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Approximation of the Huxley equation with nonstandard finite-difference scheme

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Abstract

In this paper, an explicit exact finite-difference scheme for the Huxley equation is presented based on the nonstandard finite-difference (NSFD) scheme. Afterwards, an NSFD scheme is proposed for the numerical solution of the Huxley equation. The positivity and boundedness of the scheme is discussed. It is shown through analysis that the proposed scheme is consistent, stable, and convergence. The numerical results obtained by the NSFD scheme is compared with the exact solution and some available methods, to verify the accuracy and efficiency of the NSFD scheme.

Keywords: The Huxley equation; Nonstandard finite-difference scheme; Positivity and boundedness; Consistency; Stability; Convergence.

1 Introduction

Nonlinear partial differential equations (PDEs) play an important role in the various fields of physical science and engineering such as plasma physics, fluid mechanics, optimal fibbers, solid state physics, chemical kinetics, and geochemistry. Recently, it also began to become important in various other fields of science, for example, biology and economics [14, 18]. Behaviors of many physical systems encountered in models of reaction mechanisms, convection effects, and diffusion transports give rise to the Burgers–Huxley (BH) equation. Consider the BH equation

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$$u_t + \alpha u u_x - u_{xx} = \beta u(1-u)(u-\gamma), \tag{1}$$

where $\alpha, \beta \geq 0$ and $\gamma \in (0,1)$. When $\beta = 0$, equation (1) reduces to the Burgers equation. If $\alpha = 0$, then Equation (1) recovers the Huxley equation

$$u_t - u_{xx} = \beta u(1-u)(u-\gamma), \tag{2}$$

which describes nerve pulse propagation in nerve fibers and wall motion in liquid crystals. An analytical solution to this equation subject to the following initial and boundary conditions:

$$\begin{aligned} u(x,0) &= \frac{\gamma}{1+e^{-2A_1 x}}, && 0 \leq x \leq 1, \\ u(0,t) &= \frac{\gamma}{1+e^{2A_1 A_2 t}}, && t \geq 0, \\ u(1,t) &= \frac{\gamma}{1+e^{-2A_1(1-A_2 t)}}, && t \geq 0, \end{aligned}$$

derived by Wang in [17], is given by

$$u(x,t) = \frac{\gamma}{2} + \frac{\gamma}{2}\tanh[A_1(x - A_2 t)] = \frac{\gamma}{1+e^{-2A_1(x-A_2 t)}}, \tag{3}$$

where

$$A_1 = \frac{\sqrt{8\beta}}{8}\gamma, \qquad A_2 = \frac{4-2\gamma}{\gamma}A_1.$$

In the past few years, various powerful mathematical methods such as the Adomian decomposition method (ADM) [7, 8, 10], the variational iteration method (VIM) [1, 2], the differential transform method (DTM) [3] and the Haar wavelet method [6] have been used in attempting to solve (2). Among various techniques for solving PDEs, the NSFD schemes have been proved to be one of the most efficient approaches in recent years [13]. These schemes are developed for compensating the weaknesses, such as numerical instabilities that may be caused by standard finite- difference schemes. One of the most important advantages of this scheme is that choosing a complicate denominator function instead of the stepsize h, better results can be obtained. If the stepsize h is chosen small enough, the obtained results do not change significantly but if the stepsize h gets larger, then this advantage comes into focus [13, 19–24].

This paper is organized as follows: In Section 2, we provide a brief overview of the important features of the procedures for constructing NSFD schemes for PDEs. In Section 3, we begin with proposing the exact finite-difference scheme for the Huxley equation. In Section 4, we give an NSFD scheme for numerical solution of the Huxley equation and also positivity and boundedness are studied for the proposed NSFD scheme. Moreover, a brief study of consistency is presented here, along with a Von Neumann analysis of stability and convergence. Finally, in the last section, numerical results are given to show the efficiency of the proposed NSFD scheme.

2 Nonstandard finite-difference schemes

The initial foundation of NSFD schemes came from the exact finite-difference schemes. NSFD schemes were firstly proposed by Mickens [12,13] for ordinary differential equations (ODEs) and, successively, their use has been investigated in several fields. Regarding the positivity and boundedness of solutions, NSFD schemes have a better performance over the standard finite-difference schemes, due to its flexibility to construct an NSFD scheme that can preserve certain properties and structures, which are obeyed by the original equations. The advantages of NSFD schemes have been shown in many numerical applications. Gonzalez-Parra et al. [5] and Zibaei and Namjoo [20–23] developed NSFD schemes to solve population and biological models.

This class of schemes and their formulations center on two issues. First, how should discrete representations be determined for derivatives, and second, what are the proper forms to be used for nonlinear terms. The forward Euler method is one of the simplest discretization schemes. In this method the derivative term $\frac{dy}{dt}$ is replaced by $\frac{y(t+h)-y(t)}{h}$, where h is the stepsize. However, in the Mickens schemes this term is replaced by $\frac{y(t+h)-y(t)}{\phi(h)}$, where $\phi(h)$ is an increasing continuous function of h, and the function $\phi(h)$ satisfies the following conditions:

$$\phi(h) = h + O(h^2), \qquad 0 < \phi(h) < 1, \qquad h \to 0.$$

Examples of functions $\phi(h)$ that satisfy these conditions are [12, 19–24]:

$$h, \qquad \sin h, \qquad \frac{1-e^{-\lambda h}}{\lambda}.$$

Note that in taking the $\lim h \to 0$ to obtain the derivative, the use of any of these $\phi(h)$ will lead to the usual result for the first derivative

$$\frac{dy}{dt} = \lim_{h\to 0} \frac{y\big(t+\phi_1(h)\big) - y(t)}{\phi_2(h)} = \lim_{h\to 0} \frac{y(t+h)-y(t)}{h},$$

where $\phi_1(h)$ and $\phi_2(h)$ are continuous functions of the stepsize h. A scheme is called nonstandard if at least one of the following conditions is satisfied:

1. A nonlocal approximation is used.
2. The denominator function for the discrete derivative, in general, expressed in terms of more complicated functions of the stepsize h than those conventionally used.

Nonlinear terms approximated in a nonlocal way, for example, if there are nonlinear terms in the differential equation [12, 19–24], these are replaced by

$$y^2(t) \to y_n(t)y_{n+1}(t), \qquad y^2(t) \to y_{n-1}(t)y_n(t).$$

In the two-dimensional case, nonlinear terms such as $y(t)x(t)$ are either replaced by

$$y(t)x(t) \to y_n(t)x_{n+1}(t), \qquad y(t)x(t) \to y_{n+1}(t)x_n(t).$$

One can say that there is no appropriate general method to choose the function $\phi(h)$ or to choose which nonlinear terms are to be replaced, some special techniques may be found in [15, 19–24].

3 An explicit exact finite-difference scheme

In this section, we obtain an exact finite-difference scheme for the Huxley equation. If we choose $h = A_2\Delta t$, then we easily obtain $u(x-h,t) = u(x,t+\Delta t)$, and the following equations

$$\begin{aligned} \frac{1}{u(x+h,t)} &= \frac{1}{\gamma}(1+e^{-2A_1(x+h-A_2t)}), \\ \frac{1}{u(x-h,t)} &= \frac{1}{\gamma}(1+e^{-2A_1(x-h-A_2t)}), \\ \frac{1}{u(x,t-\Delta t)} &= \frac{1}{\gamma}(1+e^{-2A_1(x-A_2(t-\Delta t))}). \end{aligned} \tag{4}$$

According to (4), we can write

$$\begin{aligned} \frac{1}{u(x,t)} - \frac{1}{u(x+h,t)} &= (\frac{1}{u(x,t)} - \frac{1}{\gamma})(1-e^{-2A_1h}), \\ \frac{1}{u(x,t)} - \frac{1}{u(x-h,t)} &= (\frac{1}{u(x,t)} - \frac{1}{\gamma})(1-e^{2A_1h}), \\ \frac{1}{u(x,t)} - \frac{1}{u(x,t-\Delta t)} &= (\frac{1}{u(x,t)} - \frac{1}{\gamma})(1-e^{-2A_1A_2\Delta t}). \end{aligned}$$

If we set the stepsize functions as

$$\psi_1 = \frac{1-e^{-2A_1h}}{2A_1}, \qquad \psi_2 = \frac{e^{2A_1h}-1}{2A_1}, \qquad \phi_1 = \frac{1-e^{-2A_1A_2\Delta t}}{2A_1A_2},$$

we will get the following forward and backward difference quotients with the special stepsize functions

$$
\begin{aligned}
\partial_x u = &\quad \frac{u(x+h,t)-u(x,t)}{\psi_1} = 2A_1 u(x+h,t)(1-\frac{u(x,t)}{\gamma}),\\
\bar{\partial}_x u = &\quad \frac{u(x,t)-u(x-h,t)}{\psi_2} = 2A_1 u(x-h,t)(1-\frac{u(x,t)}{\gamma}),\\
\bar{\partial}_t u = &\frac{u(x,t)-u(x,t-\triangle t)}{\phi_1} = 2A_1A_2 u(x,t-\Delta t)(\frac{u(x,t)}{\gamma}-1).
\end{aligned} \tag{5}
$$

Consequently, it follows that

$$
\begin{aligned}
\bar{\partial}_x\partial_x u &= \frac{\bar{\partial}_x u(x+h,t)-\bar{\partial}_x u(x,t)}{\psi_1}\\
&= \frac{2A_1}{\gamma}u(x,t)\frac{u(x-h,t)-u(x+h,t)}{\psi_1} + 2A_1\frac{u(x,t)-u(x-h,t)}{\psi_1}\\
&= -A_2\frac{u(x,t)-u(x-h,t)}{\psi_1} + \frac{2A_1}{\gamma}u(x,t)\frac{u(x-h,t)-u(x+h,t)}{\psi_1}\\
&\quad +(2A_1+A_2)\frac{u(x,t)-u(x-h,t)}{\psi_1}.
\end{aligned} \tag{6}
$$

Since $\frac{A_2}{\psi_1} = \frac{1}{\phi_1}$, according to (5), (6), and $u(x-h,t) = u(x,t+\Delta t)$, we can write

$$
\begin{aligned}
\bar{\partial}_x\partial_x u &= \frac{u(x,t+\Delta t)-u(x,t)}{\phi_1} + \frac{2A_1}{\gamma}u(x,t)\big(\frac{u(x-h,t)-u(x,t)}{\psi_1}\\
&\quad + \frac{u(x,t)-u(x+h,t)}{\psi_1}\big) + (2A_1+A_2)\frac{u(x,t)-u(x-h,t)}{\psi_1}\\
&= \frac{u(x,t+\Delta t)-u(x,t)}{\phi_1} + \frac{2A_1}{\gamma}u(x,t)\big(-2A_1u(x,t)(1-\frac{u(x-h,t)}{\gamma})\\
&\quad -2A_1u(x+h,t)(1-\frac{u(x,t)}{\gamma})\big) + (2A_1+A_2)2A_1u(x,t)(1-\frac{u(x-h,t)}{\gamma}).
\end{aligned} \tag{7}
$$

We notice that

$$
2A_1\frac{2A_1}{\gamma} = \frac{\beta\gamma}{2}, \qquad\qquad 2A_1(2A_1+A_2) = \beta\gamma,
$$

hence according to (7), we have

$$
\begin{aligned}
\bar{\partial}_x\partial_x u = &\frac{u(x,t+\Delta t)-u(x,t)}{\phi_1} + \beta(u(x,t))^2\frac{u(x+h,t)+u(x-h,t)}{2}\\
&-\beta\gamma u(x,t)\frac{u(x,t)+u(x+h,t)}{2} + \beta\gamma u(x,t) - \beta u(x,t)u(x-h,t).
\end{aligned}
$$

Now, using approximation $u_{xx} \approx \bar{\partial}_x\partial_x u$ and also assuming

$$U_j^n = u(x_j, t_n) = \frac{\gamma}{1 + e^{-2A_1(x_j - A_2 t_n)}},$$

we construct the following explicit exact finite-difference scheme:

$$\begin{aligned} \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{\psi_1 \psi_2} = \frac{U_j^{n+1} - U_j^n}{\phi_1} + \beta (U_j^n)^2 \frac{U_{j+1}^n + U_{j-1}^n}{2} \\ -\beta\gamma U_j^n \frac{U_j^n + U_{j+1}^n}{2} - \beta U_j^n U_{j-1}^n + \beta\gamma U_j^n. \end{aligned} \tag{8}$$

The following theorem is the main result of the previous discussion.

Theorem 1. *An explicit exact finite-difference scheme for the Huxley equation (2) is given by (8). The stepsize satisfies $h = A_2 \Delta t$, and the stepsize functions satisfy*

$$\psi_1 = \frac{1 - e^{-2A_1 h}}{2A_1}, \qquad \psi_2 = \frac{e^{2A_1 h} - 1}{2A_1}, \qquad \phi_1 = \frac{1 - e^{-2A_1 A_2 \Delta t}}{2A_1 A_2}.$$

4 Analysis of NSFD scheme

In this section, we present a NSFD scheme for the Huxley equation. In the classical sense, a special difference scheme of the Huxley equation can be written as

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} - \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{h^2} = \beta\Big(-(U_j^n)^3 + (1+\gamma)(U_j^n)^2 - \gamma U_j^n \Big), \tag{9}$$

where h and Δt are space and time stepsizes, respectively; and U_j^n is an approximation to $u(x_j, t_n)$. Similar to the classical difference scheme (9), we obtain a NSFD scheme for the Huxley equation as follows:

$$\frac{U_j^{n+1} - U_j^n}{\Phi} - \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{\Psi} = \beta\big(-U_j^{n+1}(U_j^n)^2 + (1+\gamma)U_j^{n+1}U_j^n - \gamma U_j^n \big), \tag{10}$$

where the functions Φ and Ψ are given by

$$\Phi = \frac{1 - e^{-2A_1 A_2 \Delta t}}{2A_1 A_2}, \qquad \Psi = \left(\frac{1 - e^{-2A_1 h}}{2A_1}\right)^2.$$

We conclude that $\Phi \to \Delta t$ and $\Psi \to h^2$ as Δt and h approach zero.

Comparing (10) with (2), we infer that the nonlinear terms on the right-hand side of (2) are in the forms

$$\big(u(x_j, t_n)\big)^3 \approx (U_j^{n+1})(U_j^n)^2, \qquad \big(u(x_j, t_n)\big)^2 \approx U_j^{n+1} U_j^n.$$

It can be easily noticed that the NSFD scheme (10) is explicit. Setting $R = \frac{\Phi}{\Psi}$, we can rewrite (10) as

$$U_j^{n+1} = \frac{(1-2R)U_j^n + R(U_{j+1}^n + U_{j-1}^n) - \beta\gamma\Phi U_j^n}{1 - \beta(1+\gamma)\Phi U_j^n + \beta\Phi(U_j^n)^2}. \tag{11}$$

We can find positivity and boundedness properties in the presented NSFD scheme. The following theorem shows these properties.

Theorem 2. *For any nonnegative initial data, if* $1 - 2R - \beta\Phi\gamma \geq 0$, *then the numerical solution* (11) *satisfies*

$$0 < U_j^n < \gamma \Longrightarrow 0 < U_j^{n+1} < \gamma$$

for all relevant values of n *and* j.

Proof. Since $1 - 2R - \beta\Phi\gamma \geq 0$ and $U_j^n > 0$, we deduce that

$$\begin{aligned}(1-2R)U_j^n &+ R(U_{j+1}^n + U_{j-1}^n) - \beta\gamma\Phi U_j^n \\ &= (1 - 2R - \beta\gamma\Phi)U_j^n + R(U_{j+1}^n + U_{j-1}^n) > 0.\end{aligned} \tag{12}$$

Moreover, $\beta\gamma\Phi \leq 1 - 2R \leq 1$ and $U_j^n < \gamma < 1$ implies

$$\begin{aligned}1 - \beta(1+\gamma)\Phi U_j^n + \beta\Phi(U_j^n)^2 &\geq \beta\gamma\Phi - \beta(1+\gamma)\Phi\gamma + \beta\Phi(U_j^n)^2 \\ &= \beta\Phi(U_j^n - 1)(U_j^n - \gamma) > 0.\end{aligned} \tag{13}$$

So the inequalities (12) and (13) imply that $U_j^{n+1} > 0$. Consider

$$\begin{aligned}(1 - 2R - \beta\gamma\Phi)U_j^n &+ R(U_{j+1}^n + U_{j-1}^n) - \gamma\big(1 - \beta(1+\gamma)\Phi U_j^n + \beta\Phi(U_j^n)^2\big) \\ &\leq (1 - 2R - \beta\gamma\Phi)\gamma + 2R\gamma - \gamma + \beta\gamma(1+\gamma)\Phi U_j^n - \beta\gamma\Phi(U_j^n)^2 \\ &= -\beta\gamma\Phi\big(\gamma - (1+\gamma)U_j^n + (U_j^n)^2\big) = -\beta\gamma\Phi\big((U_j^n - \gamma)(U_j^n - 1)\big) < 0,\end{aligned}$$

and the last inequality shows that $U_j^{n+1} < \gamma$. Hence, the solution of NSFD scheme (11) has the positivity and boundedness properties. □

Now, we establish the properties of consistency and stability of the NSFD scheme (11).

For appropriate R, setting $u_j^n = u(x_j, t_n)$ precisely, we have Taylor's formula for the solution of the Huxley equation, with appropriate $\bar{x}_j \in (x_j, x_{j+1})$ and $\bar{t}_n, \bar{\bar{t}}_n, \bar{\bar{\bar{t}}}_n \in (t_n, t_{n+1})$. For functions defined on the grid, we introduce these difference quotients

$$\partial_t u_j^n = \frac{U_j^{n+1} - U_j^n}{\Phi}, \qquad \partial_x \bar{\partial}_x u_j^n = \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{\Psi}.$$

By using the method in [4], the local truncation error τ_j^n is shown as follows:

$$\begin{aligned}\tau_j^n &= \Big(\partial_t u_j^n - u_t(x_j,t_n)\Big) - \Big(\partial_x\bar{\partial}_x u_j^n - u_{xx}(x_j,t_n)\Big) + \beta\gamma\Big(u_j^n - u(x_j,t_n)\Big)\\ &\quad -\beta(1+\gamma)\Big(u_j^{n+1}u_j^n - u^2(x_j,t_n)\Big) + \beta\Big(u_j^{n+1}(u_j^n)^2 - u^3(x_j,t_n)\Big)\\ &= (\tfrac{\Delta t}{\Phi}-1)u_t(x_j,t_n) + \tfrac{\Delta t^2}{2\Phi}u_{tt}(x_j,t_n) + \tfrac{\Delta t^3}{6\Phi}u_{ttt}(x_j,\bar{t}_n)\\ &\quad -\Big((\tfrac{h^2}{\Psi}-1)u_{xx}(x_j,t_n) + \tfrac{h^4}{12\Psi}u_{xxx}(\bar{x}_j,t_n)\Big)\\ &\quad -\beta(1+\gamma)\Big(\Delta t u(x_j,t_n)u_t(x_j,t_n) + \tfrac{\Delta t^2}{2}u(x_j,t_n)u_{tt}(x_j,t_n)\\ &\qquad\qquad + \tfrac{\Delta t^3}{3}u(x_j,t_n)u_{ttt}(x_j,\bar{\bar{t}}_n)\Big)\\ &\quad +\beta\Big(\Delta t u^2(x_j,t_n)u_t(x_j,t_n) + \tfrac{\Delta t^2}{2}u^2(x_j,t_n)u_{tt}(x_j,t_n)\\ &\qquad\qquad + \tfrac{\Delta t^3}{3}u^2(x_j,t_n)u_{ttt}(x_j,\bar{\bar{\bar{t}}}_n)\Big).\end{aligned}$$

When $\Delta t \to 0$ and $h \to 0$, we have $\Phi \approx \Delta t$ and $\Psi \approx h^2$. Therefore, $\tau_j^n = O(\Delta t) + O(h^2)$ if $\Delta t \to 0$ and $h \to 0$. This shows that the NSFD scheme (11) is consistent with the Huxley equation. In [24], Namjoo et al. considered the approximation of the BH equation using an NSFD scheme, and investigated their numerical results. In fact this scheme is consistent for the BH equation with a consistency of the order of $O(\Delta t) + O(h)$.

Von Neumann introduced a method to prove stability, using Fourier analysis, so that it can give a sufficient condition for the stability of finite-difference schemes.

Theorem 3. *Sufficient conditions for the NSFD scheme* (11) *to be stable, are that* $\gamma \in (0,1)$ *and* $\beta\Psi(1+\gamma) \leqslant 2$.

Proof. Let $U_j^n = e^{\alpha nk}e^{-ij\lambda h}$, for every $n \in \{0,1,2,\ldots,N\}$ and every $j \in \{0,1,2,\ldots,M\}$ where α is a real constant. Substituting in the NSFD scheme (11) and simplifying, we get

$$e^{\alpha k} = \frac{(1-2R) + R(e^{-i\lambda h}+e^{i\lambda h}) - \beta\gamma\Phi}{1-\beta(1+\gamma)\Phi e^{n\alpha k - ij\lambda h} + \beta\Phi e^{2(n\alpha k - ij\lambda h)}},$$

so we deduce that

$$\begin{aligned}&\beta\gamma\Phi + e^{\alpha k} - \beta(1+\gamma)\Phi e^{(n+1)\alpha k}\Big(\cos(j\lambda h) - i\sin(j\lambda h)\Big)\\ &+\beta\Phi e^{(2n+1)\alpha k}\Big(\cos(2j\lambda h) - i\sin(2j\lambda h)\Big) = 1 - 4R\sin^2(\tfrac{\lambda h}{2}).\end{aligned} \tag{14}$$

Expanding (14) into separate real and imaginary parts, we obtain the following identities:

$$\beta(1+\gamma)\Phi e^{(n+1)\alpha k}\sin(j\lambda h) - \beta\Phi e^{(2n+1)\alpha k}\sin(2j\lambda h) = 0, \tag{15}$$

$$\begin{aligned}\beta\gamma\Phi + e^{\alpha k} - \beta(1+\gamma)\Phi e^{(n+1)\alpha k}\cos(j\lambda h) + \beta\Phi e^{(2n+1)\alpha k}\cos(2j\lambda h)\\ = 1 - 4R\sin^2(\tfrac{\lambda h}{2}).\end{aligned} \tag{16}$$

Therefore, from (15), we have

$$e^{\alpha k} = (\frac{1+\gamma}{2\cos(j\lambda h)})^{\frac{1}{n}}.$$

Hence, according to the Fourier analysis, the NSFD scheme (11) is stable in the sense of Von Neumann if $|e^{\alpha k}| \leqslant 1$, or equivalently:

$$-1 \leqslant \frac{1+\gamma}{2\cos(j\lambda h)} \leqslant 1. \tag{17}$$

In order to show inequalities (17), we need to consider the following two cases.
Case1. Assume that $\cos(j\lambda h) > 0$, to prove (17), it is sufficient to show that

$$\frac{1+\gamma}{2\cos(j\lambda h)} \leqslant 1. \tag{18}$$

Since $\cos(j\lambda h) > 0$, there exists a number γ^*, such that

$$0 < \gamma^* < \cos(j\lambda h) \leqslant 1. \tag{19}$$

Now pick a number $\gamma \in (0,1)$ such that $\gamma^* = \frac{1+\gamma}{2}$. Substituting for γ^* in (19), we obtain the desired result, (18).
Case 2. Now suppose that $\cos(j\lambda h) < 0$. To prove (17), it is sufficient to show that

$$-1 \leqslant \frac{1+\gamma}{2\cos(j\lambda h)}. \tag{20}$$

Since $\cos(j\lambda h) < 0$, there exists a number γ^{**}, such that

$$-1 \leqslant \cos(j\lambda h) < \gamma^{**} < 0. \tag{21}$$

Choose a number $\gamma \in (0,1)$ such that

$$\gamma^{**} = -\frac{(1+\gamma)}{2}. \tag{22}$$

Substituting (22) into (21), we get the desired result (20).

On the other hand, by using the following inequality

$$\left|1 - 4R\sin^2(\frac{\lambda h}{2})\right| \leqslant 1 + 4R,$$

and the fact that $|e^{\alpha k}| \leqslant 1$, from (16), it follows that

$$\left|1 - 4R\sin^2(\frac{\lambda h}{2})\right| = \Big|\beta\gamma\Phi + e^{\alpha k} - \beta(1+\gamma)\Phi e^{(n+1)\alpha k}\cos(j\lambda h)$$
$$+\beta\Phi e^{(2n+1)\alpha k}\cos(2j\lambda h)\Big| \leqslant \beta\gamma\Phi + 1 + \beta(1+\gamma)\Phi + \beta\Phi = 1 + 2\beta\gamma\Phi + 2\beta\Phi.$$

Now, by imposing the condition

$$1 + 2\beta\gamma\Phi + 2\beta\Phi \leqslant 1 + 4R,$$

we obtain the following inequality

$$\frac{\beta\Phi(1+\gamma)}{2} \leqslant R,$$

where $R = \frac{\Phi}{\Psi}$. Consequently, the sufficient conditions for stability the NSFD scheme (11) are given as follows:

$$\beta\Psi(1+\gamma) \leqslant 2, \qquad \gamma \in (0,1). \tag{23}$$

□

Corollary 1. *The NSFD scheme* (11) *is convergent for* $\gamma \in (0,1)$ *and* $\beta\Psi(1+\gamma) \leqslant 2$ *with the Huxley equation* (2).

Proof. The result follows from Theorem 3 and the Lax theorem. □

5 Numerical results

To verify the effectivity of the NSFD scheme in section 4, we simulate initial-boundary value problem

$$\begin{aligned} u_t - u_{xx} &= \beta u(1-u)(u-\gamma), \qquad 0 \le x \le 1, \quad t \ge 0,\\ u(x,0) &= \frac{\gamma}{1+e^{-2A_1 x}}, \qquad 0 \le x \le 1,\\ u(0,t) &= \frac{\gamma}{1+e^{2A_1A_2 t}}, \qquad t \ge 0,\\ u(1,t) &= \frac{\gamma}{1+e^{-2A_1(1-A_2 t)}}, \qquad t \ge 0. \end{aligned} \tag{24}$$

We use the NSFD scheme (11); then we give the initial and boundary conditions

$$\begin{aligned} U_j^0 &= \frac{\gamma}{1+e^{-2A_1 x_j}}, \qquad j = 0,1,\dots,M,\\ U_0^n &= \frac{\gamma}{1+e^{2A_1A_2 t_n}}, \qquad n = 0,1,\dots,N,\\ U_M^n &= \frac{\gamma}{1+e^{-2A_1(x_M - A_2 t_n)}}, \qquad n = 0,1,\dots,N. \end{aligned} \tag{25}$$

To verify the efficiency and to measure its accuracy and the versatility of the NSFD scheme (11) for the current problem in comparison with the exact solution, absolute error for different values of β and γ is reported which is defined by

$$\big|u(x_j,t_n) - U(x_j,t_n)\big|,$$

in the point (x_j, t_n). Here $U(x_j, t_n)$ as the solution portraying the behaviors of physical system is obtained by the present scheme while $u(x_j, t_n)$ stands for the exact solution. Consider the Huxley equation in the form (24) with the initial and boundary conditions (25), and the exact solution (3). The results are compared with the exact solution. The numerical computations were performed by using uniform grids. The differences between the computed solution and the exact solution for some values of the constants β and γ are shown in Tables 1–4. As various problems of science were modeled by the Huxley equation, hence various values of the parameters have been considered in the following examples. In all numerical examples reported in this section, the Von Neumann stability conditions (23) are fulfilled for the NSFD scheme (11).

Example 1. In Table 1, the absolute errors have been shown for various values of β, x, and t with $\gamma = 0.001$. A comparison between the NSFD scheme [24] and the NSFD scheme (11) is given in Table 1. The numerical results show the high accuracy of the NSFD scheme (11).

Example 2. Table 2 shows the absolute errors for various values of γ, x, and t with $\beta = 10$. The results of the NSFD scheme (11) and the NSFD scheme [24] for different values of the parameters are shown in Table 2. Comparison of

Table 1: The absolute errors for various values of β, x, and t with $\gamma = 0.001$

x	t	NSFD [24] for $\beta = 1$	Presented NSFD for $\beta = 1$	NSFD [24] for $\beta = 10$	Presented NSFD for $\beta = 10$
	0.3	1.3158E–12	7.4809E–18	3.9633E–11	9.3750E–16
0.2	0.5	1.3753E–12	7.5894E–18	4.1390E–11	9.5875E–16
	0.9	1.3846E–12	7.6978E–18	4.1589E–11	9.1983E–16
	0.3	2.0465E–12	1.1384E–17	6.1653E–11	1.4631E–15
0.5	0.5	2.1477E–12	1.2034E–17	6.4648E–11	1.5020E–15
	0.9	2.1635E–12	1.2143E–17	6.4999E–11	1.4430E–15
	0.3	1.3159E–12	7.4809E–18	3.9651E–11	9.4271E–16
0.8	0.5	1.3755E–12	7.5894E–18	4.1408E–11	9.6363E–16
	0.9	1.3847E–12	7.9146E–18	2.3405E–11	9.2525E–16

current and exact results shows the efficiency and accuracy of the NSFD scheme (11).

Table 2: The absolute errors for various values of γ, x, and t with $\beta = 10$

x	t	NSFD [24] for $\gamma = 10^{-2}$	Presented NSFD for $\gamma = 10^{-2}$	NSFD [24] for $\gamma = 10^{-5}$	Presented NSFD for $\gamma = 10^{-5}$
	0.2	2.0599E–08	4.5578E–12	2.0529E–17	6.7762E–21
0.1	0.4	2.2927E–08	4.0248E–12	2.3054E–17	5.9292E–21
	0.6	2.3035E–08	2.8500E–12	2.3408E–17	7.6232E–21
	0.2	5.6063E–08	1.3031E–11	5.5677E–17	1.1011E–20
0.5	0.4	6.3727E–08	1.1893E–11	6.3850E–17	1.5246E–20
	0.6	6.4195E–08	8.6771E–12	6.4996E–17	2.1175E–20
	0.2	2.0724E–08	4.9406E–12	2.0527E–17	1.6940E–21
0.9	0.4	2.3063E–08	4.4080E–12	2.3053E–17	5.0821E–21
	0.6	2.3172E–08	3.2331E–12	2.3407E–17	9.3173E–21

Example 3. In Table 3 the absolute errors have been shown for various values of x, t, and γ with $\beta = 50$. The numerical results obtained by the NSFD scheme (11) show good accuracies and agreements with exact solutions.

Example 4. Table 4 shows the absolute errors for various values of x, t, and γ with $\beta = 100$. The results of the present scheme and the NSFD scheme [24] for the above values of the parameters are shown in Table 4. The numerical results reported in Table 4 indicate the high accuracy of the NSFD scheme (11).

Table 3: The absolute errors for various values of γ, x, and t with $\beta = 50$

x	t	NSFD [24] for $\gamma = 10^{-3}$	Presented NSFD for $\gamma = 10^{-3}$	NSFD [24] for $\gamma = 10^{-5}$	Presented NSFD for $\gamma = 10^{-5}$
	0.1	5.3170E–12	9.2720E–15	1.5906E–16	7.6232E–21
0.1	0.5	2.3504E–10	1.2583E–14	2.3744E–16	1.1858E–20
	1	2.3317E–10	1.2682E–14	2.3895E–16	9.3173E–21
	0.1	5.3230E–12	2.3789E–14	4.0521E–16	1.9481E–20
0.5	0.5	6.5311E–10	3.5184E–14	6.5885E–16	2.7952E–20
	1	6.4861E–10	3.0349E–14	6.6378E–16	2.8799E–20
	0.1	5.3289E–12	9.3692E–15	1.5906E–16	8.4703E–21
0.9	0.5	2.3553E–10	1.2682E–14	2.3745E–16	7.6232E–21
	1	2.3366E–10	1.0925E–14	2.3896E–16	1.1011E–20

Table 4: The absolute errors for various values of γ, x, and t with $\beta = 100$

x	t	NSFD [24] for $\gamma = 10^{-3}$	Presented NSFD for $\gamma = 10^{-3}$	NSFD [24] for $\gamma = 10^{-4}$	Presented NSFD for $\gamma = 10^{-4}$
	0.2	1.2498E–09	1.0843E–13	1.2564E–12	1.1004E–17
0.3	0.5	1.4195E–09	1.1179E–13	1.4482E–12	1.2590E–17
	0.8	1.4027E–09	9.7739E–14	1.4561E–12	1.2529E–17
	0.2	1.4802E–09	1.2868E–13	1.4865E–12	1.3030E–17
0.5	0.5	1.6913E–09	1.3352E–13	1.7241E–12	1.4989E–17
	0.8	1.6718E–09	1.1683E–13	1.7336E–12	1.4928E–17
	0.2	1.2522E–09	1.0890E–13	1.2566E–12	1.1018E–17
0.7	0.5	1.4222E–09	1.1226E–13	1.4488E–12	1.2590E–17
	0.8	1.4053E–09	9.8212E–14	1.4563E–12	1.2549E–17

Comparing the numerical results in Tables 1–4, it follows that the NSFD scheme (11) is more accurate than the NSFD scheme [24]. This follows because the local truncation error the NSFD scheme (11) is $O(\Delta t) + O(h^2)$, while the local truncation error the NSFD scheme [24] is $O(\Delta t) + O(h)$.

Figures 1 and 2 indicate the numerical solutions and the solitary wave solutions. Figures 1(a) and 11(b) compare the numerical results with the exact one for $\beta = 1$ and $\gamma = 0.001$ with stepsize $\Delta t = 0.001$ and $h = 0.1$ for a given fixed value of $x = 0.5$. Figures 2(a) and 2(b) show the error between two solutions of different formats. The two simulations show that the presented NSFD scheme is efficient. From the numerical results of this example, we conclude that the obtained results quite agreed with the exact one.

In Figures 3(a)–(d), we simulate absolute errors between the exact solution and the NSFD scheme for various stepsizes. In fact, we have compared absolute errors when $x = 0.5$, for two values of β and γ. It is noticeable

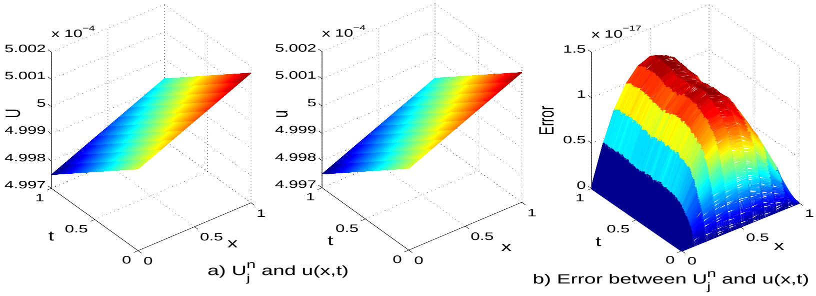

Figure 1: Simulations of the NSFD scheme for the Huxley equation with stepsize $\Delta t = 0.001$, $h = 0.1$.

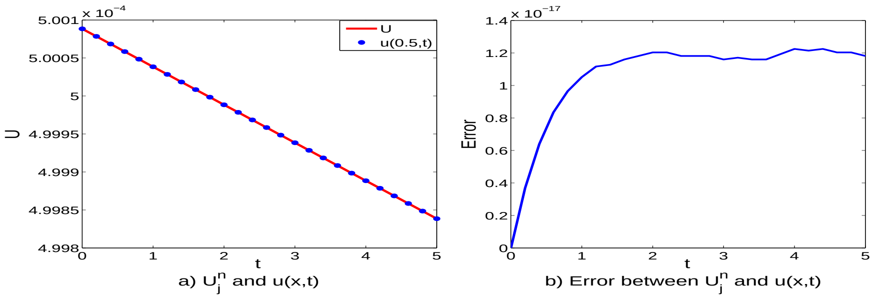

Figure 2: U and $u(x,t)$ at a fixed value $x = 0.5$ for the NSFD scheme.

that increasing stepsizes do not decrease accuracy of the scheme significantly. These comparisons show that the NSFD scheme (11) has good efficiency even for a high space stepsize.

In Table 5, we present the absolute errors of the computed solution for the Huxley equation, by ADM [7,8,10], VIM [1,2], DTM [3], Crank–Nicolson (CN) and NSFD scheme [24], with the NSFD scheme (11). Observing the numerical results, we conclude that the NSFD scheme (11) presents remarkably accurate in comparison with the other methods. In Tables 6 and 7,

Table 5: The absolute errors for $\beta = 1$ and $\gamma = 0.01$

x	t	ADM	VIM	DTM	CN	NSFD [24]	Presented NSFD
	0.05	2.4852E−06	2.4874E−06	2.4875E−06	3.7048E−07	3.5421E−10	2.6543E−14
0.1	0.1	4.9727E−06	4.9749E−06	4.9749E−06	4.0679E−07	5.2191E−10	3.9261E−14
	1 .0	4.9743E−05	4.9749E−05	4.9749E−05	4.4552E−07	7.8134E−10	4.3048E−14
	0.05	2.4763E−06	2.4875E−06	2.4874E−06	5.5746E−08	7.9198E−10	6.3115E−14
0.5	0.1	4.9638E−06	4.9749E−06	4.9749E−06	1.2894E−07	1.3305E−09	1.0499E−13
	1.0	4.9738E−05	4.9749E−05	4.9749E−05	2.4749E−07	2.1718E−09	1.2556E−13
	0.05	2.4673E−06	2.4874E−06	2.4874E−06	2.1847E−09	3.5492E−10	2.9874E−14
0.9	0.1	4.9548E−06	4.9749E−06	4.9749E−06	1.4873E−08	5.2275E−10	4.2973E−14
	1.0	4.9728E−05	4.9788E−05	4.9749E−05	4.9497E−08	7.8230E−10	4.6822E−14

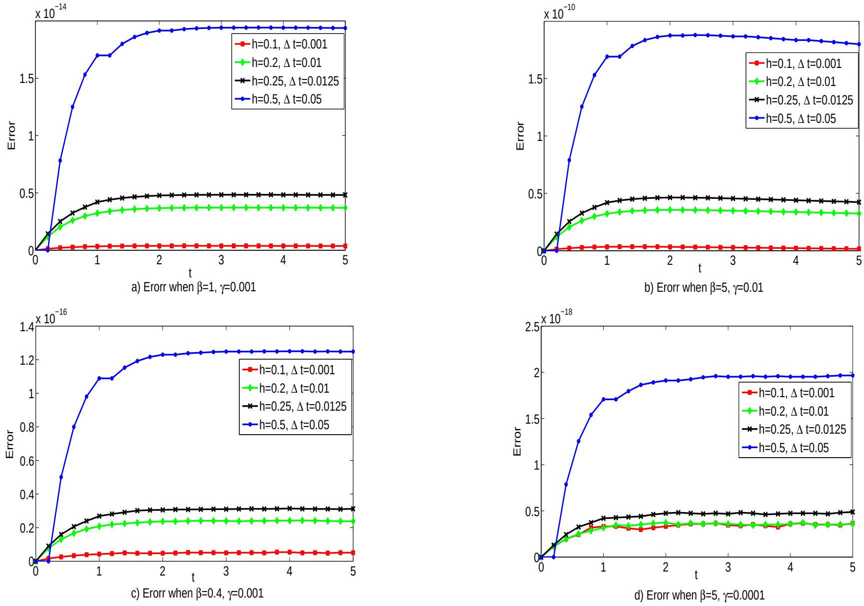

Figure 3: The absolute errors between $u(x_j, t_n)$ and U_j^n for various space and time stepsizes.

the numerical results of the presented NSFD scheme are compared with the NSFD scheme [24] and MCB–DQM, FDS4, ADM, OHAM, GCG, ECG, ELG methods in [16]. These comparisons show that the presented NSFD scheme for various values of x and t with $\beta = 1$ and $\gamma = 0.001$ offers better results than the others.

Table 8 shows comparison of the presented NSFD scheme with explicit exponential finite difference method (EFDM) in [9] for various values of x and t with $\beta = 1$ and $\gamma = 0.001$. According to the results presented in this table, the presented NSFD scheme offers high accuracy for the numerical solutions of the Huxley equation.

In Table 9, the numerical results of the meshless RBFs method in [11] are compared with the presented NSFD scheme for various values of x and t with $\gamma = 0.001$ and $\beta = 1$. The numerical results presented guaranteed the effectiveness of the presented NSFD scheme.

Table 6: Absolute errors for $\beta = 1$ and $\gamma = 0.001$

t	x	MCB-DQM	FDS4	ADM	OHAM	Presented NSFD
	0.1	1.0044E–08	2.4988E–08	1.8747E–07	2.4988E–08	1.8431E–18
0.05	0.5	2.3047E–08	2.4988E–08	1.8749E–07	2.4988E–08	4.3368E–18
	0.9	1.0044E–08	2.4987E–08	1.8751E–07	2.4988E–08	1.8431E–18
	0.1	1.4790E–08	4.9975E–08	3.7493E–07	4.9975E–08	2.9273E–18
0.1	0.5	3.8252E–08	4.9975E–08	3.7498E–07	4.9975E–08	7.2641E–18
	0.9	1.4790E–08	4.9975E–08	3.7502E–07	4.9975E–08	2.9273E–18
	0.1	2.2205E–08	4.9975E–07	3.7500E–06	4.9975E–07	4.2283E–18
1	0.5	6.2169E–08	4.9975E–07	3.7504E–06	4.9975E–07	1.1817E–17
	0.9	2.2205E–08	4.9975E–07	3.7509E–06	4.9975E–07	4.3368E–18

Table 7: Absolute errors for $\beta = 1$ and $\gamma = 0.001$

t	x	GCG	ECG	ELG	NSFD [24]	Presented NSFD
	0.1	1.0698E–08	1.0683E–08	9.2752E–09	3.5194E–13	1.8431E–18
0.05	0.5	9.2595E–09	9.2595E–09	9.2595E–09	7.8629E–13	4.3368E–18
	0.9	7.8921E–09	7.8701E–09	9.2845E–09	3.5201E–13	1.8431E–18
	0.1	2.3188E–08	2.3188E–08	2.3173E–08	5.2107E–13	2.9273E–18
0.1	0.5	2.1749E–08	2.1748E–08	2.1749E–08	1.3292E–12	7.2641E–18
	0.9	2.0382E–08	2.0381E–08	2.0360E–08	5.2115E–13	2.9273E–18
	0.1	2.4872E–07	2.4870E–07	2.4729E–07	1.3846E–12	4.2283E–18
1	0.5	2.4728E–07	2.4728E–07	2.4728E–07	2.1636E–12	1.1817E–17
	0.9	2.4591E–07	2.4585E–07	2.4530E–07	7.7895E–13	4.3368E–18

6 Conclusions

In this paper, we present an exact finite-difference scheme for the Huxley equation based on the method in Mickens papers. Moreover, we present an NSFD scheme for the Huxley equation. We also provided analysis of positivity, boundedness, consistency, stability, and convergence of the NSFD scheme. The numerical results obtained by the scheme, compared to ADM, VIM, DTM, CN, and NSFD [24] with the other methods, which reveal that the NSFD scheme is significantly more effective and accurate than the other methods in the literature.

Table 8: The absolute errors for value of γ, x, and t with $\gamma = 0.001$ and $\beta = 1$

x	t	EFDM	Presented NSFD
	0.05	1.030307E–08	1.84314E–18
0.1	0.1	1.506294E–08	2.92734E–18
	1	2.248771E–08	4.22838E–18
	0.05	2.313697E–08	4.33680E–18
0.5	0.1	3.843952E–08	7.26415E–18
	1	6.246539E–08	1.18178E–17
	0.05	1.030307E–08	1.84314E–18
0.9	0.1	1.506294E–08	2.92734E–18
	1	2.248771E–08	4.33680E–18

Table 9: The absolute errors for value of γ, x, and t with $\gamma = 0.001$ and $\beta = 1$

x	t	meshless RBFs method (MQ)	Presented NSFD
	0.1	0.0E–09	1.9E–18
0.05	0.5	1.0E–09	4.4E–18
	0.9	1.0E–09	1.8E–18
	0.1	1.0E–09	3.1E–18
0.1	0.5	0.0E–09	7.1E–18
	0.9	0.0E–09	2.8E–18
	0.1	1.0E–09	4.4E–18
1	0.5	0.0E–09	1.2E–17
	0.9	1.0E–09	4.7E–18

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تقریب معادله هاکسلی با طرح تفاضل متناهی غیراستاندارد

مهران نامجو و صادق زیبایی

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دریافت مقاله ۲۲ دی ۱۳۹۵، دریافت مقاله اصلاح شده ۴ آذر ۱۳۹۶، پذیرش مقاله ۲۳ خرداد ۱۳۹۷

چکیده : در این مقاله یک طرح تفاضل متناهی دقیق از نوع صریح برای معادله هاکسلی براساس طرح تفاضل متناهی غیراستاندارد ارائه شده است. در ادامه یک طرح تفاضل متناهی غیراستاندارد برای جواب عددی معادله هاکسلی پیشنهاد و مثبت بودن، کرانداری، سازگاری، پایداری و همگرایی طرح مورد بررسی قرار گرفته شده است. به منظورنشان دادن دقت و کارایی طرح، نتایج عددی آن با جواب دقیق و برخی روش‌های موجود مقایسه شده است.

کلمات کلیدی : معادله هاکسلی؛ طرح تفاضل متناهی غیراستاندارد؛ کرانداری و مثبت بودن؛ سازگاری، پایداری و همگرایی.