Iranian Journal of Numerical Analysis and Optimization
Vol. 7, No. 2, (2017), pp 65–85
DOI:10.22067/ijnao.v7i2.60552

The state of the art in Dynamic Relaxation methods for structural mechanics

Part 1: Formulations

M. Rezaiee-pajand*, J. Alamatian, and H. Rezaee

Abstract

In the last sixty years, the dynamic relaxation (DR) methods have evolved significantly. These explicit and iterative procedures are frequently used to solve the linear or nonlinear response of governing equations resulted from structural analyses. In the first part of this study, the common DR formulations are reviewed. Mathematical bases and also physical concepts of these solvers are explained briefly. All the DR parameters, i.e. fictitious mass, fictitious damping, fictitious time step and initial guess are described, as well. Furthermore, solutions of structural problems along with kinetic and viscous damping formulations are discussed. Analyses of the existing studies and suggestions for future research trends are presented. In the second part, the applications of dynamic relaxation method in engineering practices are reviewed.

Keywords: Dynamic Relaxation method; Formulation; Solver; Review; Iterative technique; State of the art.

---

*Corresponding author
Received 25 November 2016; revised 12 May 2017; accepted 23 April 2017
M. Rezaiee-pajand
Department of Civil Engineering, Factualy of Engineering, Ferdowsi Unversity of Mashhad, Mashhad, Iran. e-mail: rezaiee@um.ac.ir

J. Alamatian
Civil Engineering Department, Mashhad Branch, Islamic Azad Unversity, Mashhad, Iran. e-mail: alamatian@mshadiau.ac.ir

H. Rezaee
Department of Civil Engineering, Factualy of Engineering, Ferdowsi Unversity of Mashhad, Mashhad, Iran. e-mail: hasina1383@yahoo.com

Symbols

A_1^{n+1},A_2^{n+1}, A_3^{n+1},α,β	formulation parameters
c	fictitious damping factor
$[C]^n$	artificial damping matrix in nth iteration
$\{D\}$	vector of displacement
$\left\{\dot{D}\right\}^n$	artificial velocity vector in nth iteration
$\left\{\ddot{D}\right\}^n$	artificial acceleration vector in nth iteration
$\{f\}$	vector of internal forces
$\{f_I\}$	fictitious inertia force vector
$\{f_D\}$	fictitious damping force vector
$\left\{\dot{f}\right\}$	vector of internal force increment
$[I]$	unit matrix
L	Laplace operator
$[M]^n$	artificial mass matrix in nth iteration
MFT	modified fictitious time step
OFT	optimum fictitious time step
$\{P\}$	vector of external loads
$\{P\}_{EQ}^{k+1}$	equivalent load vector at $k+1^{th}$ step of numerical dynamic integration
q	number of degrees of freedom
$\{R\}$	vector of un-balance force
$\Re$	rectangular region
$[S]$	stiffness matrix
$[S]_{EQ}^{k+1}$	equivalent stiffness matrix at $k+1^{th}$ step of numerical dynamic integration
U_k	kinetic energy of fictitious dynamic system
UEF	un-balance energy function
UFF	un-balance force function
ρ	fictitious mass factor
τ	fictitious time step
ω_{min}	the lowest natural frequency of artificial dynamic system
Superscripts k	increment's number of dynamic analysis
n	iteration number of DR method
Subscripts i	each degree of freedom of structure

1 Introduction

Final stage of any structural analysis leads to the solution of a simultaneous system of equations. In other words, the responses of structures can be

computed by solving the following equations, concluding from finite element or finite difference schemes:

$$[S]\{D\} = \{f\} = \{P\} \tag{1}$$

where $[S]$ is structural stiffness matrix, $\{D\}$, $\{f\}$ and $\{P\}$ are displacement, internal force and external load vectors, respectively. In nonlinear analyses, such as geometrical, material or contact nonlinearity, internal force vector is a nonlinear function of displacement, i.e. the elements of the stiffness matrix are not constant. The analytical and iterative procedures can be used for solving Eq. (1). In the solution of linear systems, the direct or analytical methods, such as Gaussian elimination technique, Cholesky factorization and other similar procedures, are generally more effective and economic than the iterative strategies. However, these approaches cannot be utilized for nonlinear cases individually, so that they should be combined with other corrective processes to present the exact answer. It is worth emphasizing that such corrective procedures belong to the iterative methods in which nonlinear step is divided and traced by several linear domains. This approach causes some numerical errors and reduces the efficiency of the analysis. As a result, the most suitable and effective ways of solving Eq. (1), especially in the nonlinear analysis, are iterative procedures.

The first major numerical effort to utilize the finite difference method in conjunction with a relaxation scheme was performed by Allen and Southwell [5]. These researchers solved the elastic-plastic problems of a continuum. In iterative schemes, an estimation of the answer is corrected by repeating some cycles. These techniques can be classified as implicit or explicit schemes, as acknowledged by Felippa [14] and Frankel [15]. Explicit techniques use residual force to get an answer, so that all calculations are performed by vector operations. Simplicity and high efficiency in the nonlinear analysis are the main specifications of the explicit techniques. On the other hand, implicit procedures are formulated based on residual force derivatives (stiffness matrix). Because of using matrix operations, calculations of implicit strategies are complex and time consuming. For example, snap-through or snap-back points, in which stiffness matrix is zero or undefined, cause some difficulties. However, the convergence rate of implicit methods is greater than the explicit schemes.

More than six decades have been passed since the dawn of the Dynamic Relaxation (DR) methods. A lot of efforts and considerable investigations have been directed toward the formulations and applications of these powerful approaches up to now. Unfortunately, a comprehensive survey related to these solution techniques has not been published so far. The purpose of this study is to report the state of the art on different aspects of the DR schemes. This literature review will be beneficial to researchers who are interested in the field of Dynamic Relaxation method. It will provide them with an insight on how this technique was developed, how it has progressed over

the years and how it has been implemented in solving different problems. In the first part of this study, the common DR formulations are reviewed. Mathematical bases and also physical concepts of the Dynamic Relaxation are presented. These foundations are discussed to clarify the main specifications of the solvers. All the DR parameters, such as fictitious mass, fictitious damping, fictitious time step and initial guess are described. The required relationships to solve the structural problems are given. Both kinetic and viscous damping formulations are discussed. Finally, numerical steps require to take advantage of the presented formulas will be explained. In the second part, the applications of DR methods in engineering practices are reviewed.

2 Mathematical Basis

Since the development of high-speed computers, the various iterative procedures such as, the Southwell relaxation method has been successfully applied for solving partial differential equations. As it was reported by Frankel, solving the Laplace equations in a rectangular region is one of the first applications of these techniques which use finite difference approximations [15];

$$L\phi_{(x,y)} = 0, \quad x, y \in \Re, \tag{2}$$

In this relationship, $\Re$ is a rectangular region. The values of $\phi_{(x,y)}$are calculated from the iterative process so that Eq. (2) is satisfied. To simplify the operations, the finite difference is used to expand Eq. (2) in the region $\Re$. This procedure is started by a first initial guess and other successive approximations are calculated based on steps of the algorithm. One of the most elementary ways of the relaxation schemes is the Richardson method. In this technique, the correction process has the following formula [15]:

$$\phi^{n+1}_{(x,y)} = \phi^{n}_{(x,y)} + \alpha L\phi^{n}_{(x,y)}, \quad x, y \in \Re, \tag{3}$$

where α is a coefficient, which verifies the convergence criteria. It can be proven that the Richardson technique is formally equivalent to the solution of the subsequent partial differential equation, with respect to the fictitious time variable:

$$\frac{\partial \phi}{\partial t} = L\phi_{(x,y)}, \quad x, y \in \Re, \tag{4}$$

In the last relation, t is a fictitious time. The values of $\phi_{(x,y)}$ are calculated in each fictitious time. If the solution being carried to a sufficiently great t, so that the rate of change with respect to τ is negligible $\left(\frac{\partial \phi}{\partial \tau} = 0\right)$, the answer is obtained. Some other strategies, such as the Liebmann scheme, have a similar basis. These approaches are categorized as the first-order Richardson techniques. It should be added that the second- order Richardson methods have also been formulated to solve differential equations. Using these pro-

cedures for solving Laplace's equations (Eq. 2), is formally equivalent to the solution of the following second order partial differential equation, which contains the derivatives with respect to the fictitious time [15];

$$\frac{\partial^2 \phi}{\partial \tau^2} + \beta \frac{\partial \phi}{\partial \tau} = L\phi_{(x,y)}, \quad x, y \in \Re, \tag{5}$$

In the former relationship, β controls the stability and convergence rate. The values of $\phi_{(x,y)}$are calculated in each time. The answer is obtained when the first and second derivatives of $\phi_{(x,y)}$, with respect to t, come near to zero $\left(\frac{\partial^2 \phi}{\partial \tau^2} = 0,\ \frac{\partial \phi}{\partial \tau} = 0\right)$.

From mathematical point of view, the DR method is based on the second-order Richardson rule. The first use of *Dynamic Relaxation* topic in scientific literatures has been performed by Otter or Day [12, 18]. In this process, simulations system of equations (Eq. 1) is transferred to the second-order differential system of equations. This differential system of equations, in terms of the fictitious time, has the following form:

$$\frac{\partial^2 \left([S]\{D\} - \{P\}\right)}{\partial \tau^2} + \beta \frac{\partial \left([S]\{D\} - \{P\}\right)}{\partial \tau} = [S]\{D\} - \{P\} \tag{6}$$

Once again, it is clear that the answer is obtained when the first and second derivatives with respect to the fictitious time become zero.

3 Physical Concepts

The DR method can be simply demonstrated by the physical rules. From this viewpoint, the structural dynamics theories are utilized to clarify the validity of DR basis. It is well-known that the dynamic response of each structure has two discrete portions; i. e. transient response and steady-state response. If the structure has damping effect, the transient response will be zero after passing some periods of time. By damping the transient response, the steady-state response remains. When the structure is excited by a dynamic load, which is variable with respect to the time, the steady-state response is a function of time. If the external load is constant during analysis time, such as the static behavior of structure, the steady-state response is equal to the answer of static analysis. The viscous DR method is based on a similar idea. In other words, the DR procedure, which is known as Viscous Dynamic Relaxation, presents the steady-state response of a fictitious dynamic system which is excited by static loads. Based on this discussion, the steady-state response of such a system is equal to the static response. To start DR process, the static system (Eq. 1) is transferred to the dynamic space by adding fictitious inertia and damping forces, as follows:

$$\{f_I\} + \{f_D\} + [S]\{D\} = \{P\} \tag{7}$$

Here, $\{f_I\}$ and $\{f_D\}$ are vectors of fictitious inertia and damping forces, respectively. These quantities are defined by the subsequent relationships:

$$\{f_I\} = [M]\left\{\ddot{D}\right\} \tag{8}$$

$$\{f_D\} = [C]\left\{\dot{D}\right\} \tag{9}$$

where $[M]$ and $[C]$ are diagonal fictitious mass and fictitious damping matrices, respectively. In the last equations, the dots denote the derivatives with respect to the fictitious time.

On the other hand, there is the Kinetic DR method which has been proposed on a different and interesting physical idea. Based on this concept, the total energy of each structure, which combines from the potential and kinetic energies, is constant. In the static equilibrium state, the potential energy is minimized, and hence the kinetic energy should be maximum. The kinetic Dynamic Relaxation method utilizes this idea and traces some successive peak in the kinetic energy of structure, which finally leads to the static equilibrium. The fundamental relationships of kinetic DR approach are very similar to the viscous version. The main difference is that kinetic DR iterations trace the un-damped vibrations of the dynamic system. The related fictitious dynamic equation has the following form:

$$\{f_I\} + [S]\{D\} = \{P\} \tag{10}$$

In the last relationship, inertia force vector is defined by Eq. (8). As discussed earlier, there are two general types of damping, which can be utilized in DR process: *Kinetic damping* and *Viscous damping*. The DR formulation depends on the type of damping model. In other words, the different fundamental DR relations can be obtained based on two types of damping model. Each of these methods has a different specification with many interesting performances. In the following sections, the fundamental DR formulations for kinetic damping and viscous damping are reviewed. All of these approaches utilize the explicit iterative relationships.

3.1 Viscous Damping Formulation

Several common DR algorithms, such as Papadrakakis [19], Underwood [33], are based on the viscous damping process. This kind of modeling is closer to the real dynamic system than the kinetic damping one. To establish the related equation, damping force is defined by utilizing the viscous damping theory, as it was shown in Eq. (9). By this definition, DR procedure leads to a common dynamic problem, which can be solved by numerical integrations.

To take advantage of the vector operations' simplicity, explicit integrations are utilized to derive DR iterative relationships [16]. The main required assumption to fulfill this goal is the diagonal property of the fictitious mass and damping matrices, which are used in Eqs. (8) and (9). As a result, Eq. (7) can be simplified as follows [34]:

$$m_{ii}\ddot{D}_i^n + c_{ii}\dot{D}_i^n + f_i^n = p_i, \quad i = 1, 2, \cdots, q \tag{11}$$

It is assumed that the structure has q degrees of freedom and m_{ii} and c_{ii} are i^{th} diagonal elements of the fictitious mass and damping matrices, respectively. To perform numerical integration, the most common approach utilizes the standard central finite difference expressions. For this purpose, Brew and Brotton developed DR formulations so that a vector form was achieved [8]. This approach is the most common formulation for iterative DR procedure. In fact, the derivatives of displacement, with respect to the fictitious time, are determined by the succeeding relationships [7, 8]:

$$\{D\}^{n+1} = \{D\}^n + \tau^{n+1}\left\{\dot{D}\right\}^{n+\frac{1}{2}} \tag{12}$$

$$\left\{\ddot{D}\right\}^n = \frac{\left\{\dot{D}\right\}^{n+\frac{1}{2}} - \left\{\dot{D}\right\}^{n-\frac{1}{2}}}{\tau^n} \tag{13}$$

where τ^n is the fictitious time step in n^{th} increment of numerical time integration. Furthermore, the velocity of n^{th} time step is estimated, as follows:

$$\left\{\dot{D}\right\}^n = \frac{\left\{\dot{D}\right\}^{n+\frac{1}{2}} + \left\{\dot{D}\right\}^{n-\frac{1}{2}}}{2} \tag{14}$$

Substituting Eqs. (14) and (13) into (11) prepares the velocity of the next time step [33]:

$$\dot{D}_i^{n+\frac{1}{2}} = \frac{2m_{ii} - \tau^n c_{ii}}{2m_{ii} + \tau^n c_{ii}}\dot{D}_i^{n-\frac{1}{2}} + \frac{2\tau^n}{2m_{ii} + \tau^n c_{ii}} r_i^n, \quad i = 1, 2, \cdots, q \tag{15}$$

In the last relation, r_i^n is the residual force of the i^{th} degree of freedom, i.e., $\{R\}^n$ in the n^{th} iteration:

$$\{R\}^n = [M]\left\{\ddot{D}\right\}^n + [C]\left\{\dot{D}\right\}^n = \{P\} - \{f\}^n \tag{16}$$

In the following lines, the next estimation of the displacement can be calculated from Eq. (12). The viscous DR procedure is performed by iterating Eqs. (15) and (12), successively, until convergence criterions are satisfied. At this stage, the residual force or kinetic energy is vanished. In addition to this common formulation, Rezaiee-Pajand and Taghavian-Hakkak proposed a new iterative formula for displacement by use of Taylor's series [23], as

follows:

$$\{D\}^{n+1} = \{D\}^n + \tau^n \left\{\dot{D}\right\}^n + \frac{(\tau^n)^2}{2} \left\{\ddot{D}\right\}^n \tag{17}$$

Acceleration vector is found from Eq. (16) and replaced in Eq. (17). Then, the following iterative relation is obtained:

$$\{D\}^{n+1} = \{D\}^n + \left(\tau^n - \frac{c(\tau^n)^2}{2}\right) \left\{\dot{D}\right\}^n + \frac{(\tau^n)^2}{2} [M]^{-1} \{R\}^n \tag{18}$$

In each step, the velocity vector is calculated from next equation:

$$\left\{\dot{D}\right\}^n = \frac{\{D\}^n - \{D\}^{n-1}}{\tau^n} \tag{19}$$

It should be noted that the DR iterations are usually unstable because of using numerical time integrations to solve the differential equations of motion. To overcome this drawback, the fictitious parameters, such as time step and diagonal mass and damping matrices, are determined so that the stability conditions are satisfied. It should be added that the slow convergence rate of the viscous DR method has always been a critical issue.

The most common way to study the stability and convergence rate of a viscous DR algorithm is an error analysis between two successive iterations. One of the first classified error analyses was performed by Papadrakakis [19]. He utilized the following assumptions:

$$[M] = \rho[D] \tag{20}$$

$$[C] = c[D] \tag{21}$$

In the former relations, $[D]$, ρ and c, are the main diagonal terms of stiffness matrix, fictitious mass and damping factors, respectively. The optimum values of mass and damping factors are formulated based on maximum and minimum eigen-value of the fictitious dynamic system. The estimation of these eigen-values is mostly calculated from Gerschgrin's circle theory. On the other hand, one of the famous error analyses for viscous DR iterations was performed by Underwood [33]. He assumed a linear reduction for displacement error between two successive iterations. This analysis leads to following inequality for fictitious mass:

$$m_{ii} \geq \frac{(\tau^n)^2}{4} \sum_{j=1}^{q} |s_{ij}|, \quad i = 1, 2, \cdots, q \tag{22}$$

In another study, Rezaiee-Pajand and Alamatian revised the error analysis of the viscous DR method. Two discrete regions were considered for eigen-value in Gerschgörin's theory by these researchers. The related formulation led to a simple and effective way of calculating the fictitious mass, as follows [21]:

$$m_{ii} = \frac{(\tau^n)^2}{4} \max\left[2s_{ii}, \sum_{j=1}^{q} |s_{ij}|\right], \quad i = 1, 2, \cdots, q \tag{23}$$

Fictitious damping is another parameter in viscous DR iterations. This factor has a great effect on convergence rate of DR procedure. The reason for this subject can be found in physical basis of the DR method, which tries to obtain the steady-state answer of a fictitious dynamic system. Based on the structural dynamics theories, the critical damping causes rapid convergence rate to the steady-state response. This concept is commonly used to calculate the fictitious damping. By utilizing the critical damping theory, Zhang et al. presented an estimation of the critical damping for viscous DR iterations, as follows [37]:

$$c_{ii} = 2m_{ii}\omega_{min}, \quad i = 1, 2, ..., q \tag{24}$$

In the last relationship, ω_{min} is the lowest natural frequency of a structure in the free vibration case. This factor can be calculated from the following Rayleigh's principle [37]:

$$\omega_{min}^2 \approx \frac{(\{D\}^n)^T \{f\}^n}{(\{D\}^n)^T [M]^n \{D\}^n} \tag{25}$$

In the former equation, Rayleigh's principle was applied to the whole structure. Another way of finding the lowest natural frequency was proposed, in which this can separately be calculated for each node [36]. This approach leads to succeeding result:

$$(\omega_{min}^i)^2 \approx \frac{\left(\left\{D^i\right\}^n\right)^T \left\{f^i\right\}^n}{\left(\left\{D^i\right\}^n\right)^T [M^i]^n \left\{D^i\right\}^n}, \quad i = 1, 2, ..., q \tag{26}$$

where ω_{min}^i, $\left\{D^i\right\}^n$, $\left\{f^i\right\}^n$ and $\left[M^i\right]^n$ are the lowest frequency, displacement, internal force vector and fictitious mass matrix of the i^{th} node, respectively. The parameter h indicates the total number of nodes of the structure. It should be added that Rezaiee-Pajand and Alamatian proposed a modified relationship for fictitious damping, as follows [21]:

$$c_{ii} = m_{ii}\sqrt{4\omega_{min}^2 - \omega_{min}^4}, \quad i = 1, 2, \cdots, q \tag{27}$$

It is evident, when the lowest natural frequency approaches zero, Eq. (27) reduces to Eq. (24). All the previous methods lack accuracy, because they exploit the Rayleigh principle to estimate the lowest natural frequency and eventually to find the fictitious damping. To overcome this drawback, the structural dynamics schemes can be utilized so that the lowest natural frequency of the fictitious dynamic system is calculated with suitable accuracy. The Stodela iterative process is a technique which has been successfully applied in DR iterations to calculate ω_{min} [24]. In addition, Rezaiee-Pajand

and Sarafrazi combined the power iteration method with DR algorithm and obtained a more accurate value for the lowest eigen-value. They utilized one step of power iterative scheme for matrix $[M^{-1}S - aI]$ in each step of DR procedure [27]. Here, factor a stand for the upper bound of the highest eigen-value, which can be set to 4.

3.2 Kinetic damping formulation

It is not required to calculate the viscous damping terms in the kinetic formulation. This technique needs only the time step and the fictitious nodal masses. In fact, the number of necessary DR parameters is reduced. In this way, time interval may be fixed [31]. As a result, the kinetic damping provides an alternative approach for DR method. Cundall proposed this procedure for application to unstable rock mechanism [11]. The high stability and rapid convergence rate for structures with large displacement are the main specification of the kinetic damping formulation. In such analyses, the un-damped motion of a structure is traced until a local peak in total kinetic energy is detected. After that, all current velocities are reset to zero and DR process is restarted with current geometry and repeated through further peaks. This procedure is generally decreasing until the energies of all modes of vibration are dissipated, and the static equilibrium is reached. The method of kinetic damping is found to be very stable and rapidly convergent when dealing with large displacements [31]. The stability of this approach depends on the time-mass relation [27]. One of the important applications of kinetic damping DR algorithm is parallel processing in the finite element analysis [13, 32]. The parallel DR scheme is undertaken in two stages. First, the overall mesh is divided into a different number of sub domains corresponding to the number of available processors. Then, DR strategy, usually based on kinetic damping algorithm, is used over the sub domains and converged results from each sub domain are received and compiled to obtain the overall results for the domain. In the case of the viscous damping factor in Eq. (15) is equal to zero ($c_{ii} = 0$), the next fundamental iterative relationships of the kinetic DR strategy could be obtained [13]:

$$\dot{D}_i^{n+\frac{1}{2}} = \dot{D}_i^{n-\frac{1}{2}} + \frac{\tau^n}{m_{ii}} r_i^n, \quad i = 1, 2, \cdots, q \tag{28}$$

$$D_i^{n+1} = D_i^n + \tau^{n+1} \dot{D}_i^{n+\frac{1}{2}}, \quad i = 1, 2, \cdots, q \tag{29}$$

By exploiting Eqs. (28) and (29) successively, the total kinetic energy of un-damped structure is traced during iterations. The position of structure with maximum kinetic energy could represent the static equilibrium, because in this condition, potential energy will be minimized. The kinetic DR algorithm uses the aforementioned idea to approach the static equilibrium position. In

each iteration, the total kinetic energy, U_k, is calculated by the following relationship:

$$U_k = \sum_{i=1}^{q} m_{ii} \left(\dot{D}_i^{n+\frac{1}{2}} \right)^2 \tag{30}$$

It is worth emphasizing to mention that the DR iterations run sequentially, until an energy peak is detected. At this stage, all current velocities are reset to zero. This process should be continued through further peaks until the convergence criterions are satisfied. To establish this algorithm, it is assumed that peak of kinetic energy could occur between times τ^n and τ^{n+1}, i.e. a fall in the total kinetic energy would be taking placed at time τ^{n+1}. As described by Topping and Ivanyi [31], the displacement, which peak of the kinetic energy belongs to it, is calculated from the following equation [13]:

$${}^{*}D_i^n = D_i^{n+1} - \frac{3}{2}\tau^{n+1}\dot{D}_i^{n+\frac{1}{2}} + \frac{(\tau^n)^2}{2m_{ii}} r_i^n, \quad i = 1, 2, \cdots, q \tag{31}$$

here ${}^{*}D_i^n, i = 1, 2, \cdots, q$ is the displacement in which the kinetic energy is maximized. The analysis can be restarted by ${}^{*}D_i^n, i = 1, 2, \cdots, q$ as the initial displacement. For fully convergence, velocities of the first time-step should be calculated at the mid point, as follows:

$$\dot{D}_i^{n+\frac{1}{2}} = \frac{\tau^n}{2m_{ii}} r_i^n, \quad i = 1, 2, \cdots, q \tag{32}$$

In the last relationship, r_i^n is calculated from displacement's components, presented by Eq. (31). Having this parameter, DR iterations can be restarted from Eqs. (28) and (29). The mentioned process will be iterated for some successive peaks of the kinetic energy until the convergence criteria are satisfied. It should be noted that the diagonal elements of the mass matrix, which control the stability of kinetic DR algorithm, are determined from the subsequent formula [13]:

$$m_{ii} \geq \frac{(\tau^n)^2}{2} \sum_{j=1}^{q} |s_{ij}|, \quad i = 1, 2, \cdots, q \tag{33}$$

In the latest study, a new formulation was proposed for the fictitious mass of the kinetic DR method. The transformed Gerschgörin theory was also utilized in the mentioned research [3].

4 Other fictitious parameters

As it was discussed earlier, there are four groups of unknown parameters in DR iterations; i. e. fictitious mass, fictitious damping, pseudo time step

and initial guess of displacement. The most common approach to estimate fictitious mass and damping matrices is based on error analysis and stability verification of DR iterations. The general form of such procedure was presented in section 3. Before starting DR iterations, fictitious time step and initial guess of displacement should be determined. The researchers have presented various methods to determine these quantities. In the following lines, these approaches are reviewed, separately.

4.1 Fictitious time step

One of the important parameters in the DR techniques is the fictitious time step. This factor is utilized in the numerical integration of the fictitious system. The subsequent sections explain several ways of calculating this vital factor:

4.1.1 Constant time step

In the most common DR algorithms, the fictitious time step is kept constant and equal to one. This assumption leads to the next simple relationship:

$$\tau^n = 1.0 \tag{34}$$

4.1.2 Minimum residual force

Minimization of the out-of-balance force helps to improve DR convergence rate. In this scheme, the out-of-balance force function is defined as follows [1, 4, 17, 20]:

$$\mathrm{UFF} = \sum_{i=1}^{\mathrm{q}} \left(r_i^{n+1}\right)^2 \tag{35}$$

where, UFF is the un-balance force function in the $n+1^{th}$ iteration of DR method. By minimizing this function, the next modified fictitious time step (MFT) is calculated:

$$\frac{\partial\ \mathrm{UFF}}{\partial \tau^{n+1}} = 0 \Rightarrow \tau_{\ \mathrm{MFT}} = \frac{\sum_{i=1}^{\mathrm{q}} r_i^n \dot{f}_i^{n+\frac{1}{2}}}{\sum_{i=1}^{\mathrm{q}} \left(\dot{f}_i^{n+\frac{1}{2}}\right)^2} \tag{36}$$

In this equation, $\dot{f}_i^{n+\frac{1}{2}}$ is the rate of internal force of the i^{th} degree of freedom. It is possible to find this parameter by using the chain rule (differentiation), as follows [1, 4, 17, 20]:

$$\dot{f}_i^{n+\frac{1}{2}} \approx \sum_{j=1}^{q} s_{ij}^n \dot{D}_j^{n+\frac{1}{2}}, \quad i = 1, 2, \cdots, q \tag{37}$$

By using the second derivative test, it can be proven mathematically that the last time step minimizes the out-of-balance force function, in each iteration of the DR algorithm. As a result, the convergence rate will increase, and the analysis time will decrease.

4.1.3 Minimum residual energy

In the steady-state response of an artificial dynamic system, the kinetic energy is zero. Therefore, convergence to the steady-state response could be increased by minimizing the energy function. This function can be defined in the following form [2, 25]:

$$\text{UEF} = \sum_{i=1}^{\text{q}} \left(\delta D_i^{n+1} r_i^{n+1}\right)^2 \tag{38}$$

In the last equation, UEF is the un-balance energy function in the $n+1^{th}$ iteration of the DR process. If the out-of-balance force is minimized, the succeeding optimum fictitious time step (OFT) can be obtained [2, 25]:

$$\tau_{\text{OFT}}^{n+1} = \frac{-A_2^{n+1} \pm \sqrt{(A_2^{n+1})^2 - 4A_1^{n+1}A_3^{n+1}}}{2A_1^{n+1}} \tag{39}$$

where, A_1^{n+1}, A_2^{n+1} and A_3^{n+1} are constant factors and defined as below:

$$A_1^{n+1} = 2\sum_{i=1}^{q} \left(v_i^{n+\frac{1}{2}} \dot{f}_i^{n+\frac{1}{2}}\right)^2 \tag{40}$$

$$A_2^{n+1} = -3\sum_{i=1}^{\text{q}} \left[\left(v_i^{n+\frac{1}{2}}\right)^2 r_i^n \dot{f}_i^{n+\frac{1}{2}}\right] \tag{41}$$

$$A_3^{n+1} = \sum_{i=1}^{\text{q}} \left(v_i^{n+\frac{1}{2}} r_i^n\right)^2 \tag{42}$$

It is clear that minimization of the out-of-balance energy function will be a conditional procedure. If Eq. (39) could not result in a real value for the fictitious time step, the presented approach is useless.

4.1.4 Zero damping method

In a published article, Rezaiee-Pajand and Sarafrazi set the damping factor to zero and obtained below equation for the fictitious time step [28, 30]:

$$\tau^{n+1} = \frac{1}{(1+\omega_{\min})^2}\tau^n \tag{43}$$

4.1.5 Error analysis in kinetic damping method

The error between two consecutive steps in kinetic DR technique was calculated by Rezaiee-Pajand and Rezaee. It resulted in formulating a new time step for this method. [24]:

$$\tau^{n+1} = \frac{(\frac{2}{\tau^n} + \tau^n\,\omega_{\min}^2) - \sqrt{(\frac{2}{\tau^n} + \tau^n\,\omega_{\min}^2)^2 - 4\,(\frac{2}{\tau^n} + \tau^n\,\omega_{\min}^2)^2}}{2(\frac{1}{\tau^n} - \tau^n\,\omega_{\min}^2)^2} \tag{44}$$

4.2 Initial guess of displacement

The DR calculations are started by an initial guess of displacement. In the iteration process, this quantity is corrected successively. The more suitable guess of displacement leads to a more rapid convergence rate of the solver. The following sections explain some approaches to estimate initial displacement in the first iteration.

4.2.1 Constant vector

The simplest approach to start DR process is using a constant vector for the initial guess of displacement. In this case, the unit or null vectors are the most common selection for the initial displacement. It should be added that the convergence rate of such techniques may not be suitable.

4.2.2 Alwar's approach

An exponential function was proposed by Alwar et al. to determine the steady-state response [6]. For each degree of freedom of structure, the aforementioned function is assumed for damped oscillation. The suggested function has the following form:

$$D_i = \alpha_i e^{-\beta_i \tau} \tag{45}$$

here α_i and β_i are unknown coefficients, which are determined by solving two equations obtained from two successive maximum in the oscillation curve. In this method, it is necessary to calculate two first successive maximum domains of the oscillation curve of each degree of freedom.

4.2.3 Zhang's method

In this strategy, the un-damped vibrations of structure are traced until first minimum and maximum displacements of each degree of freedom are obtained. After that, the initial displacement can be calculated by the next formula [37]:

$$D_i^0 = \frac{D_i^{\max} + D_i^{\min}}{2} \tag{46}$$

where $D_i^{\max}$and $D_i^{\min}$ are the first maximum and minimum displacements of i^{th} degree of freedom in the un-damped vibration.

4.2.4 Modified Zhang's method

This scheme is a modified version of Zhang's approach and is based on the concept of constant total energy of structure. In this way, the un-damped vibrations are started with initial displacement and by obtaining first minimum or maximum. Then, the initial guess is corrected by the following relationship [36]:

$$D_i^0 = \frac{D_i^0 + D_i^{\min}}{2} \quad \text{or} \quad D_i^0 = \frac{D_i^0 + D_i^{\max}}{2} \tag{47}$$

In each cycle, the initial guess is corrected so that by iterating Eq. (46), a suitable initial guess is obtained. It should be added that there are some other papers, which deal with DR stability and its convergence rate [9,10,35].

5 DR algorithms

Dynamic Relaxation, which is an iterative method, can be performed by some simple steps. As discussed in the previous sections, the basic foundation of these steps is given by mathematical formulations. For better clarification, the main algorithms of DR process are presented in this part. These algorithms can be classified based on two DR formulations; i.e. Viscous and Kinetic damping theories. It should be noted that there are several similarities between these two main procedures. In the following, the viscous and kinetic DR algorithms will be briefly explained.

5.1 Viscous DR steps

The viscous DR method requires the following steps:

(a) Assume initial values for artificial velocity (null vector), displacement (available methods presented in section 4.2 or null vector or convergence displacement on the previous increment), fictitious time step ($\tau = 1$) and convergence criterion for out of balance force and kinetic energy ($e_R = 1.0E-6$ and $e_K = 1.0E-12$).

(b) Construct the tangent stiffness matrix and internal force vector.

(c) Apply boundary conditions.

(d) Calculate out of balance (residual) force vector (Eq. 16).

(e) If $\sqrt{\sum_{i=1}^{q} \left(r_i^n\right)^2} \leq e_R$, go to (m), otherwise, continue.

(f) Construct the artificial diagonal mass matrix (section 3.1).

(g) Form artificial damping matrix (section 3.1).

(h) Update artificial velocity vector using Eq. (15).

(i) If $\sum_{i=1}^{q} \left(\dot{D}_i^{n+\frac{1}{2}}\right)^2 \leq e_K$, go to (m), otherwise, continue.

(j) Calculate fictitious time step (section 4.1)

(k) Update displacement vector using Eq. (12).

(l) Set $n = n + 1$ and go to (b).

(m) Print results of the current increment.

(n) If increments are not complete, go to (a), otherwise, stop.

5.2 Kinetic DR steps

The subsequent steps are needed to obtain the solution by the kinetic DR method:

(a) Assume values for initial fictitious velocity (null vector), initial displacement (available methods presented in section 4.2 or null vector or convergence displacement on the previous increment), fictitious time step (1), initial kinetic energy ($KE^0 = 0$) and a convergence criterion for the out-of-balance force and the kinetic energy ($e_R = 1.0E-6$ and $e_K = 1.0E-12$),

(b) Construct the tangent stiffness matrix and internal force vector,

(c) Calculate the out-of-balance force vector using Eq. (16),

(d) If $\sqrt{\sum_{i=1}^{q}\left(r_i^n\right)^2} \leq e_R$, go to (q), otherwise continue,

(e) Construct the fictitious diagonal mass matrix using (section 3.2),

(f) Update the fictitious velocity vector using Eq. (28),

(g) If $\sum_{i=1}^{q}\left(\dot{d}_i^{n+\frac{1}{2}}\right)^2 \leq e_K$, go to (q), otherwise, continue,

(h) Calculate the kinetic energy from Eq. (30),

(i) If $KE^{n+1} \leq KE^n$, go to (m), otherwise, continue,

(j) $\tau^{n+1} = \tau^n$

(k) Update the displacement vector using Eq. (29),

(l) Go to (b),

(m) Calculate the initial displacements of the restarted analysis (section 3.2),

(n) Construct the tangent stiffness matrix and the internal force vector,

(o) Calculate the initial velocities of the restarted analysis (section 3.2),

(p) Go to (g)

(q) Print results of the current increment,

(r) If increments are not complete, go to (a), otherwise, stop.

As it was demonstrated in the related steps, the kinetic and viscous DR algorithms have many similarities. However, they use different relationships for mass and damping matrices. Based on this fact, Rezaiee-Pajand and et al. selected twelve well-known DR procedures in order to analyze the geometric nonlinear behavior of several frames and trusses. They graded these methods in terms of the number of iterations and the analysis time. In this way, the five most efficient algorithms were introduced by these investigators [29]. Moreover, Rezaiee-Pajand and Estiri compared these different approaches in the geometric nonlinear analysis of 3D frames and found the three most efficient algorithms [22].

6 Conclusion

In this paper, a comprehensive review of the Dynamic Relaxation (DR) methods was presented. At the first stage, the most common DR schemes, which include the kinetic and viscous damping formulations, were briefly explained. To utilize these techniques, several parameters are required. All the related factors were also described. Simplicity, full explicit vector operations and no sensitivity to nonlinear behaviors are the main specifications of the DR procedure. Moreover, solution methodologies for the structural problems were presented. The strategies reported in this paper could be used to evaluate the responses of different structures. It is worth emphasizing that DR techniques have the great ability to be implemented in all branches of structural analyses. They are very simple and also eliminate the round-off errors.

As it was mentioned by the researchers, the interest in obtaining more accurate and efficient results, using a variety of DR strategies remains strong, even in the presence of some simplifying assumptions. By analyzing the outcomes of presented studies, it can be seen that several investigations on the different aspects of these solvers are needed. For instance, no attempt has been made for considering a broad comparison study, so far. Consequently, no recommendations are made as to the most appropriate technique selection for a particular problem. The principal concepts of the various DR formulations can be summarized in two main categories, i.e. the numerical stability and the convergence rate. In other words, researchers try to guarantee the DR stability, especially in the case of intense non-linearity. From the future research point of view, the DR formulation could be enhanced by modification of its parameters. On the other hand, improving the convergence rate of DR iterations may be one of the interesting topics for further studies. The second part of the article will concentrate on the DR applications.

References

1. Alamatian, J. *Structural analysis using dynamic relaxation method*, MS Dissertation, Department of Civil Engineering, Ferdowsi University, Mashhad, Iran, (2002).

2. Alamatian, J. *Numerical integration for structural analysis*, PhD Thesis, Department of Civil Engineering, Ferdowsi University, Mashhad, Iran, (2007).

3. Alamatian, J. *A new formulation for fictitious mass of the Dynamic Relaxation method with kinetic damping*, Computers and Structures 90-91 (2012), 42-54.

4. Alamatian, J. and Kadkhodayan, M. *Optimum time step for Dynamic Relaxation method*, Amirkabir Journal of Science & Technology 17(65-C) (2006), 43-52.

5. Allen, D.N.G. and Southwell, R.V. *Dynamic Relaxation*, Phil. Trans. 42 (1950), 379-414.

6. Alwar, R.S., Rao, N.R. and Rao, M.S. *An altenative procedure in Dynamic Relaxation*. Computers and Structures 5(2) (1975), 271-274.

7. Barnes, M.R. *Form finding and analysis of tension structures by Dynamic Relaxation*, International Journal of Space Structures 14(2) (1999), 89-104.

8. Brew, J.S. and Brotton, M. *Non-linear structural analysis by Dynamic Relaxation method*, International Journal of Numerical Methods Engineering 3(4) (1971), 463-483.

9. Bunce, J.W. *A note on estimation of critical damping in Dynamic Relaxation*, International Journal of Numerical Methods Engineering 4(1972), 301-304.

10. Cassell, A.C. and Hobbs, R.E. *Numerical stability of Dynamic Relaxation analysis of non-linear structures*, International Journal of Numerical Methods Engineering 10 (1976), 1407-1410.

11. Cundall, P.A. *Explicit finite-difference methods in geomechanics*, Proceeding 2nd International Conference on Numerical Methods in Geomechanics. Blacksburg, Virginia, USA, 1 (1976), 132–150.

12. Day, A.S. *An introduction to Dynamic Relaxation*, The Engineer 219 (1965), 218-221.

13. Ivanyi, P. and Topping, B.H.V. *Parallel Dynamic Relaxation form-finding*, In: Papadrakakis M, Topping BHV, editors. *Innovative Computational Methods in Structural Mechanics.* Edinburgh: Saxe-Coburg Publications, 1999.

14. Felippa, C.A. *Nonlinear finite element methods*, ASEN 5017, Course Material, (1999), http://kaswww.colorado.edu /courses.d /NFEMD/.

15. Frankel, S.P. *Convergence rates of iterative treatments of partial differential equations*, Mathl Tabl Natn Res Coun 4 (1950), 65-75.

16. Gosling, P.D. and Lewis, W.J. *Optimal structural membranes - II. Form-finding of prestressed membranes using a curved quadrilateral finite element for surface definition*, Computers and Structures 61(5) (1996), 885-895.

17. Kadkhodayan, M., Alamatian, J. and Turvey, G.J. *A new fictitious time for the Dynamic Relaxation (DXDR) method*, International Journal of Numerical Methods Engineering 74 (2008), 996-1018.

18. Otter, J.R.H., Cassell, A.C. and Hobbs, R.E. *Poisson. Dynamic Relaxation*, Institute of Civil Engineersrs, ICE proceedings Journal, USA, 35(4) (1966), 633-656.

19. Papadrakakis, M.A. *Method for automatic evaluation of the Dynamic Relaxation parameters*, Computational Methods in Applied Mechanics and Engineering 25 (1981), 35-48.

20. Rezaiee-Pajand, M. and Alamatian, J. *Superior time step for geometrical nonlinear analysis by Dynamic Relaxation method*, Modarres Technical and Engineering 19 (2005), 61-74.

21. Rezaiee-Pajand, M. and Alamatian, J. *The Dynamic Relaxation method using new formulation for fictitious mass and damping*, Journal of Structural Engineering and Mechanics 34(1) (2010), 109-133.

22. Rezaiee-Pajand, M. and Estiri, H. *Comparative analysis of three-dimensional frames by dynamic relaxation methods*, *Mechanics of Advanced Materials and Structures* (2017).

23. Rezaiee-Pajand, M. and Hakkak, M.T. *Nonlinear analysis of truss structures using Dynamic Relaxation*, International Journal of Engineering, Transactions B: Applications 19(1) (2006), 11-22.

24. Rezaiee-Pajand, M., Kadkhodayan, M., Alamatian, J. and Zhang, L.C. *A new method of fictitious viscous damping determination for the Dynamic Relaxation method*, Computers and Structures 89 (2011), 783–794.

25. Rezaiee-Pajand, M., Kadkhodayan, M. and Alamatian, J. *Timestep Selection for Dynamic Relaxation Method*, Mechanics Based Design of Structures and Machines 40 (2012), 42–72.

26. Rezaiee-Pajand, M. and Rezaee, H. *Fictitious time step for the kinetic dynamic relaxation method*, Mechanics of Advanced Materials and Structures 21 (2014), 631–644.

27. Rezaiee-Pajand, M. and Sarafrazi, S.R. *Nonlinear structural analysis using dynamic relaxation method with improved convergence rate*, International Journal of Computational Methods 7 (2010), 1-28.

28. Rezaiee-Pajand, M. and Sarafrazi, S.R. *Nonlinear dynamic structural analysis using dynamic relaxation with zero damping*, Computers and Structures 89 (2011), 1274-1285.

29. Rezaiee-Pajand, M., Sarafrazi, S.R. and Rezaiee, H. *Efficiency of dynamic relaxation methods in nonlinear analysis of truss and frame structures*, Computers and Structures 112-113 (2012), 295-310.

30. Sarafrazi, S.R. *Numerical integration for structural dynamic analysis*, PhD Thesis, Department of Civil Engineering, Ferdowsi University, Mashhad, Iran, (2010).

31. Topping, B.H.V.and Ivanyi, P. *Computer aided design of cable-membrane structures*, Saxe-Coburg Publications on Computational Engineering, Chapter 3, 39–82, 2007.

32. Topping, B.H.V. and Khan, A.I. *Parallel schemes for dynamic relaxation, engineering computations*, International Journal of Computer Aided Engineering Software 11 (1994), 513–548.

33. Underwood, P. *Dynamic Relaxation*, In: Belytschko T, Hughes TJR, Editors. Computational Methods for transient analysis. Computational Methods in Mechanics, 1 (1983), Elsevier Science, 245-256.

34. Wakefield, D.S. *Engineering analysis of tension structures: Theory and practice*, Engineering Structures 21(8) (1999), 680-690.

35. Wood, W.L. *Note on Dynamic Relaxation*, International Journal of Numerical Methods Engineering 3 (1971), 145-147.

36. Zhang, L.C., Kadkhodayan, M. and Mai, Y.W. *Development of the maDR method*, Computers and Structures 52 (1994), 1-8.

37. Zhang, L.C. and Yu, T.X. *Modified adaptive Dynamic Relaxation method and its application to elastic-plastic bending and wrinkling of circular plates*, Computers and Structures 34 (1989), 609-614.

مروری بر روش های رهایی پویا در سازه های مکانیکی
بخش اول : رابطه سازی ها

محمد رضایی پژند[1] ، *جواد علامتیان*[2] و *حسینه رضایی*[3]

[1] دانشگاه فردوسی مشهد، دانشکده مهندسی، گروه مهندسی عمران
[2] دانشگاه آزاد اسلامی، واحد مشهد، گروه مهندسی عمران
[2] دانشگاه فردوسی مشهد، دانشکده مهندسی، گروه مهندسی عمران

دریافت مقاله ۵ آذر ۱۳۹۵، دریافت مقاله اصلاح شده ۲۲ اسفند ۱۳۹۵، پذیرش مقاله ۳ اردیبهشت ۱۳۹۶

چکیده : در شصت سال گذشته، روش های رهایی پویا تحول های فراوانی یافته اند. در بیشتر زمان ها، این فرایند های صریح و تکراری برای حل معادله های خطی و ناخطی حاکم بر رفتار سازه ها به کار می روند. در بخش اول این مطالعه، رابطه سازی های معمول مرور می شوند. پایه های ریاضی و نیزمفهوم های فیزیکی به طور خلاصه شرح داده خواهند شد. سپس، شرح همه ی عامل های رهایی پویا، مانند: جرم ساختگی، میرایی ساختگی، گام زمانی ساختگی و حدس نخستین، می آیند. افزون بر این ها، به چگونگی حل مساله های سازه ها با شیوه های جنبشی و لزجی پرداخته می شود. سرانجام، روش های کنونی و راستاهای پژوهش آیندگان واکاوی می گردند. دربخش دوم، کاربردهای روش های رهایی پویا در رشته های مهندسی مرور خواهند شد. این موضوع، توانایی بالای فرایند رهایی پویا در حل مساله های مهندسی را آشکار می سازد.

کلمات کلیدی : روش های رهایی پویا؛ رابطه سازی؛ حل کننده؛ مرور؛ فن های تکراری.