

# Introducing a secondary goal for evaluating DMUs by cross efficiency in data envelopment analysis

R.Mehrjoo<sup>a</sup>, N.Shoja<sup>b\*</sup>

<sup>(a)</sup> *Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran*

<sup>(b)</sup> *Associate Professor, Islamic Azad University, Firoozkuh Branch, Firoozkuh, Iran*

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## Abstract

One way to rank DMUs in DEA is the cross efficiency method. In this method, the efficiency of each DMU is calculated by other DMUs optimum weights, which makes the ranking more acceptable for managers. Existing alternative optimum weights in cross efficiency method lead to several ranks for DMUs. Several secondary goals have introduced to avoid this problem, till now. In this paper, a new model is presented, that would be satisfying and acceptable for all DMUs. Therefore, by solving this model, the optimum weights are agreeable and fairly for DMUs.

**Keywords:** *Data Envelopment Analysis, Ranking, Cross Efficiency, Secondary Goal.*

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\*Corresponding author: [Shoja@iaufb.ac.ir](mailto:Shoja@iaufb.ac.ir)

## **1. Introduction**

Ranking DMUs in data development analysis models was introduced by Sexton et al (1986), which is an effective way to rank DMUs. However, in this method, existence of alternative weights provides several ranks for the DMUs, makes the model unable to rank them correctly. In fact, existence of alternative weights caused to several rank. After that, many different methods have been presented to find the most appropriate weight among optimum weights, in each a secondary goal has been considered as an objective function. Regarding the secondary goal, a weight would be selected which achieve those goals. Doyle and Green (1994) introduced aggressive and benevolent models. Wang et al (2011) discussed the ways to define weights in cross efficiency. Furthermore, Wang et al (2011) applied the Ideal and Anti-Ideal DMUs for combination of DMUs' cross efficiency. Liang et al (2008) used Game Theory in cross efficiency and generated game efficiency. Rudder and Reucher (2011) represented a DEA model by using Peer-DMU which is optimized cross efficiency. More information would be found in Jahanshahloo et al (2008), and Alder et al (2002).

In this paper, a model is introduced to deal with such problem, in which the difference between cross efficiencies of DMUs obtained by DMUO would be minimized simultaneously. In this regard, by solving one model for DMUO, among all weights related to that, the one would be selected that minimizes the difference. Therefore, the obtained weight can be an equitable criterion for ranking the other DMUs, which makes the ranking process complete, satisfying, and acceptable. In fact, the weights extracted in this model, minimize the difference of upper-band and lower-band. The upper-band is the maximum efficiency score which could be given to other DMUs by the optimum weight of DMUO. On the other hand, the lower-band is the minimum efficiency score. Initially, the model obtaining in this way, is non-linear that would be illustrated that some constraints are redundant, so they can be omitted. Then the presented model would be solved for each DMU. Regarding the resulted weight for each DMU, the cross efficiency for all DMUs would be implied, and then ranking is calculated by combination of them.

**2. CCR model**

Suppose that we have N determining unit of DMU that use input M's to produce output S's . The vector of  $X_j$  ( $j = 1, \dots, n$ ) is the corresponded vector of DMU<sub>j</sub>'s inputs that  $X_j=(X_{1j}, \dots, X_{mj})$  is an m vector and the vector of  $Y_j$  is a correspondent vector of DMU that  $Y_j= (Y_{1j}, \dots, Y_{sj})$  is a kind of vector with S and vector relating to DMU<sub>j</sub> is defined in the form of  $(X_j, Y_j)$   $j=1, \dots, n$  . In self – assessment method with solving the following model, an efficiency amount for DMU<sub>0</sub> is obtained. The following model is a CCR multiple model.

$$\begin{aligned} \text{Max } \theta_o &= \frac{u^t y_o}{v^t x_o} \\ \text{s. t. } \quad \frac{u^t y_j}{v^t x_j} &\leq 1 \quad j = 1, \dots, n \quad (1) \\ u &\geq 0 \quad v \geq 0 \end{aligned}$$

Which is fractional model of CCR, but with changes of Charnes & Cooper (1962) this model is turned into the following linear model which is called CCR model with input nature.

$$\begin{aligned} \theta_o^* &= \text{Max } u^t y_o \\ v^t x_o &= 1 \\ u^t y_j - v^t x_j &\leq 0 \quad j = 1, \dots, n \quad (2) \\ (u, v) &\geq 0. \end{aligned}$$

$\theta_o^*$  the obtained optimum amount from above – mentioned model, is the efficiency

amount of DMU<sub>0</sub> which has been obtained through self – assessment method.

**3. Cross efficiency**

For obtaining cross efficiency of other DMUs we use optimum multiplications of the model and place vector of  $X_j$  ,  $Y_j$  in it and then they are defined.

$$E_{jk} = \frac{\sum_{r=1}^S u_{rk}^* y_{rj}}{\sum_{i=1}^m v_{ik}^* x_{ij}} \quad j = 1, \dots, n$$

It is obvious that  $\Theta_0^* = E_{kk}$  and also because  $\sum_{i=1}^m v_{ik}^* x_{ik} = 1$  is one offeature of this model , so  $E_{rk} = \sum_{r=1}^S u_{rk}^* y_{rk}$ . The intersection efficiency relating to DMU has been defined as follows:  $E_j = \frac{1}{n} \sum_{k=1}^n E_{jk}$

**4. propose model**

It is assumed that all DMUs were evaluated by CCR model, so the efficiency of them were obtained, where  $\theta_o^*$  is the efficiency of DMOU. Consider the following model:

$$\begin{aligned} \text{Min } \quad &\theta_o^u - \theta_o^l \\ \text{s. t. } \quad &\frac{UY_o}{VX_o} = \theta_o^* \\ \frac{UY_j}{VX_j} &\leq 1 \quad j = 1, \dots, n \quad (3) \end{aligned}$$

$$\theta_o^u = \max \left\{ \frac{UY_1}{VX_1}, \frac{UY_2}{VX_2}, \dots, \frac{UY_n}{VX_n} \right\} \quad (*)$$

$$\theta_o^l = \min \left\{ \frac{UY_1}{VX_1}, \frac{UY_2}{VX_2}, \dots, \frac{UY_n}{VX_n} \right\} \quad (**)$$

$$(U, V) \geq 0.$$

It should be mentioned that if  $\frac{UY_o}{VX_o}$  exists in (\*) and (\*,\*) constraints,  $\theta^l_o = \theta^u_o$  and the value of objective function would be zero. Since the existence of DMU is not necessary, in (\*) and (\*,\*) expression,  $\frac{UY_o}{VX_o}$  would be removed. Also, (\*) and (\*,\*) constraints in (3) model converted in the following in model (4).

$$\begin{aligned} \min \quad & \theta^u_o - \theta^l_o \\ \text{s.t.} \quad & \frac{UY_o}{VX_o} = \theta^*_o \\ & \frac{UY_j}{VX_j} \leq 1 \quad j = 1, \dots, n \\ & \theta^u_o \geq \frac{UY_j}{VX_j} \quad j = 1, \dots, n, \quad j \neq 0 \quad (4) \\ & \theta^u_o = \frac{UY_1}{VX_1} \text{ or } \theta^u_o = \frac{UY_2}{VX_2} \text{ or } \dots \text{ or} \\ & \theta^u_o = \frac{UY_n}{VX_n} \quad (*) \\ & \theta^l_o \leq \frac{UY_j}{VX_j} \quad j = 1, \dots, n, \quad j \neq 0 \\ & \theta^l_o = \frac{UY_1}{VX_1} \text{ or } \theta^l_o = \frac{UY_2}{VX_2} \text{ or } \dots \text{ or} \\ & \theta^l_o = \frac{UY_n}{VX_n} \quad (**) \\ & (U, V) \geq 0. \end{aligned}$$

**Theorem1:** in model (4), (\*) and (\*,\*) constraints are redundant in the optimum solution.

**Proof:**

Since the  $\theta^u_o$  is minimized in objective function, so one of the equations in (\*) constraint applies in optimum solution definitely, and because  $\theta^l_o$  is maximized in

objective function, then one of the equations in (\*\*) constraint applies in optimum solution.

Therefore, according to Theorem, model (3-2) and following model have the same optimum solution.

$$\begin{aligned} \min \quad & \theta^u_o - \theta^l_o \\ \text{s.t.} \quad & \frac{UY_o}{VX_o} = \theta^*_o \\ & \frac{UY_j}{VX_j} \leq 1 \quad j = 1, \dots, n \quad (*) \\ & \theta^u_o \geq \frac{UY_j}{VX_j} \quad j = 1, \dots, n, \quad j \neq 0 \quad (5) \\ & \theta^l_o \leq \frac{UY_j}{VX_j} \quad j = 1, \dots, n, \quad j \neq 0 \\ & (U, V) \geq 0. \end{aligned}$$

**Theorem2:** The model (5) and the following one are equivalent.

$$\begin{aligned} \min \quad & \theta^u_o - \theta^l_o \\ \text{s.t.} \quad & \frac{UY_o}{VX_o} = \theta^*_o \\ & \theta^u_o \geq \frac{UY_j}{VX_j} \quad j = 1, \dots, n, \quad j \neq 0 \quad (6) \\ & \theta^l_o \leq \frac{UY_j}{VX_j} \quad j = 1, \dots, n, \quad j \neq 0 \\ & \theta^u_o \leq 1 \quad (**) \\ & (U, V) \geq 0. \end{aligned}$$

In fact, in model (5) the (\*) constraint is replaced by (\*\*) constraint in model (6).

**Proof:**

It is obvious that (\*) and (\*,\*) constraints in model (6) cause to application of (\*) constraint in model (5). So, by solving the model (6) the obtained optimum solution makes the difference of cross efficiency

minimum. At least , the obtained weight results that one of the DMUs gets the highest efficiency score, and the other one has the minimum reduction.

### 5. Numerical example

This example was resented in several papers till now. we compose this example for illustrating our model. Information of the problem is shown in table (1).

The weights of each DMU after solving model is given by table (2).

So in this case because of obtaining several efficient DMUs, A method to rank all DMUs is necessary. We used our method and solve the model for each DMU and the unique weights were obtained that shown in table (3).

Table (1)

DMU	$x_1$	$x_2$	$y_1$	$y_2$	<b>DEA score</b>
1	1.5	0.2	1.4	0.35	1
2	4	0.7	1.4	2.1	1
3	3.2	1.2	4.2	1.05	1
4	5.2	2	2.8	4.2	1
5	3.5	1.2	1.9	2.5	0.9775
6	3.2	0.7	1.4	1.5	0.8674

Table (2)

	$\theta$	$v_1$	$v_2$	$u_1$	$u_2$
DMU1	1	0	5.00	0.71	0
DMU2	1	0	1.43	0	0.48
DMU3	1	0.31	0	0.24	0
DMU4	1	0.19	0	0	0.24
DMU5	0.98	0.11	0.51	0.12	0.3
DMU6	0.87	0.15	0.72	0.16	0.43

Table (3)

	$\theta_u - \theta_l$	$\theta_l$	$\theta_u$	$v_1$	$v_2$	$u_1$	$u_2$
DMU1	0.14	0.86	1	0.2	0.8	0.2	0.49
DMU2	0.17	0.83	1	0.18	0.82	0.18	0.49
DMU3	0.2	0.8	1	0.32	0.68	0.31	0.51
DMU4	0.14	0.86	1	0.18	0.8	0.2	0.49
DMU5	0.17	0.83	1	0.18	0.82	0.19	0.49
DMU6	0.18	0.82	1	0.17	0.83	0.18	0.49

**Theorem3.** If the under evaluating DMU is inefficient, then for the optimum solution of model (6) always  $\theta_j^{u*} = 1$ .

**Proof.** According to the constraints  $\theta^{u_o} \leq 1$  and  $\theta^{u_o} \geq \frac{UY_j}{VX_j}$ :

$$\frac{UY_j}{VX_j} \leq 1, j = 1, \dots, n, j \neq o$$

And also because  $\theta_0^* \leq 1$  then  $\frac{UY_o}{VX_o} \leq 1$  and the constraint  $(U, V) \geq 0$  causes each feasible solution of model (3-4) is a feasible solution for CCR model in evaluating DMU<sub>o</sub>, and  $\theta_0^* = \frac{UY_o}{VX_o}$ , so each feasible solution of model (3-4) is an optimum solution in CCR model in evaluating DMU<sub>o</sub>. Because at least one of constraints  $\frac{UY_j}{VX_j} \leq 1$  is tight, then  $\theta_j^{u*} = 1$ .

## 6. Conclusion

In this paper, A new method is introduced for ranking DMUs as a secondary goal. The model minimizes the differences between upper and lower bounds of solutions. So, all DMUs is ranked by this method. Also we present a simpler model for inefficient DMUs.

A numerical example is given in this paper, solved by the new method, which ranked all the DMUs. This method can be developed in other cases, for example : DMUs by interval data, negative data and etc.

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## معرفی یک هدف ثانویه برای ارزیابی DMUها با کارایی متقاطع در تحلیل پوششی داده‌ها

راضیه مهرجو<sup>۱</sup>، نقی شجاع<sup>۲\*</sup>

<sup>۱</sup> گروه ریاضی، دانشگاه آزاد اسلامی واحد علوم و تحقیقات، تهران، ایران  
<sup>۲</sup> دانشیار، دانشگاه آزاد اسلامی واحد فیروزکوه، فیروزکوه، ایران

### چکیده

یک روش رتبه‌بندی DMUها روش کارایی متقاطع می‌باشد. در این روش کارایی هر DMU با وزن‌های بهیبه بقیه DMUها محاسبه می‌شود، که رتبه بندی را برای مدیران قابل قبول تر می‌سازد. وجود وزن‌های بهیبه دگرین در کارایی متقاطع منجر به وجود رتبه‌های مختلف برای DMUها می‌گردد. اهداف ثانویه گوناگونی برای حل این مشکل تاکنون معرفی گردیده است. در این مقاله روش جدیدی ارائه می‌شود که راضی کننده و قابل قبول برای همه DMUها می‌باشد. بنابراین با حل این مدل وزن‌های بهینه مورد محاسبه رتبه‌بندی برای همه DMUها راضی کننده و منصفانه است.

**کلید واژه:** تحلیل پوششی داده‌ها، رتبه بندی، کارایی متقاطع، هدف دوم.