JₙRₘ

Utilizing Monte Carlo Method for Ranking Extreme Efficient Units in Data Envelopment Analysis

Gh-R.Jahanshahloo[a] , M.Zahedi-Seresht[b*]

[(a,b)] *Department of Mathematics, Kharazmi University, Tehran, Iran*

Abstract

Data envelopment analysis (DEA) is a mathematical programming method for calculating efficiency of decision making units (DMU). In calculating the efficiency score of units through DEA we may come up with some efficient units. But the question is among these efficient units which of them is better. As we know, it is possible to rank inefficient units through efficiency score; however, for ranking efficient units it is not helpful and other methods should be developed in these regards. To obviate this problem there have been so many attempts in the literature which have their pros and cons. Cross-efficiency method was first introduced by Sexon et al. for ranking efficient units. The major problem of this method is alternative optimal solutions in each model which must be solved for each DMU. Another problem of this method is dependency of obtained solutions on the solution obtained by other units. Another method which has widely been used is super efficiency, presented by Anderson and Petersen. There are several flaws in their suggested method. Infeasibility, instability, dependency of the model on the input and output orientation and non-zero slack variables are the weaknesses of this method which may occur in specific problems. This article is an attempt to present a method which does not have the aforementioned problems and can be utilized to calculate the rank of extreme efficient units through using the Hit or Miss Monte Carlo method. At the end of the article some examples are made in order to show the efficiency of the presented method.

---

[*] Corresponding author: Zahedi_s@yahoo.com

1- Introduction

Data Envelopment Analysis (DEA) was first introduced by Charnes et al. [1] through presenting CCR model and it was developed further by Banker et al. [2] through introducing BCC model. DEA is an efficient method for calculating peer decision making units with some input and outputs based on the observed data. The importance of DEA lies in this fact that it can be utilized for evaluating decision making units such as banks, universities, public units, etc. Consider n number of DMUs with input vectors of $x_j = (x_{1j}, x_{2j}, \ldots, x_{mj})$ and output vectors of $y_j = (y_{1j}, y_{2j}, \ldots, y_{sj})$ which j=1,..,n such that x_{ij} and y_{ij} represent i input and i output for DMU_j j= 1,…,n respectively which all of them are nonnegative. The efficient score of DMU_d is calculated by the equation $e_d(u,v) = y_d u^t / x_d v^t$ in which u and v are the weight vectors of inputs and outputs. $E_d(u,v)$ is called the relative efficiency of DMU_d and is defined as follows:

$$E_d(u,v) = e_d(u,v)/\max \{ e_j(u,v) \mid j = 1,\ldots,n\} \qquad (1)$$

It is clear that E_d^* is not beyond 1 and each unit whose optimal objective value equals 1 is considered as efficient. For each n units, the model (1) should be solved once and the optimum solution of model (1) introduces input and output weights which is different from optimum solutions of other DMUs. As we know, it is possible the more than one DMU be efficient and their efficient scores equal to 1. It could be intriguing to study DEA in order to find how efficient units could be discriminated. There has been so much research in this area, each of them presents a method for ranking efficient units. Abello et al. [3] discussed special qualities and properties. Sexton et al. [4] proposed cross-efficiency method. This method uses weights obtained by solving each of n linear programming problems. The cross-efficiency method calculates the efficiency score of each DMU n times and save the data in a matrix. The cross-efficiency score of each DMU is included in each row of this matrix. Then the average of these rows should be calculated. This number is the ranking score of DMU. There are some problems in this method. The major one appears when the DEA models give alternative solutions. Sungmook Lim [5] replaced the secondary goal through minimizing (or maximizing) the best (or worst) cross-efficiency of peer DMUs.

Another important model for ranking extreme efficient units has been proposed

by Andersen and Petersen [6]. This method removes the extreme DMU from observed set and rank ranks it based on the effect of this unit on the production possibility set. Instability and infeasibility in some cases are the main problems of this model that will be completely explained in the next section. Then, Mehrabian et al. (MAJ) [7] have improved the AP model. However in some circumstances, this model is infeasible. Saati et al. [8] have modified MAJ model and solved its infeasibility and Jahanshahloo et al.[9] have changed the type of data normalization to achieve better result. Jahanshahloo et al. [10] proposed a ranking method based on a full-inefficient frontier. This method can be used for ranking DMUs to get information about the system, and also for ranking only efficient DMUs.

Some papers based on specific norms have proposed some models to remove the difficulties of these models. For instance, Jahanshahloo et al. [11] have used L1 norm and Amirteimori et al. [12] have proposed L2 norm for ranking DMUs. Jahanshahloo et al. [13] proposed Gradient line to rank efficient units. The most advantage of this method is that it is always feasible. The L∞ norm (Tchebycheff norm) for ranking extreme

efficient units was proposed by Rezai Balf et al. [14]. This method was always feasible as well. Moreover, Jahanshahloo et al. [15] proposed two ranking methods. First, they defined an ideal line and determined a common set of weights for efficient DMUs to obtain a new efficiency score and then efficient units were ranked by this score. Then they defined a special line and compared all efficient DMUs with this line for ranking. These studies continued. Chen, Y. [16] suggested new super-efficiency model to recognize the super-efficiency score when infeasibility occurs. This model can be used for ranking the efficient DMUs even if infeasibility occurs. Adler, N. et al., [17] divided the all ranking methods into six groups, somewhat overlapping areas. Abri et al. [18] proposed convex combination of extreme DMUs for ranking non-extreme DMUs.

In this article the effect of DMU on the product possibility set is to be calculated through the Hit or Miss Monte Carlo method in order to rank the intended unit. Like super-efficiency, in this paper a method is presented which is based on omitting extreme efficient DMU from observed set. The presented method obtains the rank of this unit through simple calculation without any problems

experienced in other methods, especially in super-efficiency. In part 2, we will state two widely used methods and define their uses. In part 3, production possibility set (PPS) will be introduced. Also, in this part a method will be presented can check whether the newly created artificial DMU belonging PPS created by n DMU or not. Also in this section, through examples with 2 inputs and 1 output, the rank of units is investigated by the super-efficiency method, cross-efficiency method and our presented method.

In part 4, the efficiency of our method is compared with other methods through a practical example with 3 inputs and 3 outputs. Like super-efficiency method, our method can also obtain the rank of extreme efficient units without any drawbacks of other methods, specifically the super-efficiency method. At the end, a synopsis of the conducted procedure in the study is presented along with some suggestions for further studies.

2- Background

In this section, CCR model and its efficiency and the two widely used methods of super-efficiency and cross-efficiency are to be explained.

2-1- The CCR model and efficiency

The CCR model which was presented by Charnes et al. [1] is an input oriented standard model for n DMUs. Each DMU is specified by some inputs $x_j = (x_{1j}, x_{2j}, \ldots, x_{mj})$ and outputs $y_j = (y_{1j}, y_{2j}, \ldots, y_{sj})$ in which all inputs and outputs are nonnegative and all DMUs are independent and homogenous. The efficient score of DMU_d is obtained by the following model known as CCR model:

$$\text{Min} \quad \theta_d \tag{2}$$
$$\text{s.t.} \quad \sum_{j=1}^{n} \lambda_j \, x_{ij} \leq \theta \, x_{id} \qquad i=1,\ldots,m$$
$$\sum_{j=1}^{n} \lambda_j \, y_{rj} \geq y_{rd} \qquad r=1,\ldots,s$$
$$\lambda_j \geq 0 \qquad j=1,\ldots,n$$

$\theta=1, \lambda_j = 0 \ (j{\neq}d)$ and $\lambda_d = 1$ is feasible for model (2). So, the optimal solution for this model which is presented by θ^* is less than 1 or equals to 1. It is clear that if θ_d^* equals to 1, DMU_d will be efficient and if θ_d^* less than 1, DMU_d will be inefficient. Dual of the model (2) is as follows:

$$\text{Max} \quad E_{dd}=\sum_{r=1}^{s} u_{rd} \, y_{rd} \tag{3}$$
$$\text{s.t.} \quad \sum_{i=1}^{m} v_{id} \, x_{id} = 1$$
$$\sum_{r=1}^{s} u_{rj} \, y_{rj} - \sum_{i=1}^{m} v_{ij} \, x_{ij} \leq 0 \quad j=1,\ldots,n$$
$$u_{rd} \geq 0 \, , v_{id} \geq 0 \qquad r=1,\ldots,s \quad i=1,\ldots,m$$

In this model E_{dd}^* equals to θ_d^* in CCR model for DMU_d.

2-2- Cross-Efficiency and Supper-Efficiency

As it was mentioned, usually among evaluating units we may encounter some efficient units in which it is not possible to rank them according to the efficiency scores. Therefore, it was deemed necessary to devise new models for ranking efficient units. Scholars in this field have proposed various models for ranking efficient unit which some of them will be stated below. Sexton et al. [4] presented cross-efficiency method based on optimum weights of each unit. It should be mentioned that model (3) is equal to the following model:

$$E_{dd}^* = \text{Max} \qquad E_{dd} = \frac{\sum_{r=1}^{s} u_{rd} y_{rd}}{\sum_{i=1}^{m} v_{id}^* x_{id}} \qquad (4)$$

$$\text{s.t.} \qquad E_{dj} = \frac{\sum_{r=1}^{s} u_{rd}^* y_{rj}}{\sum_{i=1}^{m} v_{id}^* x_{ij}} \leq 1 \qquad j=1,\ldots,n$$

$$u_{rd} \geq 0 \qquad r=1,\ldots,s$$

$$v_{id} \geq 0 \qquad i=1,\ldots,m$$

In which v_{id} are the i input weight and u_{rd} of the r output weight of DMU_d. Then, the cross-efficiency for DMU_j of the DMU_d weights which is obtained from model (4) is as follows:

$$E_{dj} = \frac{\sum_{r=1}^{s} u_{rd}^* y_{rj}}{\sum_{i=1}^{m} v_{id}^* x_{ij}} \qquad d,j=1,\ldots,n \qquad (5)$$

In which u_{rd}^* and v_{id}^* are the optimum weights of the model (4).

For DMUj (j = 1,2,… ,n) the average of all E_{dj} (d= 1,…, n) that is $\overline{E}_J$ is considered as cross-efficiency score as follows:

$$\overline{E}_j = \frac{1}{n} \sum_{d=1}^{n} E_{dj} \qquad (6)$$

As it was mentioned above, this method has some weaknesses. The major weakness of this method is alternative optimum solutions for each DMU which is obtained through model (4). Also, the solution of equation (5) directly depends on the optimal solution of model (4).

Another widely used method for efficient units is super-efficiency presented by Anderson and Petersen [6]. This method was created based on model (3) or CCR model in which we omit a DMU from the set of observations and rank it between the remained DMUs. Super-efficiency model is as follows:

$$\text{Min} \quad \theta \qquad (7)$$

$$\text{s.t.} \qquad \sum_{\substack{j=1 \\ j \neq o}}^{n} \lambda_j \, x_{ij} \leq \theta x_{io} \qquad i=1,\ldots,m$$

$$\sum_{\substack{j=1 \\ j \neq o}}^{n} \lambda_j \, y_{rj} \geq y_{ro} \qquad r=1,\ldots,s$$

$$\lambda_j \geq 0 \qquad j=1,\ldots,n , \qquad j \neq o$$

Although this model is useful to rank extreme efficient DMUs, Thrall [19] pointed out that it may be infeasible and instable in CCR model. However, it is not yet clear under which conditions the model is inefficient or instable. Another weakness of this model is that it just can be utilized for ranking extreme efficient DMUs, though most of the efficient units in real problems are extreme DMU.

3- Proposed Method

Consider n DMUs and assume some of them are efficient and extreme units. As mentioned before, super-efficiency and cross-efficiency methods have some drawbacks for ranking.

In this section we propose a method for ranking them through generating random DMUs in the production of possibility set (PPS).

The set of feasible activities is called PPS which is denoted by T_c. We postulate the PPS of the proposition of T_c

P1-The observed activities (x_j, y_j) (j=1,…,n) belong to T_c.

P2-If an activity $(x, y) \in T_c$ then $(\lambda x, \lambda y) \in T_c$ for every positive scalar λ.

P3-For an activity $(x, y) \in T_c$, any nonnegative activity $(\bar{x}, \bar{y}) \in T_c$ with $\bar{x} \geq x$ and $\bar{y} \leq y$.

P4-Any nonnegative linear combination of activities in T_c belong to T_c.

Using P1, P2, P3 and P4 for n DMUs we can conclude as follows:

$$T_C = \left\{ (x, y) \left| \begin{array}{l} \sum_{j=1}^{n} \lambda_j x_j \leq x, \\ \sum_{j=1}^{n} \lambda_j y_j \geq y, \\ \lambda_j \geq 0 \text{ for } j = 1, \dots, n \end{array} \right. \right\}$$

Where λ is a nonnegative vector in R^n.

Fig.(1) represents PPS for n unit with one input and output which was created through using 4 propositions mentioned above. As you can see in the Fig.(1), all observed DMUs are present and in addition there are other artificial DMUs.

3-1 calculating the ratio of the volume of old PPS and new PPS resulting from omitting the extreme efficient DMU

Here the question is whether one new DMU belongs to old PPS or not. How can we determine whether this DMU belongs to the PPS of the old DMUs or not?

Consider model (8):

$$\text{Min} \quad \theta_{new} \tag{8}$$

s.t.
$$\sum_{j=1}^{n} \lambda_j x_{ij} \leq \theta x_{i\,new} \qquad i=1,\dots,m$$
$$\sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{r\,new} \qquad r=1,\dots,s$$
$$\lambda_j \geq 0 \qquad j=1,\dots,n$$

In which (x_{new}, y_{new}) is new DMU. In calculating model (8), if $\theta^*_{new} > 1$ or model (8) is infeasible, then new DMU does not belong to PPS and if $\theta^* \leq 1$, new DMU belongs to PPS which was created by n DMUs.

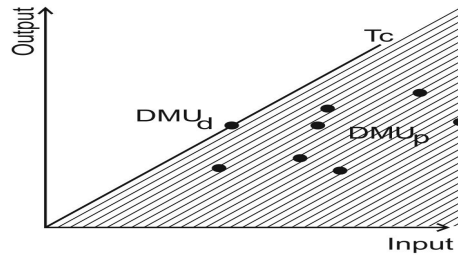For instance, consider 3 DMUs with their PPS in Fig. (2) below:
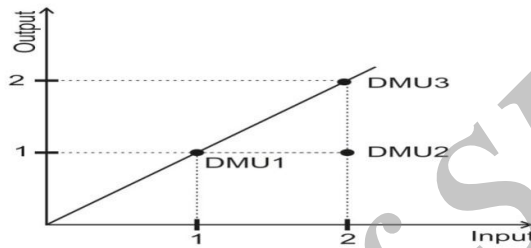
Fig.1. The Production Possibility Set (PPS).

Fig. 2. The PPS of 3 DMUs.

Set of observations includes DMU1, DMU2 and DMU3. Now through using model (8), we want to see whether DMU_{new} belongs to the PPS created by the 3 DMUs or not. Consider Fig. (3) and model (9) for this question:

$$\text{Min} \quad \theta_{new} \quad (9)$$
$$\text{s.t.} \quad \lambda_1 + 2\lambda_2 + 2\lambda_3 \leq 1/2\theta$$
$$\lambda_1 + \lambda_2 + 2\lambda_3 \geq 1$$
$$\lambda_1, \lambda_2, \lambda_3 \geq 0$$

In model (9), $\theta_{new}^* = 2$ which represents DMU_{new} doesn't belong to the PPS created by 3 DMUs.

Now consider Fig. (4) and model (10) for new DMU:

$$\text{Min} \quad \theta_{new} \quad (10)$$
$$\text{s.t.} \quad \lambda_1 + 2\lambda_2 + 2\lambda_3 \leq 3/2\theta$$
$$\lambda_1 + \lambda_2 + 2\lambda_3 \geq 1$$
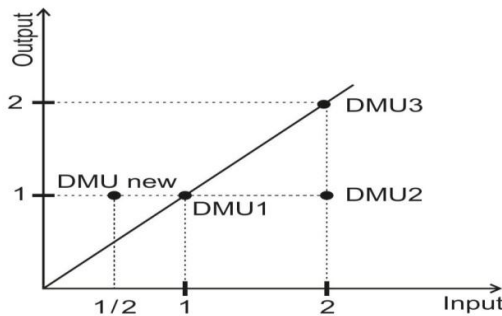$$\lambda_1, \lambda_2, \lambda_3 \geq 0$$

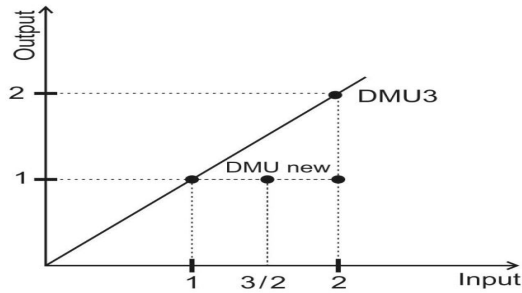Fig. 3. The PPS of 3 DMUs and new DMU out of the PPS.

Fig.4. The PPS of 3DMUs and a DMU into the PPS.

In model (16), $\theta^*_{new} = 2/3$ represents that DMU_{new} belongs to the PPS created by 3 DMUs.

Now consider a PPS which is made by n DMU. Through omitting extreme efficient DMU_d , from set of observation it is clear that T_C changes to T'_C. It is clear that T'_C is feasible as yet n-1 DMUs remain.

$$T'_C = \left\{ (x,y) \middle| \begin{array}{l} \sum_{\substack{j=1\\j\neq d}}^{n} \lambda_j\ x_j \leq x, \\ \sum_{\substack{j=1\\j\neq d}}^{n} \lambda_j\ y_j \geq y\ , \\ \lambda_j \geq 0\ \text{for}\ j = 1,...,n\ , j \neq d \end{array} \right\}$$

Consider T= $T_c - T'_c$ is the effect of DMU_d on PPS and the result of n DMUs. T equals to the share of DMU_d in making PPS and it is possible to utilize it for ranking extreme efficient units. Consider Fig.(5):

It is obvious that DMU_1 is the extreme efficiency. After omitting it from set of observation, T'_C are made from two other DMUs. See Fig.(6) :
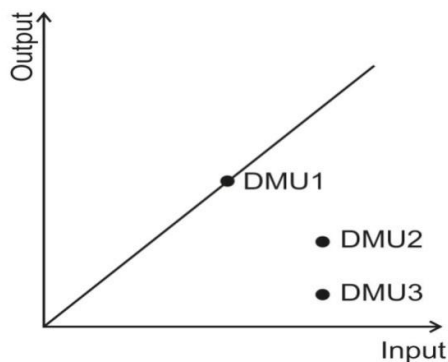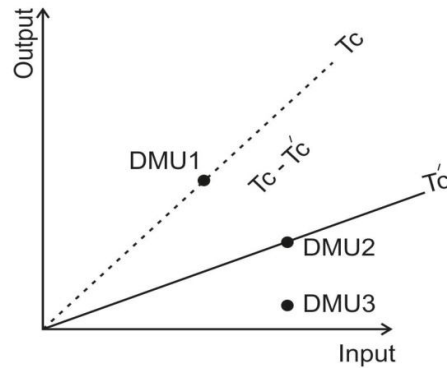

Fig. 5.The PPS for 3 DMUs.

Fig. 6.The PPS before and after removing a DMU.

As shown, DMU_1 has a contribution in making half of the PPS before the change. In order to calculate such a change after omitting, the super-efficiency method should solve model (7) which may cause some problems.

Infeasibility and instability are the major problems which model (7) represents. In this section, we will present a method which does not have any of aforementioned problems and can be used for ranking extreme efficient units. Then, a method will be presented which calculate the contribution of extreme efficient DMU in making PPS and we can use it for ranking extreme efficient units. In the next section the presented method ranks units without any problems mentioned for other methods.

Consider model (2) for n DMU. Assume $\bar{x}_i = \max\{x_{ij}, j = 1, ..., n\}$ for

i=1,…, m and $\bar{y}_r = \max\{y_{rj}, j = 1, ..., n\}$ for r=1,…,s. The first category of these constraints is divided by $\bar{x}_{ij}$ (i=1,…,m) and the second one is divided by $\bar{y}_r = \max\{y_{rj}, j = 1, ..., n\}$ for r=1,…,s. Through this, data are normalized without any problem for the overall structure of the problem and data.

$$\text{Min} \quad \theta \tag{11}$$
$$\text{s.t.} \quad \sum_{j=1}^{n} \lambda_j \frac{x_{ij}}{\bar{x}_i} \leq \theta \frac{x_{id}}{\bar{x}_i} \qquad \text{i=1,…,m}$$
$$\sum_{j=1}^{n} \lambda_j \frac{y_{ij}}{\bar{y}_r} \geq \frac{y_{rd}}{\bar{y}_r} \qquad \text{r=1,…,s}$$
$$\lambda_j \geq 0 \qquad \text{j=1,…,n}$$

This change does not affect CCR and PPS models. Now model (11) is rewritten into model (12):

$$\text{Min} \quad \theta \tag{12}$$
$$\text{s.t.} \quad \sum_{j=1}^{n} \lambda_j \hat{x}_{ij} \leq \theta \hat{x}_{io} \qquad \text{i=1,…,m}$$
$$\sum_{j=1}^{n} \lambda_j \hat{y}_{rj} \geq \hat{y}_{ro} \qquad \text{r=1,…,s}$$
$$\lambda_j \geq 0 \qquad \text{j=1,…,n}$$

For all r,s. for all i,j and $\quad 0 \leq \hat{y}_{rj} \leq 1$ In model (12) , $\quad 0 \leq \hat{x}_{ij} \leq 1$

Consider model (13) and (14) as follows:

$$\text{Min} \quad \theta_{T_C} \qquad\qquad (13)$$

s.t. $\quad \sum_{j=1}^{n} \lambda_j\, x_{ij} \leq \theta\, x_{i\,new} \qquad$ i=1,…,m

$\qquad \sum_{j=1}^{n} \lambda_j\, y_{rj} \geq y_{r\,new} \qquad$ r=1,…,s

$\qquad\qquad \lambda_j \geq 0 \qquad\qquad$ j=1,…,n

$$\text{Min} \quad \theta_{T_C'} \qquad\qquad (14)$$

s.t. $\quad \sum_{\substack{j=1 \\ j\neq d}}^{n} \lambda_j\, x_{ij} \leq \theta x_{i\,new} \qquad$ i=1,…, m

$\qquad \sum_{\substack{j=1 \\ j\neq d}}^{n} \lambda_j\, y_{rj} \geq y_{r\,new} \qquad$ r=1,…,s

$\qquad\qquad \lambda_j \geq 0 \qquad$ j=1,…,n $\quad$, j ≠ d

In which model (14) is obtained as the result of omitting DMU_d from model (13). Generate new DMU with new inputs and outputs between zero and one randomly and make vectors of input and output (x_{new}, y_{new}). Then follow these steps:

1-Consider R=1 and N′ =1.

2-Solve model (14).

3-If $\theta_{T_C'}^* \leq 1$, then $(x_{new}, y_{new}) \in T_C'$,Change N′ to N′ + 1 and again generate new random DMU and go to step2 otherwise go to step4.

4-If $\theta_{T_C'}^* > 1$, then solve model (13) and go to step5.

5-If $\theta_{T_C}^* \leq 1$, then $(x_{new}, y_{new}) \in$ T= $T_C - T_C'$,change R to R+1 and N′ to N′ + 1 and again generate new random

DMU and go to step2 otherwise go to step6.

6-If $\theta^* > 1$, then $(x_{new}, y_{new}) \notin T_C$. Again generate new random DMU and go to step2.

Where R equals to the number of random generated artificial input and output vectors in T= $T_C - T_C'$ and it represents the contribution of DMU_d in making PPS. Also consider N′ as generated artificial input and output vectors which are in T_C. Consider $r_d = N'/(N'-R) \geq 1$ as scores for raking Extreme efficient DMUs. R is the effect and the contribution of DMU_d in making PPS and r_d is the efficiency score of this extreme unit. It is obvious that if R equals to zero, then $r_d = 1$ and DMU_d is not extreme.

As it was mentioned, due to the weaknesses of cross-efficiency and super-efficiency methods, we proposed a new method. (a) In super-efficiency method through omitting an extreme efficient unit it is possible that infeasibility may occur in model (7). This problem is absent in our proposed method. (b) The second problem in this method was instability which is not present in proposed method (c) Our method is not dependent on the input and output orientation while super-efficiency method depends on the orientation. (d) Our

method does not have to do with nonzero slack variables while in super-efficiency method it is one of the problems (e) alternative optimum solution is one of the drawbacks of the cross-efficiency method which does not exist in our presented method since a unique value of objective function, i.e. θ, is used. (f) The efficiency score of cross-efficiency is related to the optimum solutions of other units while our proposed method is not related to other decision making units.

Consider 3 DMUs with 2 inputs and one output in table 1.

As it can be seen in table 2 , all of them are efficient.

According to the obtained results in cross-efficiency and super-efficiency methods, it is seen that the rank of DMUs is different from each other. This indicates that these methods are not reliable for ranking.

Since the rank of each efficient unit is directly related to its effect on the PPS, our suggested method of ranking can calculate the rank of extreme efficient units through considering their effects on the PPS. To rank available efficient units in this question we generate 1000 random artificial DMUs with inputs and outputs between zero and one.

Through solving model (13) and (14) and the aforementioned steps, R and N′ are achieved for ranking. Then, we calculate r_j(j=1,…,n).

Table 1. Inputs and outputs of 3 DMUs.

DMUs	DMU_A	DMU_B	DMU_C
Output y	1	1	1
Input x_1	1	4	6
Input x_2	6	2	2

Table 2. Ranking by super-efficiency and cross-efficiency.

	Efficiency	Super Efficiency score	Ranking by Super Efficiency	Cross-Efficiency score	Ranking by Cross-Efficiency
DMU_A	1.00	4	1	0.55	3
DMU_B	1.00	1.3	2	0.75	1
DMU_C	1.00	1	3	0.72	2

From the information of table 3, A has the first rank, next is B and then C between 3 efficient DMUs.

As shown in table (4), if we increase random generated input and output vectors to 2000, the rank does not change. For 2000 generated vectors in table (4).

It can be seen from tables (3) and (4) that among efficient units A has the first rank;

B has second rank and C is the last. As R=0 for C, it can be concluded that it is not extreme DMU.

Now consider 3 new DMUs in table (5).

The result of ranking by the supper-efficiency and the cross efficiency are shown in table (6)

Table 3. The value of R and r_j for 1000 artificial generated random DMUs.

DMUs	A	B	C
R	152	40	0
N'	677	718	723
r_j	1.29	1.06	1

Table 4. The value of R and r_j for 2000 artificial generated random DMUs.

DMUs	A	B	C
R	274	70	0
N'	1395	1430	1425
r_j	1.24	1.05	1

Table 5. Inputs and outputs of 3 DMUs.

DMUs	DMU_A	DMU_B	DMU_C
Output y	0	1	1
Input x_1	1	4	6
Input x_2	6	2	2

Table 6. Ranking by super-efficiency and cross-efficiency.

	Efficiency	Super Efficiency score	Ranking by Super Efficiency	Cross-Efficiency score	Ranking by Cross-Efficiency
DMU_A	1.00	Infeasible	Infeasible	0.65	3
DMU_B	1.00	1.3	1	0.75	1
DMU_C	1.00	1	2	0.72	2

As can be seen in table (6), model (7) is infeasible for first DMU. Now, see table (7) and table (8) which is ranked all 3 DMUs by our proposed method.

As can be seen, infeasibility does not occurred in our method.

4- Numerical example

Jahanshahloo et al., [20] used empirical example to rank 20 DMUs .This example has three inputs and outputs as it is shown in Table (9) below:

In table (10) for all inefficient DMUs it is possible to use their efficient scores and there is no problem to rank. For ranking efficient units in table (9) the new method presented in this article has been utilized which their ranks are shown in table (10). Since R was not zero for all of them, all of the efficient units were extreme. In table (10) the ranks of all units are calculated by supper-efficiency method and cross-efficiency method. It can be seen that ranks are different in two methods while in our method there is no change in ranks with different random generated vectors.

To assess the correlation between the new methods presented in this study with those widely used and discussed in the relevant literatures, we have used the Spearman rank-order correlation test, the results of which are presented in Table 11.

Table 7. The value r_j for 1000 artificial generated random DMUs.

DMUs	A	B	C
R	205	46	0
N'	730	775	767
r_j	1.28	1.05	1

Table 8. The value r_j for 2000 artificial generated random DMUs.

DMUs	A	B	C
R	384	63	0
N'	1505	1524	1509
r_j	1.34	1.04	1

Table 9. Data of 20 Iranian bank branches (Empirical Example).

DMUs	Inputs			Outputs		
	Staff	Computer terminals	Space(m^2)	Deposits	Loans	Charges
1	0.950	0.700	0.155	0.190	0.521	0.293
2	0.7096	0.600	1.000	0.277	0.627	0.462
3	0.798	0.750	0.513	0.228	0.970	0.261
4	0.865	0.550	0.210	0.193	0.632	1.000
5	0.815	0.850	0.268	0.233	0.722	0.246
6	0.842	0.650	0.500	0.207	0.603	0.569
7	0.719	0.600	0.350	0.182	0.900	0.716
8	0.785	0.750	0.120	0.125	0.234	0.298
9	0.476	0.600	0.135	0.080	0.364	0.244
10	0.678	0.550	0.510	0.082	0.184	0.049
11	0.711	1.000	0.305	0.212	0.318	0.403
12	0.811	0.650	0.255	0.123	0.923	0.628
13	0.659	0.850	0.340	0.175	0.645	0.261
14	0.976	0.800	0.540	0.144	0.514	0.243
15	0.685	0.950	0.450	1.000	0.262	0.098
16	0.613	0.900	0.525	0.115	0.402	0.464
17	1.000	0.600	0.205	0.090	1.000	0.161
18	0.634	0.650	0.235	0.059	0.349	0.068
19	0.372	0.700	0.238	0.039	0.190	0.111
20	0.583	0.550	0.500	0.110	0.615	0.764

Table 10. Ranking By our new method, super-efficiency and cross-efficiency.

	Efficiency	Ranking With 1000 random DMUs	Ranking With 2000 random DMUs	Super-efficiency ranking	Cross-efficiency ranking
DMU 1	1.000	7	7	7	9
DMU 2	0.901	9	9	9	12
DMU 3	0.991	8	8	8	6
DMU 4	1.000	2	2	2	2
DMU 5	0.897	10	10	10	8
DMU 6	0.748	14	14	14	11
DMU 7	1.000	5	5	5	1
DMU 8	0.797	12	12	12	16
DMU 9	0.787	13	13	13	13
DMU10	0.289	20	20	20	20
DMU11	0.604	16	16	16	14
DMU12	1.000	6	6	6	3
DMU13	0.816	11	11	11	10
DMU14	0.469	18	18	18	17
DMU15	1.000	1	1	1	4
DMU16	0.639	15	15	15	15
DMU17	1.000	3	3	3	5
DMU18	0.472	17	17	17	18
DMU19	0.408	19	19	19	19
DMU20	1.000	4	4	4	7

Table 11. Spearman rho for proposed method and other methods.

methods	Spearman rank-order correlation
Our method & Super-efficiency	1
Our method & Cross-efficiency	0.925

As can be seen in Table 11, there is a strong correlation between this new method and other widely used ones and scatterplots are as follow:

As the scatter plots Fig (7) and Fig (8) and Table 11 suggest, there is a strong correlation between the new proposed method for ranking and other widely used methods. However, our method can do without such complications as infeasibility, alternative optimal solutions, and complex computations, associated with the conventional ones.
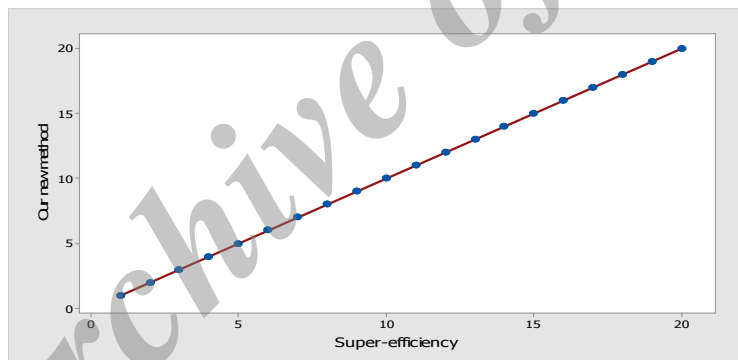

Fig. 7. Scatterplot of Our new method vs Super-efficiency.
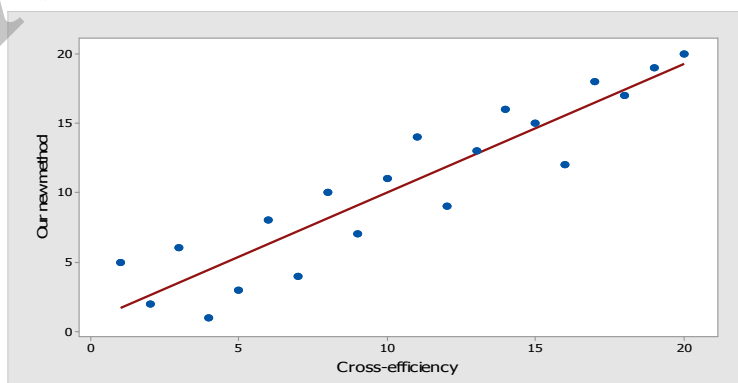

Fig. 8. Scatterplot of Our new method vs Cross-efficiency.

However, some questions might arise regarding this new method: how many random numbers should be generated for computing ? Or at what point can one be assured that the ranking of the DMUs would not change anymore? To assure that the ranking would not change anymore when using the our new method, go through the following steps :

1- Let C=1000.

2- Repeat the algorithmic steps presented in Section3.

3- Let C=C+a (where a can be defined by the user)

4- Repeat the algorithmic steps presented in Section3.

5- If the ranking with C random numbers is the same as the ranking with C-a, proceed to step 6; otherwise, return to step 3.

6- Now, for measuring validity, use the Spearman rank-order correlation test for ranking with our new method and other conventional methods.

As in Table 10, we can increase the number of random numbers until the ranking offered by this method turns out to be the same in two consecutive steps.

5- Conclusion

In this article 2 well-known and highly used methods, i.e. supper-efficiency and cross-efficiency were presented and their drawbacks were mentioned. Then, a new method was presented which through utilizing Monte Carlo method the effect of each of extreme efficient DMUs on PPS is analyzed without any of the drawbacks of previous methods and the rank of each DMU is obtained. An important question which was raised here is that how many random vectors should be built for input and output in order to calculate the rank of each unit. The answer is up to the point that ranks do not change anymore. This number can be called as the reliability point for the generated ranks.

References

[1] Charnes A., Cooper W.W. , Rhodes E. , Measuring the efficiency of decision making units. European Journal of Operational Research, 2(1978), 429–441

[2] Banker R.D., Charnes A., Cooper W.W., Some models for estimating technical and scale efficiencies in data envelopment analysis, Management Science ,30(1984), 1078–1092.

[3] Abello J.M., Pardalos P.M., Resende M.G.C. , Handbook of massive data sets. Kluwer, Dordrecht(2002).

[4] Sexton T.R., Silkman R.H , Hogan A.J, Data Envelopment Analysis: Critique and Extensions. in Measuring Efficiency: An Assessment of Data Envelopment Analysis. San Francisco: Jossey-Bass,(1986),73-105.

[5] Sungmook Lim,Minimax and maximin formulations of cross-efficiency in DEA, Computers & Industrial Engineering 62(2012), 726-731.

[6] Andersen P., Petersen N.C., A procedure for ranking efficient units in data envelopment analysis, Management Science,39(10)(1993), 1261-1294.

[7] Mehrabian S. , Alirezaee M.R., Jahanshahloo G.R.,Acomplete efficiency ranking of decisionmaking units in data envelopment analysis. Comput. Optim. Appl. ,14(1999), 261–266.

[8] Saati M.S., Zarafat Angiz M., Jahanshahloo G.R.,A model for ranking decision making units in data envelopment analysis. Ricerca Operativa 31(97) (2001), 47–59.

[9] Jahanshahloo G.R., Hosseinzadeh Lotfi F., Zhiani Rezai H., F. Rezai Balf, UsingMont Carlo method for ranking efficient DMUs. Appl. Math. Comput. 162(1) (2005), 371–379.

[10] Jahanshahloo,G.R., Afzalinejad,M, A ranking method based on a full-inefficient frontier, Applied Mathematical Modelling,30(2005), 248–260.

[11] Jahanshahloo G.R., Hosseinzadeh Lotfi F.,, Shoja. N., Tohidi G., Razavian S.,Ranking using L1 norm in data envelopment analysis. Appl. Math. Comput. 153(1) (2004), 215–224 .

[12] Amirteimoori. A., Jahanshahloo G.R., Kordrostami S.,Ranking of decision making units in data envelopment analysis. A distance-based approach. Appl. Math. Comput. 171(2005), 122–135 .

[13] Jahanshahloo G.R, Sanei M., Hosseinzadeh Lotfi F., Shoja N.,Using the gradient line for ranking DMUs in DEA. Appl. Math. Comput. 151(1) (2004), 209–219 .

[14] Rezai Balf F., Zhiani Rezai H., Jahanshahloo G.R., Hosseinzadeh Lotfi F., Ranking efficient DMUs

using the Tchebycheff norm. Appl. Math. Model. 36(2012), 45–56.

[15] Jahanshahloo G.R., Hosseinzadeh Lotfi F., Khanmohammadi M., Kazemimanesh M., Rezai V., Ranking of units by positive ideal DMU with common weights. Expert Syst. Appl. 37 (2010), 7483–7488 .

[16] Chen, Y.,"Ranking efficient units in DEA", Omega, 32(2004), 213-219.

[17] Adler N., Friedman L., Sinuany-Stern Z.,Review of ranking methods in the data envelopment analysis context, European Journal of Operational Research, 140 (2) (2002), 249-265.

[18] Abri G.R., Jahanshahloo G.R., Hosseinzadeh Lotf F., Shoja N., Fallah Jelodar M.,A new method for ranking non-extreme efficient units in data envelopment analysis, Optim Lett, 7(2013), 309–324.

[19] Thrall R.M., Duality, classification and slacks in data envelopment analysis, Journal of Operations *Research, 66*(1996),109-138.

[20] Jahanshahloo G.R., Hosseinzadeh Lotfi F. , rezaie V., Khanmohammadi M. ,Ranking DMUs by ideal points with interval data in DEA, Applied

Mathematical Modelling, 35(2011), 218-229.

استفاده از روش مونت کارلو در رابطه با ارزیابی DMU های کارای راسی

غلامرضا جهانشاهلو[۱]، مازیار زاهدی سرشت[۲]*

۲،۱ گروه ریاضیات، دانشگاه خوارزمی، تهران، ایران

چکیده

تحلیل پوششی داده‌ها یک روش برنامه‌ریزی ریاضی برای محاسبه کارایی واحدهای تصمیم‌گیری می‌باشد. زمانی که تحلیل پوششی داده‌ها نمره کارایی واحدها را بدست می‌آورد ممکن است تعدادی از آنها کارا شوند. حال این سوال پیش می‌آید از بین این واحدهای کارا کدام بهترین می‌باشد. واحدهای ناکارا را می‌توان توسط نمره کاراییشان رتبه‌بندی کرد ولی برای واحدهای کارا باید روشی را ارایه کرد که آنها را رتبه‌بندی کند.

روش‌هایی زیادی برای رتبه‌بندی واحدهای کارا ارایه گردیده است که هر کدام از آنها دارای معایب و مزایایی می‌باشند. Sexton روش Cross-efficiency را برای رتبه‌بندی واحدهای کارا ارایه کرد که یکی از مشکلات بزرگ این روش جواب بهینه چندگانه در هرکدام از مدهایی می‌باشد که باید برای هر DMU حل شوند. ایراد دیگر این روش، وابستگی جواب مدل به جواب‌های بدست آمده توسط واحدهای دیگر می‌باشد. یکی از روش‌های پر کاربرد دیگر، Super-efficiency می‌باشد که توسط Anderson and Petersen ارایه گردید. این روش هم دارای معایب زیادی می‌باشد. نشدنی بودن، ناپایداری، وابستگی مدل به ماهیت ورودی یا خروجی و متغیرهای کمکی s مشکلات این روش می‌باشند که در مسایل خاص ممکن است اتفاق بیافتد. در این مقاله ما روشی را ارایه کرده‌ایم که هیچ‌کدام از این مشکلات را ندارد و می‌تواند رتبه واحدهای کارای راسی را با محاسباتی ساده و با استفاده از روش The Hit or Miss Monte Carlo Method بدست آورد. در انتهای این مقاله برای نشان دادن کارایی روش خودمان مثالی کاربردی را ذکر کرده‌ایم.

کلید واژه: تحلیل پوششی داده‌ها، بهره‌وری، رتبه‌بندی، شبیه‌سازی مونت کارلو، روش Cross-efficiency، روش Super-efficiency

* عهده‌دار مکاتبات: Zahedi_s@yahoo.com