^2 \longrightarrow [0,+\infty)$ is a continuous mapping such that $\theta(x,y) = 0$ if and only if $x = y = 0$.
If $s_1 \leq s_2$ and $t_1 \leq t_2$ then $1/2(t_1+s_1) - \theta(t_1,s_1) \leq 1/2(t_2+s_2) - \theta(t_2,s_2)$, for all $s_1, t_1, s_2, t_2 > 0$ and $\theta(\beta t, \beta s) \leq \beta\theta(t,s)$, for all $t, s > 0$, $\beta \in [0,1]$. Define a fuzzy norm $N$ as follows:

$$N(x,t) = \begin{cases} t/\|x\| &, \quad 0 < t \leq \|x\| \\ 1 &, \quad \|x\| < t \\ 0 &, \quad t \leq 0. \end{cases}$$

Suppose that $x, y \in X$, $t > 0$, $\alpha \in (0,1]$, $N(x-f(y),t) \geq \alpha$ and $N(f(x)-y,s) \geq \alpha$.
Case1: Let $\|x-f(y)\| < t$ and $\|f(x)-y\| < s$. Then

$$\begin{aligned}\|f(x)-f(y)\| &\le 1/2(\|x-f(y)\|+\|f(x)-y\|)-\theta(\|x-f(y)\|,\|f(x)-y\|)\\ &\le 1/2(t+s)-\theta(t,s).\end{aligned}$$

Thus $N(f(x)-f(y),1/2(t+s)-\theta(t,s))=1\ge\alpha$.
Case2: Let $0<t\le\|x-f(y)\|$ and $\|f(x)-y\|<s$. So $t/\|x-f(y)\|=N(x-y,t)\ge\alpha$ and $\|x-f(y)\|<s$. Hence $\alpha\|x-f(y)\|\le t$ and $\|f(x)-y\|<s$. Therefore

$$\begin{aligned}\alpha\|f(x)-f(y)\| &\le \alpha(1/2(\|x-f(y)\|+\|f(x)-y\|)-\theta(\|x-f(y)\|,\\ &\quad \|f(x)-y\|))\\ &\le 1/2(\alpha\|x-f(y)\|+\alpha\|f(x)-y\|)-\theta(\alpha\|x-f(y)\|,\\ &\quad \alpha\|f(x)-y\|)\\ &\le 1/2(t+s)-\theta(t,s).\end{aligned}$$

Thus

$$N(f(x)-f(y),1/2(t+s)-\theta(t,s))=(1/2(t+s)-\theta(t,s))/(\|f(x)-f(y)\|)\ge\alpha.$$

Case3: Let $0<t\le\|x-f(y)\|$ and $0<s\le\|f(x)-y\|$. So $t/\|x-y\|=N(x-y,t)\ge\alpha$ and $s/\|f(x)-y\|=N(f(x)-y,s)\ge\alpha$. Hence $\alpha\|x-f(y)\|\le t$ and $\alpha\|f(x)-y\|<s$. Therefore

$$\begin{aligned}\alpha\|f(x)-f(y)\| &\le \alpha(1/2(\|x-f(y)\|+\|f(x)-y\|)-\theta(\|x-f(y)\|,\\ &\quad \|f(x)-y\|))\\ &\le 1/2(\alpha\|x-f(y)\|+\alpha\|f(x)-y\|)-\theta(\alpha\|x-f(y)\|,\\ &\quad \alpha\|f(x)-y\|)\\ &\le 1/2(t+s)-\theta(t,s).\end{aligned}$$

Thus

$$N(f(x)-f(y),1/2(t+s)-\theta(t,s))=(1/2(t+s)-\theta(t,s))/(\|f(x)-f(y)\|)\ge\alpha.$$

Case4: Let $\|x-f(y)\|<t$ and $0<s\le\|f(x)-y\|$. Similar to case2, we obtain that $N(f(x)-f(y),1/2(t+s)-\theta(t,s))\ge\alpha$.
By Theorem 5, $f$ has a unique fixed point in $X$.

## 3 Conclusion

We have introduced four contractive conditions in fuzzy normed linear spaces and proved some results about fixed point theorem. In fact, the established properties are the extended fuzzy forms of some classical contractive properties. To reveal this fact, some examples have been studied.

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**توابع انقباض روی فضاهای نرم دار فازی**

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**چکیده :** در این مقاله، ما از تعریف فضای نرم دار فازی که توسط بگ و سامانتا ارایه شده است استفاده کرده ایم و چهار نوع از توابع انقباض فازی را معرفی کرده ایم . ما نشان داده ایم که این توابع روی فضاهای نرم دار فازی لزوما دارای نقطه ثابت منحصر به فرد می باشد و نشان خواهیم داد که این قضایای ارایه شده در واقع توسیع فازی قضایای کلاسیک می باشند

**کلمات کلیدی :** نرم فازی؛ فضای نرم دار فازی؛ نقطه ثابت؛ $\alpha$-نیم نرم؛ شرایط انقباض.