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# Subpullbacks and coproducts of $S$-posets

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**Abstract.** In 2001, S. Bulman-Fleming et al. initiated the study of three flatness properties (weakly kernel flat, principally weakly kernel flat, translation kernel flat) of right acts $A_S$ over a monoid $S$ that can be described by means of when the functor $A_S \otimes -$ preserves pullbacks. In this paper, we extend these results to $S$-posets and present equivalent descriptions of weakly kernel po-flat, principally weakly kernel po-flat and translation kernel po-flat $S$-posets. Moreover, we show that most of flatness properties of $S$-posets can be transferred to their coproducts and vice versa.

## 1   Introduction and preliminaries

Let $S$ be a pomonoid. A poset $A$ is called a *right $S$-poset* (denoted by $A_S$) if there exists a right action $A \times S \to A$, $(a, s) \mapsto as$, which satisfies (i) the action is monotone in each variable, (ii) $a(st) = (as)t$ and $a1 = a$ for all $a \in A$ and $s, t \in S$. Left $S$-posets are defined analogously. The notation $A_S$ (respectively, $_S A$) will often be used to denote a right (respectively, left) $S$-poset, and $\Theta_S = \{\theta\}$ is the one-element right $S$-poset. All right (respectively, left) $S$-posets form a category, denoted **Pos**-$S$ (respectively, $S$-**Pos**) (see [4]), whose *morphisms* are the functions that preserve both

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the action and the order. In these categories, as in the category **Pos** of posets, the monomorphisms and epimorphisms are the injective and surjective morphisms, respectively. In **Pos**-$S$ and $S$-**Pos**, a morphism $g : A \to B$ is called an *order embedding* if $g(a) \leq g(a')$ implies $a \leq a'$ for all $a, a' \in A$. A surjective order embedding is called an *order isomorphism.*

A nonempty subset $I$ of a pomonoid $S$ is called an *ordered right ideal* of $S$ if (i) $IS \subseteq I$ and (ii) $a \leq b \in I$ implies $a \in I$ for all $a, b \in S$. An $S$-subposet $B_S$ of a right $S$-poset $A_S$ is called *strongly convex* if $a \leq b$ implies $a \in B_S$ for any $a \in A_S$ and $b \in B_S$. Clearly, if $I$ is an ordered right ideal of a pomonoid $S$, then $I$ is a strongly convex $S$-subposet of the $S$-poset $S$. A pomonoid $S$ is called *weakly right reversible* if for any $s, s' \in S$, there exist $u, v \in S$ such that $us \leq vs'$.

Preliminary work on flatness properties of $S$-posets, was done by Fakhruddin in [6, 7], and continued in recent papers [1, 3, 9, 12] etc.

To define the tensor product $A \otimes_S B$ of a right $S$-poset $A_S$ and a left $S$-poset $_SB$ (see [12]), we first equip the Cartesian product $A \times B$ with component-wise order. Let $A \otimes_S B = (A \times B)/\rho$, where $\rho$ is the order-congruence on the right $S$-poset $A \times B$ (on which $S$ acts trivially) generated by the relation $H = \{((as, b), (a, sb)) \mid a \in A_S, b \in {_SB}, s \in S\}$. The equivalence class of $(a, b)$ in $A \otimes_S B$ is denoted $a \otimes b$. The order relation on $A \otimes_S B$ will be described in Lemma 2.1. Similar to $S$-acts, it is easy to see that $A \otimes_S S$ can be equipped with a natural right $S$-action, and $A \otimes_S S \cong A$ for all $S$-posets $A_S$. It can be seen that $a \otimes s \leq a' \otimes t$ in $A \otimes_S S$ if and only if $as \leq a't$ in $A_S$.

Subpullbacks and subequalizers in the category $S$-**Pos** are defined in [1]. The categories $S$-**Pos** and **Pos** are poset-enriched concrete categories, where the order relation on morphism sets is defined pointwise (i.e. $f \leq g$ for $f, g : A \to B$ if and only if $f(a) \leq g(a)$ for every $a \in A$). In such categories, a diagram

$$
\begin{array}{ccc}
{_SP} & \xrightarrow{\ p_2\ } & {_SN} \\
{\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle g} \\
{_SM} & \xrightarrow{\ f\ } & {_SQ}
\end{array}
$$

$$(\mathbf{P_1})$$

is called *the subpullback diagram for $f$ and $g$* if

(1) the diagram ($\mathbf{P_1}$) is subcommutative (i.e. $fp_1 \leq gp_2$), and
(2) if

$$
\begin{array}{ccc}
{}_SP' & \xrightarrow{\ p_2'\ } & {}_SN \\
{\scriptstyle p_1'}\downarrow & & \downarrow{\scriptstyle g} \\
{}_SM & \xrightarrow{\ f\ } & {}_SQ
\end{array}
$$

is a diagram in $S$-**Pos** such that $fp_1' \leq gp_2'$, then there exists a unique morphism $\varphi : {}_SP' \to {}_SP$ such that $p_1\varphi = p_1'$ and $p_2\varphi = p_2'$.

In $S$-**Pos** or **Pos**, ${}_SP$ may in fact be realized as

$$
P = \{(m,n) \in M \times N \mid f(m) \leq g(n)\}
$$

with restrictions $p_1$ and $p_2$ of the projections of $M \times N$ onto ${}_SM$ and ${}_SN$ (note that $P$ is possibly empty). The subpullback diagram ($\mathbf{P_1}$) is denoted by $P(M, N, f, g, Q)$ and tensoring it by any right $S$-poset $A_S$ one gets the subcommutative diagram

$$
\begin{array}{ccc}
A \otimes_S P & \xrightarrow{\ id_A \otimes p_2\ } & A \otimes_S N \\
{\scriptstyle id_A \otimes p_1}\downarrow & & \downarrow{\scriptstyle id_A \otimes g} \\
A \otimes_S M & \xrightarrow{\ id_A \otimes f\ } & A \otimes_S Q
\end{array}
$$

in **Pos**. For the subpullback of mappings $id_A \otimes f$ and $id_A \otimes g$, we may take

$$
P' = \{(a \otimes m, a' \otimes n) \in (A \otimes_S M) \times (A \otimes_S N) \mid a \otimes f(m) \leq a' \otimes g(n)\}
$$

with $p_1', p_2'$ being the restrictions of the projections.

From the definition of subpullbacks it follows that there exists a unique monotonic mapping $\phi : A \otimes_S P \to P'$ such that, in the diagram

$$(\mathbf{P_2})$$

we have $p_i'\phi = id_A \otimes p_i$ for $i = 1, 2$. This mapping is called *the $\phi$ corresponding to the subpullback diagram $P(M, N, f, g, Q)$ for $A_S$.* It can be seen that the mapping $\phi$ in diagram ($\mathbf{P_2}$) is given by

$$\phi(a \otimes (m, n)) = (a \otimes m, a \otimes n)$$

for all $a \in A_S$ and $(m, n) \in_S P$. Note that for $\phi$ to be surjective requires

$$(\forall a, a' \in A_S)(\forall m \in_S M)(\forall n \in_S N)[a \otimes f(m) \le a' \otimes g(n) \Rightarrow$$

$$(\exists a'' \in A_S)(\exists m' \in_S M)(\exists n' \in_S N)$$

$$(f(m') \le g(n') \wedge a \otimes m = a'' \otimes m' \wedge a' \otimes n = a'' \otimes n')],$$

and for $\phi$ to be order embeddable requires

$$(\forall a, a' \in A_S)(\forall m, m' \in_S M)(\forall n, n' \in_S N)$$

$$[f(m) \le g(n) \wedge f(m') \le g(n') \wedge a \otimes m \le a' \otimes m' \wedge a \otimes n \le a' \otimes n' \Rightarrow$$

$$a \otimes (m, n) \le a' \otimes (m', n') \text{ in } A \otimes_S P].$$

Moreover, if the mapping $\phi$ is both a surjection and an order embedding, then $\phi$ is an order isomorphism.

Similar to $S$-acts, coproducts of $S$-posets are disjoint unions, with $S$-action and order defined componentwise.

If $A = \bigcup\limits_{i \in I}^{\cdot} A_i$, where $A_i$ are strongly convex right $S$-subposets of $A_S$, then by the mapping corresponding to the subpullback diagram $P(M, N, f, g, Q)$ for $A_i$, $i \in I$, we mean the unique monotonic mapping $\phi_i$ which makes, in the diagram



$$(\mathbf{P_{2i}})$$
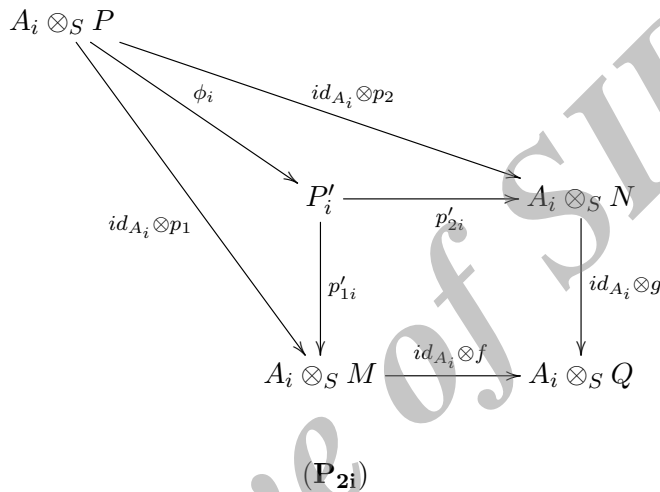
we have $p'_{ji}\phi_i = id_{A_i} \otimes p_j$ for $j = 1, 2$, where

$$P'_i = \{(a \otimes m, a' \otimes n) \in (A_i \otimes_S M) \times (A_i \otimes_S N) \mid a \otimes f(m) \leq a' \otimes g(n)\}$$

and $p'_{1i}, p'_{2i}$ are the restrictions of projections to $P'_i$.

It is shown in [2, 10] that, if we require either bijectivity or surjectivity of $\phi$ for pullback diagram of certain types, we not only recover most of the well-known forms of flatness, but obtain some new properties of acts as well. Furthermore, some of these results are extended to $S$-posets, and the classes of right $S$-posets corresponding to all of the cells in the first and second columns of Figure 1 are considered in [9]. This paper continues the investigation of the classes of right $S$-posets $A_S$ over $S$ for which the functor $A_S \otimes -$ has certain subpullback preservation properties. The variations of the types of subpullbacks considered in [9] and this paper are of the following types:
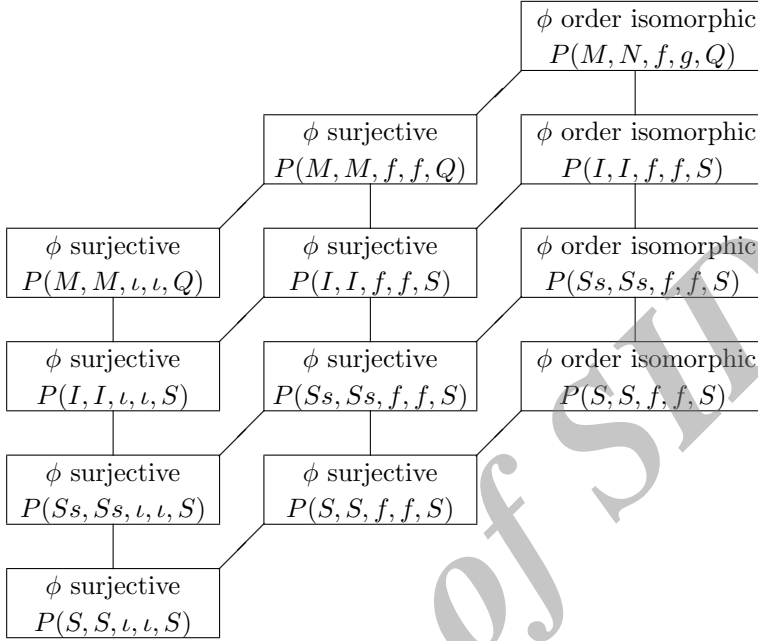
Figure 1.

Where $I$ ($Ss$) stands for a (principal) left ideal of $S$, and $\iota$ for a monomorphism of left $S$-posets. Every rectangle stands for a class of right $S$-posets that is defined by the property it contains. In the second and third columns, for instance, a rectangle with the text "$\phi$ order isomorphic $P(M, N, f, g, Q)$" denotes the class of all right $S$-posets $A_S$ such that the mapping $\phi$ is order isomorphic corresponding to every subpullback diagram $P(M, N, f, g, Q)$. But in the first column, for instance, a rectangle with the text "$\phi$ surjective $P(Ss, Ss, \iota, \iota, S)$" denotes the class of all right $S$-posets $A_S$ such that the mapping $\phi$ is surjective corresponding to every pullback diagram $P(Ss, Ss, \iota, \iota, S)$. A line between two rectangles indicates that the class of right $S$-posets corresponding to the rectangle at the upper end of the line is contained in the class corresponding to the rectangle at the lower end.

An $S$-poset $A_S$ is called *subpullback flat* (respectively, *subequalizer flat*) if the functor $A_S \otimes -$ takes subpullbacks (respectively, subequalizers) in $S$-**Pos** to subpullbacks (respectively, subequalizers) in **Pos**. Clearly, $A_S$ is subpullback flat if and only if the mapping $\phi$ is order isomorphic corresponding to every subpullback diagram $P(M, N, f, g, Q)$ in $S$-**Pos**.

It is proved in [1] that an $S$-poset $A_S$ is subpullback flat and subequalizer flat if and only if $A_S$ satisfies the following conditions:

(P): $(\forall a, a' \in A_S)(\forall u, v \in S)(au \leq a'v$

$$\Rightarrow (\exists a'' \in A_S)(\exists s, t \in S)(a = a''s \wedge a' = a''t \wedge su \leq tv));$$

(E):  $(\forall a \in A_S)(\forall u, v \in S)(au \leq av$
$\Rightarrow (\exists a' \in A_S)(\exists s \in S)(a = a's \wedge su \leq sv)).$

It is shown in [9] that an $S$-poset $A_S$ satisfies condition $(P)$ if and only if the mapping $\phi$ is surjective corresponding to every subpullback diagram $P(M, N, f, g, Q)$. Conditions $(WP)$ and $(PWP)$ are also introduced in [9]. An $S$-poset $A_S$ is said to satisfy *condition* $(WP)$ if the mapping $\phi$ is surjective corresponding to every subpullback diagram $P(I, I, f, f, S)$, where $I$ is a left ideal of $S$. An $S$-poset $A_S$ is said to satisfy *condition* $(PWP)$ if the mapping $\phi$ is surjective corresponding to every subpullback diagram $P(Ss, Ss, f, f, S), s \in S$.

In Section 2 of this paper, we introduce three additional flatness properties (weakly kernel po-flat, principally weakly kernel po-flat, translation kernel po-flat) of $S$-posets by means of subpullback preservation, and present equivalent descriptions of them (both for arbitrary and for cyclic $S$-posets).

It is shown in [10] that most of flatness properties of acts over a monoid $S$ are equivalent to the surjectivity or bijectivity of mappings corresponding to the pullback diagrams in special cases. Furthermore, it is shown in [8] that these flatness properties can be transferred to their coproducts. The purpose of Section 3 of this paper is to carry over these results to the setting of $S$-posets, and we show that flatness properties introduced in [9] can be transferred from $S$-posets over a pomonoid $S$ to their coproducts.

Although much of our work follows directly from the unordered case, some care is needed. Moreover, the results need to be stated and justified, which is the aim of this article.

## 2   Subpullbacks and flatness

In this section, we discuss the classes of right $S$-posets $A_S$ corresponding to the three lowest cells in the third column of Figure 1. We give an alternative description of a right (cyclic, one-element) $S$-poset having the corresponding property.

We begin with the following result used by many authors and formulated in [12, Theorem 5.2].

**Lemma 2.1.** *Let $A_S$ be a right $S$-poset, $_S B$ a left $S$-poset, $a, a' \in A_S$, $b, b' \in {}_S B$. Then $a \otimes b \leq a' \otimes b'$ in $A \otimes_S B$ if and only if there exist $a_1, a_2, \cdots, a_n \in A_S$,*

$b_2, \cdots, b_n \in {}_S B$ *and* $s_1, t_1, \cdots, s_n, t_n \in S$ *such that*

$$
\begin{array}{llll}
a & \leq a_1 s_1 & & \\
a_1 t_1 & \leq a_2 s_2 & s_1 b \leq t_1 b_2 \\
a_2 t_2 & \leq a_3 s_3 & s_2 b_2 \leq t_2 b_3 \\
& \vdots & & \vdots \\
a_n t_n & \leq a' & s_n b_n \leq t_n b'.
\end{array}
$$

**Definition 2.2.** A right $S$-poset $A_S$ is called

  (i)  *weakly kernel po-flat* if the mapping $\phi$ is order isomorphic corresponding to every subpullback diagram $P(I, I, f, f, S)$, where $I$ is a left ideal of $S$;

 (ii)  *principally weakly kernel po-flat* if the mapping $\phi$ is order isomorphic corresponding to every subpullback diagram $P(Ss, Ss, f, f, S), s \in S$;

(iii)  *translation kernel po-flat* if the mapping $\phi$ is order isomorphic corresponding to every subpullback diagram $P(S, S, f, f, S)$.

From Figure 1 and Theorems 2.1, 2.2, 2.3, 2.4, 3.2, 4.1, 5.3 of [9], we see that the new properties just defined are related to properties already studied as shown in Figure 2.
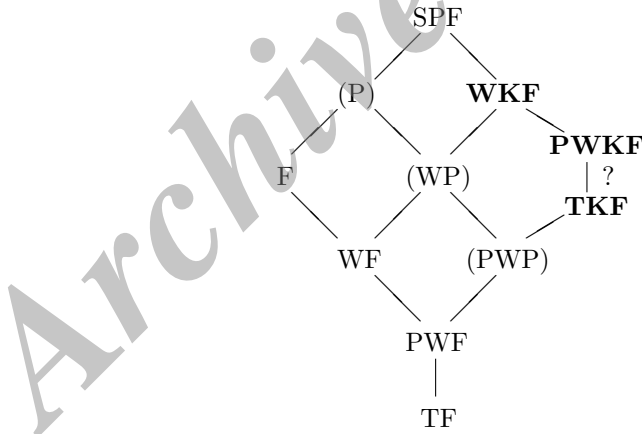


Figure 2.

**Note 2.3.** SPF = subpullback flatness, F = flatness, WF = weak flatness, PWF = principal weak flatness, WKF = weak kernel po-flatness, PWKF = principal weak kernel po-flatness, TKF = translation kernel po-flatness, TF = torsion freeness.

If $S$ is a pomonoid and $t \in S$, then $\rho_t : S \to S$ will denote the *right translation* by $t$, that is, $\rho_t(s) = st$ for every $s \in S$.

Recall that a binary relation $\sigma$ on an $S$-poset $A_S$ is called a *compatible quasi-order* on $A_S$ if it is transitive, compatible with the $S$-action, and contains the relation $\leq$ on $A_S$. The relationship between order-congruences and compatible quasi-order on $A_S$ is given in [13].

Suppose that $\rho$ is a right order congruence on a pomonoid $S$. Define a relation $\widehat{\rho}$ by

$$s \ \widehat{\rho} \ t \Leftrightarrow [s]_\rho \leq [t]_\rho \quad \text{in} \quad S/\rho.$$

It is clear that $\widehat{\rho}$ is a compatible quasi-order on $A_S$.

The *subkernel or directed kernel* of an $S$-poset morphism $f : A_S \to B_S$ is defined by $\overrightarrow{\ker} f = \{(a, a') \in A \times A \mid f(a) \leq f(a')\}$ (see [5]). It is shown in [13] that $\overrightarrow{\ker} f$ is a compatible quasi-order on $A_S$. Furthermore, we first give equivalent characterizations of weak kernel flatness, principal weak kernel flatness and translation kernel flatness, both for arbitrary and for cyclic $S$-posets. If $\rho$ is an equivalence relation on $S$ and $s \in S$, then $\bar{s}$ denotes the equivalence class of $s$ modulo $\rho$.

**Proposition 2.4.** *A right $S$-poset $A_S$ is weakly kernel po-flat if and only if $A_S$ satisfies condition $(WP)$ and for every left ideal $I$ of $S$ and every morphism $f : {}_S I \to {}_S S$ the following condition holds:*

$$(\forall a, a' \in A_S)(\forall s, s', t, t' \in I)$$

$$\left. \begin{array}{l} a \otimes s \leq a' \otimes s' \ in \ A \otimes I, \ \ f(s) \leq f(t) \\ a \otimes t \leq a' \otimes t' \ in \ A \otimes I, \ \ f(s') \leq f(t') \end{array} \right\} \Longrightarrow$$

$$a \otimes (s, t) \leq a' \otimes (s', t') \ in \ A \otimes_S \overrightarrow{\ker} f.$$

*Proof.* It follows from Lemma 5.1 of [9] and Definition 2.2. $\qquad\square$

As a direct consequence, we have

**Corollary 2.5.** *A cyclic right $S$-poset $S/\rho$ is weakly kernel po-flat if and only if $S/\rho$ satisfies condition $(WP)$ and for every left ideal $I$ of $S$ and every morphism $f : {}_S I \to {}_S S$ the following condition holds:*

$$(\forall s, s', t, t' \in I)$$

$$\left. \begin{array}{l} \bar{1} \otimes s \leq \bar{1} \otimes s' \ in \ S/\rho \otimes I, \ \ f(s) \leq f(t) \\ \bar{1} \otimes t \leq \bar{1} \otimes t' \ in \ S/\rho \otimes I, \ \ f(s') \leq f(t') \end{array} \right\} \Longrightarrow$$

$$\bar{1} \otimes (s, t) \leq \bar{1} \otimes (s', t') \ in \ S/\rho \otimes_S \overrightarrow{\ker} f.$$

**Proposition 2.6.** *A right $S$-poset $A_S$ is principally weakly kernel po-flat if and only if $A_S$ satisfies condition $(PWP)$ and the following condition holds:*

$$(\forall a, a' \in A_S)(\forall s, s', t, t', z, x \in S \ such \ that \ \overrightarrow{\ker}\rho_x \subseteq \overrightarrow{\ker}\rho_z)$$

$$\left.\begin{array}{ll} asx \le a's'x, & sz \le tz \\ atx \le a't'x, & s'z \le t'z \end{array}\right\} \Longrightarrow a \otimes (sx, tx) \le a' \otimes (s'x, t'x) \ in \ A \otimes_S P,$$

*where $_SP = \{(ux, vx) \mid u, v \in S, uz \le vz\}$.*

*Proof.* **Necessity.** Let $A_S$ be principally weakly kernel po-flat. Then $A_S$ satisfies condition $(PWP)$. Suppose that

$$\begin{array}{ll} asx & \le a's'x, \quad sz \le tz, \\ atx & \le a't'x, \quad s'z \le t'z \end{array}$$

for some $a, a' \in A_S$ and $s, s', t, t', z, x \in S$ such that $\overrightarrow{\ker}\rho_x \subseteq \overrightarrow{\ker}\rho_z$. Define a mapping $f : {}_S(Sx) \to {}_S S$ by $f(x) := z$. Since $\overrightarrow{\ker}\rho_x \subseteq \overrightarrow{\ker}\rho_z$, $f$ is well-defined. Clearly, $f$ is a morphism of left $S$-posets. Using Theorem 4.1 of [9], from the inequality $asx \le a's'x$ we obtain that there exist $b \in A_S$ and $u, v \in S$ such that $as = bu$, $a's' = bv$ and $ux \le vx$. Hence we have

$$\begin{array}{ll} a \otimes sx & = as \otimes x = bu \otimes x = b \otimes ux \le b \otimes vx \\ & = bv \otimes x = a's' \otimes x = a' \otimes s'x \end{array}$$

in $A \otimes_S Sx$. Analogously, $a \otimes tx \le a' \otimes t'x$ in $A \otimes_S Sx$. Because $A_S$ is principally weakly kernel po-flat, the mapping $\phi$ is an order embedding corresponding to the subpullback diagram $P(Sx, Sx, f, f, S)$. Then the inequalities

$$\begin{array}{ll} a \otimes sx & \le a' \otimes s'x, \quad f(sx) \le f(tx), \\ a \otimes tx & \le a' \otimes t'x, \quad f(s'x) \le f(t'x) \end{array}$$

imply

$$a \otimes (sx, tx) \le a' \otimes (s'x, t'x) \ in \ A \otimes_S P,$$

where

$$\begin{array}{ll} P & = \{(ux, vx) \in Sx \times Sx \mid f(ux) \le f(vx)\} \\ & = \{(ux, vx) \mid u, v \in S, uz \le vz\}. \end{array}$$

**Sufficiency.** Let $\phi$ be the canonical mapping corresponding to the subpullback diagram $P(Sx, Sx, f, f, S)$ for $A_S$, where $s \in S$ and $f : {}_S(Sx) \to {}_S S$ is a morphism. Because $A_S$ satisfies condition $(PWP)$, the mapping $\phi$ is surjective corresponding to every subpullback diagram $P(Sx, Sx, f, f, S)$ by Theorem 4.1 of [9]. Next wet

show that $\phi$ is also an order embedding corresponding to every subpullback diagram $P(Sx, Sx, f, f, S)$. Suppose that

$$
\begin{aligned}
a \otimes sx &\leq a' \otimes s'x \text{ in } A \otimes_S Sx, \quad f(sx) \leq f(tx), \\
a \otimes tx &\leq a' \otimes t'x \text{ in } A \otimes_S Sx, \quad f(s'x) \leq f(t'x)
\end{aligned}
$$

for some $a, a' \in A_S$ and $s, t, s', t', x \in S$. Then

$$
\begin{aligned}
asx &\leq a's'x, \quad sz \leq tz, \\
atx &\leq a't'x, \quad s'z \leq t'z,
\end{aligned}
$$

where $z = f(x)$. By assumption

$$
a \otimes (sx, tx) \leq a' \otimes (s'x, t'x) \text{ in } A \otimes_S P,
$$

where $_S P = \{(ux, vx) \mid u, v \in S, uz \leq vz\} = \overrightarrow{\ker} f$. Hence $\phi$ is an order embedding, and so $A_S$ is principally weakly kernel po-flat. $\qquad\square$

Using Proposition 2.6, we have the following description for a principally weakly kernel po-flat cyclic $S$-poset.

**Corollary 2.7.** *A cyclic right $S$-poset $S/\rho$ is principally weakly kernel po-flat if and only if $S/\rho$ satisfies condition $(PWP)$ and the following condition holds:*

$$(\forall s, s', t, t', z, x \in S \text{ such that } \overrightarrow{\ker}\rho_x \subseteq \overrightarrow{\ker}\rho_z)$$

$$
\left.\begin{array}{ll}
sx \;\widehat{\rho}\; s'x, & sz \leq tz \\
tx \;\widehat{\rho}\; t'x, & s'z \leq t'z
\end{array}\right\} \Longrightarrow \bar{1} \otimes (s, t) = \bar{1} \otimes (s', t') \text{ in } S/\rho \otimes_S P,
$$

*where $_S P = \{(ux, vx) \mid u, v \in S, \ uz \leq vz\}$.*

**Proposition 2.8.** *A right $S$-poset $A_S$ is translation kernel po-flat if and only if $A_S$ satisfies condition $(PWP)$ and the following condition holds:*

$$(\forall a, a' \in A_S)(\forall s, s', t, t', z \in S)$$

$$
\left.\begin{array}{ll}
as \leq a's', & sz \leq tz \\
at \leq a't', & s'z \leq t'z
\end{array}\right\} \Longrightarrow a \otimes (s, t) \leq a' \otimes (s', t') \text{ in } A \otimes_S \overrightarrow{\ker}\rho_z.
$$

*Proof.* It is similar to that of Proposition 2.6. $\qquad\square$

For a cyclic right $S$-poset, Proposition 2.8 yields the following

**Corollary 2.9.** *A cyclic right $S$-poset $S/\rho$ is translation kernel po-flat if and only if $S/\rho$ satisfies condition $(PWP)$ and the following condition holds:*

$$(\forall s, s', t, t', z \in S)$$

$$
\left.\begin{array}{ll}
s \;\widehat{\rho}\; s', & sz \leq tz \\
t \;\widehat{\rho}\; t', & s'z \leq t'z
\end{array}\right\} \Longrightarrow \bar{1} \otimes (s, t) \leq \bar{1} \otimes (s', t') \text{ in } S/\rho \otimes_S \overrightarrow{\ker}\rho_z.
$$

We now consider whether a one-element $S$-poset $\Theta_S = \{\theta\}$ satisfies each of our new properties. In preparation, we need to give the definition of connectedness for $S$-posets.

**Definition 2.10.** An $S$-poset $_S B$ is called *connected* if for all $b, b' \in {}_S B$ there exist elements $s_1, t_1, \cdots, s_n, t_n \in S$ and $b_2, \cdots, b_n \in {}_S B$ such that

$$
\begin{aligned}
s_1 b &\leq t_1 b_2 \\
s_2 b_2 &\leq t_2 b_3 \\
&\vdots \\
s_n b_n &\leq t_n b'.
\end{aligned}
$$

The foregoing sequence of inequalities will be called *a scheme of length $n$* connecting $b$ and $b'$.

**Proposition 2.11.** *For any pomonoid $S$, the following statements are equivalent:*

(i) *$\Theta_S$ is principally weakly kernel po-flat;*

(ii) *$\Theta_S$ is translation kernel po-flat;*

(iii) *For every $z \in S$, $\overrightarrow{\ker}\rho_z$ is connected as a left $S$-poset.*

*Proof.* (i)$\Rightarrow$(ii) is clear.

(ii)$\Rightarrow$(iii). Take $(s, t), (s', t') \in \overrightarrow{\ker}\rho_z$. Using translation kernel po-flatness,

$$
\begin{aligned}
\theta s &\leq \theta s', \quad sz \leq tz, \\
\theta t &\leq \theta t', \quad s'z \leq t'z
\end{aligned}
$$

imply $\theta \otimes (s, t) \leq \theta \otimes (s', t')$ in $\Theta \otimes_S \overrightarrow{\ker}\rho_z$. By Lemma 2.1, there exist $s_1, t_1, \cdots, s_n, t_n \in S$ and $b_2, \cdots, b_n \in \overrightarrow{\ker}\rho_z$ such that

$$
\begin{aligned}
\theta &\leq \theta s_1 \\
\theta t_1 &\leq \theta s_2 & s_1(s, t) &\leq t_1 b_2 \\
\theta t_2 &\leq \theta s_3 & s_2 b_2 &\leq t_2 b_3 \\
&\vdots & &\vdots \\
\theta t_n &\leq \theta & s_n b_n &\leq t_n (s', t').
\end{aligned}
$$

The right hand part of a scheme corresponding to the latter inequality shows that $\overrightarrow{\ker}\rho_z$ is connected.

(iii)$\Rightarrow$(i). Note first from Theorem 4.1 of [9] that $\Theta_S$ always satisfies condition $(PWP)$. Consider any $x, z \in S$ such that $\overrightarrow{\ker}\rho_x \subseteq \overrightarrow{\ker}\rho_z$. Because $\overrightarrow{\ker}\rho_z$ is connected, there exists a scheme corresponding to the inequality $\theta \otimes (s, t) \leq \theta \otimes (s', t')$

in $\Theta \otimes_S \overrightarrow{\ker}\rho_z$ for any $(s,t), (s',t') \in \overrightarrow{\ker}\rho_z$. Multiplying each inequality in the right hand column of this scheme (on the right) by $x$ establishes

$$\theta \otimes (sx, tx) \leq \theta \otimes (s'x, t'x) \text{ in } \Theta \otimes_S P,$$

where $_SP = \{(ux, vx) \mid u, v \in S, uz \leq vz\}$, and so, by Proposition 2.6, $\Theta_S$ is principally weakly kernel po-flat and the proof is complete. $\qquad\square$

From Corollary 5.4 of [9], it follows that $\Theta_S$ satisfies condition $(WP)$ if and only if $S$ is weakly right reversible. So, using Proposition 2.4, we have

**Proposition 2.12.** *$\Theta_S$ is weakly kernel po-flat if and only if $S$ is weakly right reversible, and for every left ideal $I$ of $S$, $\overrightarrow{\ker}f$ is connected for every homomorphism $f : {}_SI \to {}_S S$.*

The following example from [2, Proposition 26] illustrates that principal weak kernel flatness does not imply weak kernel flatness.

**Example 2.13.** Let $S$ be a right zero semigroup $K$ with 1 adjoined and $|K| > 1$. The order of $S$ is discrete. Then $S$ is not weakly right reversible, and so by Proposition 2.12, $\Theta_S$ is not weakly kernel po-flat. Now we show that $\Theta_S$ is principally weakly kernel po-flat. By Proposition 2.11, we need to check that $\overrightarrow{\ker}\rho_z$ is connected as a left $S$-poset for every $z \in S$. Since the order of $S$ is discrete, $\overrightarrow{\ker}\rho_z = \ker\rho_z$ and connectedness only involves equalities. Thus, we could directly apply Proposition 26 from [2] and obtain the result.

**Note 2.14.** From the preceding example we obtain that there exists a principally weakly kernel po-flat right $S$-poset, but does not satisfy conditions $(WP)$, $(WP)_w$, $(P)$ or $(P)_w$, and is not subpullback flat, flat, po-flat, weakly flat, or weakly po-flat by Theorem 6.2 of [9].

We have been unable so far to answer the question of whether principally weakly kernel po-flat and translation kernel po-flat are equivalent, we also have not yet been able to provide a suitable example to distinguish them. But, if $S$ is an ordered lpp monoid, then all translation kernel po-flat $S$-posets are principally weakly kernel po-flat.

Recall that a pomonoid $S$ is called an *ordered lpp monoid* if the $S$-subposet $Sx$ is projective for all $x \in S$. By Proposition 4.8 of [12], a pomonoid $S$ is an ordered lpp monoid if and only if for every $a \in S$ there exists an idempotent $e$ of $S$ such that $a = ea$ and $sa \leq ta$ implies $se \leq te$ for $s, t \in S$. These pomonoids comprise quite an extensive class, including all $I$-regular pomonoids and all right po-cancellable pomonoids (See [12], for more information).

**Theorem 2.15.** *If $S$ is an ordered lpp monoid, then all translation kernel po-flat $S$-posets are principally weakly kernel po-flat.*

*Proof.* Suppose $S$ is an ordered lpp monoid and $A_S$ is translation kernel po-flat. To show that $A_S$ is principally weakly kernel po-flat, we check the condition of Proposition 2.6. Suppose that $a, a' \in A_S$ and $s, s', t, t', z, x \in S$ are such that $\overrightarrow{\ker \rho_x} \subseteq \overrightarrow{\ker \rho_z}$ and

$$\begin{aligned} asx &\le a's'x, & sz &\le tz, \\ atx &\le a't'x, & s'z &\le t'z. \end{aligned}$$

Because $S$ is an ordered lpp monoid, there exists $e \in E(S)$ such that $ex = x$ (and hence $ez = z$), and $px \le qx$ implies $pe \le qe$ for all $p, q \in S$. Because $A_S$ satisfies condition $(PWP)$, from $(as)x \le (a's')x$, we obtain $c \in A_S$ and $p, q \in S$ such that $as = cp, a's' = cq$ and $px \le qx$ by Theorem 4.1 of [9]. From $(at)x \le (a't')x$, we obtain $d \in A_S$ and $g, h \in S$ such that $at = dg, a't' = dh$ and $gx \le hx$. Because $S$ is an ordered lpp monoid, we have $pe \le qe$ and $ge \le he$. We can now calculate

$$\begin{aligned} ase &= cpe \le cqe = a's'e, \\ ate &= dge \le dhe = a't'e. \end{aligned}$$

Therefore, we have $ase \le a's'e$ and $ate \le a't'e$. Moreover,

$$sez = sz \le tz = tez \text{ and } s'ez = s'z \le t'z = t'ez.$$

Using translation kernel po-flatness of $A_S$, we know that $a \otimes (se, te) \le a' \otimes (s'e, t'e)$ in $A \otimes_S \overrightarrow{\ker \rho_z}$ by Proposition 2.8. Using Lemma 2.1, there exist $a_1, \cdots, a_n \in A_S$, $(x_2, y_2), \cdots, (x_n, y_n) \in \overrightarrow{\ker \rho_z}$, and $s_1, t_1, \cdots, s_n, t_n \in S$ such that

$$\begin{aligned} a &\le a_1 s_1 \\ a_1 t_1 &\le a_2 s_2 & s_1(se, te) &\le t_1(x_2, y_2) \\ a_2 t_2 &\le a_3 s_3 & s_2(x_2, y_2) &\le t_2(x_3, y_3) \\ &\vdots & &\vdots \\ a_n t_n &\le a' & s_n(x_n, y_n) &\le t_n(s'e, t'e). \end{aligned}$$

Multiplication of each inequality in the right-hand column on the right by $x$ produces the scheme

$$\begin{aligned} a &\le a_1 s_1 \\ a_1 t_1 &\le a_2 s_2 & s_1(sx, tx) &\le t_1(x_2 x, y_2 x) \\ a_2 t_2 &\le a_3 s_3 & s_2(x_2 x, y_2 x) &\le t_2(x_3 x, y_3 x) \\ &\vdots & &\vdots \\ a_n t_n &\le a' & s_n(x_n x, y_n x) &\le t_n(s'x, t'x), \end{aligned}$$

where each $(x_i x, y_i x)$, $i = 2, \cdots, n$, belongs to $P = \{(ux, vx) \mid u, v \in S, \ uz \le vz\}$. In other words,

$$a \otimes (sx, tx) \le a' \otimes (s'x, t'x) \text{ in } A \otimes_S P,$$

as was to be shown. $\qquad\qquad\square$

## 3   Flatness and coproducts

In this section, we will show that most of flatness properties of $S$-posets over a pomonoid $S$ can be transferred to their coproducts.

Recall from [12] that an $S$-poset $A_S$ is called *decomposable* if there exist nonempty strongly convex $S$-subposets $A_1, A_2 \subseteq A$ such that $A = A_1 \dot\cup A_2$ (i.e. $A = A_1 \cup A_2$ and $A_1 \cap A_2 = \emptyset$). Otherwise $A_S$ is called *indecomposable*.

**Lemma 3.1.** *Let* $A = \dot{\bigcup_{i \in I}} A_i$, *where* $A_i$, $i \in I$, *are right $S$-subposets of $A_S$. Let* $_S B$ *be a left $S$-poset. If $a \otimes b \le a' \otimes b'$ in $A_i \otimes_S B$, then $a \otimes b \le a' \otimes b'$ in $A \otimes_S B$.*

*Proof.* It is obvious. $\qquad\qquad\square$

The next result will be useful for the remainder of this section.

**Lemma 3.2.** *Let* $A = \dot{\bigcup_{i \in I}} A_i$, *where* $A_i$, $i \in I$, *are right strongly convex $S$-subposets of $A_S$. Let $_S B$ be a left $S$-poset and suppose that $a \otimes b \le a' \otimes b'$ in $A \otimes_S B$. Then $a \in A_i$ for some $i \in I$, if and only if $a' \in A_i$.*

*Proof.* **Necessity.** Let $a \otimes b \le a' \otimes b'$ in $A \otimes_S B$. Using Lemma 2.1, there exist $a_1, \cdots, a_n \in A_S$, $b_2, \cdots, b_n \in {}_S B$ and $s_1, t_1, \cdots, s_n, t_n \in S$ such that

$$
\begin{array}{llll}
a & \le a_1 s_1 & & \\
a_1 t_1 & \le a_2 s_2 & s_1 b \le t_1 b_2 & \\
a_2 t_2 & \le a_3 s_3 & s_2 b_2 \le t_2 b_3 & \quad (*) \\
& \vdots & \vdots & \\
a_n t_n & \le a' & s_n b_n \le t_n b'. &
\end{array}
$$

Since $a \in A_i$, we have $a_1 \in A_i$. Otherwise there exists $j \ne i \in I$ such that $a_1 \in A_j$, and so $a_1 s_1 \in A_j$. The inequality $a \le a_1 s_1$ and the fact that $A_j$ is strongly convex imply $a \in A_j$ which is a contradiction. Thus $a_1 \in A_i$ and $a_1 t_1 \in A_i$. Again the inequality $a_1 t_1 \le a_2 s_2$ implies $a_2 \in A_i$. Otherwise there exists $j \ne i \in I$ such that $a_2 \in A_j$, and so $a_2 s_2 \in A_j$. Applying strong convexity of $A_j$ to the inequality $a_1 t_1 \le a_2 s_2$, we obtain $a_1 t_1 \in A_j$, and so this implies that $a_1 t_1 \in A_i \cap A_j = \emptyset$

which is again a contradiction. By continuing this process we get $a_n \in A_i$. Since $a_n t_n \leq a'$ and $A_i$ is strongly convex, we have $a' \in A_i$, as required.

**Sufficiency.** Suppose that $a \otimes b \leq a' \otimes b'$ in $A \otimes_S B$. By Lemma 2.1, we get the above system of inequalities (∗). Since $a' \in A_i$ and $A_i$ is strongly convex, we have $a_n t_n \in A_i$, and so $a_n \in A_i$. Applying strong convexity of $A_i$ to the inequality $a_{n-1} t_{n-1} \leq a_n s_n$, we obtain $a_{n-1} t_{n-1} \in A_i$, and so $a_{n-1} \in A_i$. By continuing this process we get $a_1 s_1 \in A_i$. Again applying strong convexity of $A_i$ to the inequality $a \leq a_1 s_1$, we have $a \in A_i$, as required.                            $\square$

Using Lemmas 3.1 and 3.2, we immediately get the following

**Corollary 3.3.** *Let* $A = \dot{\bigcup}_{i \in I} A_i$, *where* $A_i$, $i \in I$, *are right strongly convex S-subposets of* $A_S$. *Let* $_S B$ *be a left S-poset. If* $a \in A_i$, *then* $a \otimes b \leq a' \otimes b'$ *in* $A \otimes_S B$ *if and only if* $a \otimes b \leq a' \otimes b'$ *in* $A_i \otimes_S B$.

**Lemma 3.4.** *Let* $A = \dot{\bigcup}_{i \in I} A_i$, *where* $A_i$, $i \in I$, *are right strongly convex S-subposets of* $A_S$. *Let* $_S B$ *be a left S-poset and* $a \otimes b \in A \otimes_S B$ *for* $a \in A_S$ *and* $b \in {}_S B$. *If* $a \in A_i$ *for some* $i \in I$, *then* $a \otimes b \in A_i \otimes_S B$.

*Proof.* If there exists $j \neq i$ such that $a \otimes b \in A_j \otimes_S B$, then $a \otimes b = a' \otimes b'$ for some $a' \in A_j$ and $b' \in {}_S B$. By Lemma 3.2, $a' \in A_i$ which is a contradiction. Hence $a \otimes b \in A_i \otimes_S B$, as required.                            $\square$

**Corollary 3.5.** *Let* $A = \dot{\bigcup}_{i \in I} A_i$, *where* $A_i$, $i \in I$, *are right strongly convex S-subposets of* $A_S$. *Let* $\phi : A \otimes_S P \to P'$ *be the mapping corresponding to the subpullback diagram* $P(M, N, f, g, Q)$ *for* $A_S$. *If* $\phi_i = \phi|_{A_i \otimes_S P}$, *then* $\phi_i : A_i \otimes_S P \to P'_i$.

*Proof.* Because $\phi(a \otimes (m, n)) = (a \otimes m, a \otimes n)$ for all $a \in A_S$ and $(m, n) \in_S P$, it suffices to show that $a \otimes m \in A_i \otimes_S M$ and $a \otimes n \in A_i \otimes_S N$ for $a \in A_i$, $m \in_S M$ and $n \in_S N$. But these are true by Lemma 3.4.                            $\square$

**Lemma 3.6.** *Let* $A = \dot{\bigcup}_{i \in I} A_i$, *where* $A_i$, $i \in I$, *are right strongly convex S-subposets of* $A_S$. *Let* $\phi : A \otimes_S P \to P'$ *be a mapping and* $\phi_i = \phi|_{A_i \otimes_S P}$. *Then* $\phi$ *is the mapping corresponding to the subpullback diagram* $P(M, N, f, g, Q)$ *for* $A_S$, *if and only if* $\phi_i$ *is the mapping corresponding to the subpullback diagram* $P(M, N, f, g, Q)$ *for* $A_i$.

*Proof.* **Necessity.** Suppose that $\phi$ is the mapping corresponding to the subpull-back diagram $P(M, N, f, g, Q)$ for $A_S$. We need to prove that $p'_{1i}\phi_i = id_{A_i} \otimes p_1$ and $p'_{2i}\phi_i = id_{A_i} \otimes p_2$ in the diagram ($\mathbf{P_{2i}}$). By assumption, the lower square

$$
\begin{array}{ccc}
P'_i & \xrightarrow{\;p'_{2i}\;} & A_i \otimes_S N \\
\downarrow{\scriptstyle p'_{1i}} & & \downarrow{\scriptstyle id_{A_i}\otimes g} \\
A_i \otimes_S M & \xrightarrow{\;id_{A_i}\otimes f\;} & A_i \otimes_S Q
\end{array}
$$

in diagram ($\mathbf{P_{2i}}$) is subcommutative. Next to show $p'_{1i}\phi_i = id_{A_i} \otimes p_1$. Let $a_i \otimes (m, n) \in A_i \otimes_S P$. Then

$$p'_{1i}\phi_i(a_i \otimes (m, n)) = p'_{1i}(a_i \otimes m, a_i \otimes n) = a_i \otimes m$$

$$= id_{A_i}(a_i) \otimes p_1(m, n) = (id_{A_i} \otimes p_1)(a_i \otimes (m, n)).$$

It can also be seen that $p'_{2i}\phi_i = id_{A_i} \otimes p_2$. Since $\phi_i$ makes $p'_{ji}\phi_i = id_{A_i} \otimes p_j$ for $j = 1, 2$, in the diagram ($\mathbf{P_{2i}}$), then by uniqueness, $\phi_i$ is the mapping corresponding to the subpullback diagram $P(M, N, f, g, Q)$ for $A_i$.

**Sufficiency.** Let $\phi_i$ be the mapping corresponding to the subpullback diagram $P(M, N, f, g, Q)$ for $A_i, i \in I$. Since the lower square

$$
\begin{array}{ccc}
P' & \xrightarrow{\;p'_2\;} & A \otimes_S N \\
\downarrow{\scriptstyle p'_1} & & \downarrow{\scriptstyle id_A\otimes g} \\
A \otimes_S M & \xrightarrow{\;id_A\otimes f\;} & A \otimes_S Q
\end{array}
$$

in the diagram ($\mathbf{P_2}$) is subcommutative, it suffices to show that $p'_1\phi = id_A \otimes p_1$ and $p'_2\phi = id_A \otimes p_2$. Let $(a \otimes (m, n)) \in A \otimes_S P$. Then there exists $i \in I$ such that $a \in A_i$. Thus we have

$$p'_1\phi(a \otimes (m, n)) = p'_1\phi_i(a \otimes (m, n)) = p'_1(a \otimes m, a \otimes n) = a \otimes m$$

$$= id_{A_i}(a) \otimes p_1(m, n) = id_A(a) \otimes p_1(m, n) = (id_A \otimes p_1)(a \otimes (m, n)).$$

The same argument shows that $p'_2\phi = id_A \otimes p_2$. $\qquad\square$

The following two theorems are our main results in this section.

**Theorem 3.7.** *Let $\phi$ be the mapping corresponding to the subpullback diagram $P(M, N, f, g, Q)$ for $A_S$ and let $\phi_i, i \in I$, be as in Lemma 3.6. Then $\phi$ is surjective if and only if $\phi_i$ is surjective for every $i \in I$.*

*Proof.* **Necessity.** Let $\phi$ be surjective. Since $a_i, a_i' \in A_i$, we have $a_i, a_i' \in A_S$. By Lemma 3.1, $a_i \otimes f(m) \leq a_i' \otimes g(n)$ in $A_i \otimes_S Q$ implies $a_i \otimes f(m) \leq a_i' \otimes g(n)$ in $A \otimes_S Q$. Using surjectivity of $\phi$,

$$(\exists a_i'' \in A_S)(\exists m' \in_S M)(\exists n' \in_S N)$$

$$(f(m') \leq g(n') \wedge a_i \otimes m = a_i'' \otimes m' \wedge a_i' \otimes n = a_i'' \otimes n').$$

By Lemma 3.2, $a_i \otimes m = a_i'' \otimes m'$ in $A \otimes_S M$ and $a_i \in A_i$ imply $a_i'' \in A_i$. Hence $\phi_i$ is surjective.

   **Sufficiency.** Let $\phi_i$ be surjective for every $i \in I$ and suppose that $a \otimes f(m) \leq a' \otimes g(n)$ in $A \otimes_S Q$ for $a, a' \in A_S$, $m \in {}_S M$, $n \in {}_S N$. Since $a \in A_S$, there exists $i \in I$ such that $a \in A_i$. By Corollary 3.3, we have $a \otimes f(m) \leq a' \otimes g(n)$ in $A_i \otimes_S Q$. Using surjectivity of $\phi_i$,

$$(\exists a'' \in A_i)(\exists m' \in_S M)(\exists n' \in_S N)$$

$$(f(m') \leq g(n') \wedge a \otimes m = a'' \otimes m' \wedge a' \otimes n = a'' \otimes n').$$

By Lemma 3.1, $a \otimes m = a'' \otimes m'$ in $A_i \otimes_S M$ and $a' \otimes n = a'' \otimes n'$ in $A_i \otimes_S N$ imply that $a \otimes m = a'' \otimes m'$ in $A \otimes_S M$ and $a' \otimes n = a'' \otimes n'$ in $A \otimes_S N$, respectively. Thus $\phi$ is surjective, as required. □

**Theorem 3.8.** *Let $\phi$ be the mapping corresponding to the subpullback diagram $P(M, N, f, g, Q)$ for $A_S$ and let $\phi_i, i \in I$, be as in Lemma 3.6. Then $\phi$ is an order embedding if and only if $\phi_i$ is an order embedding for every $i \in I$.*

*Proof.* **Necessity.** Let $\phi$ be an order embedding and suppose that

$$f(m) \leq g(n) \wedge f(m') \leq g(n') \wedge (a \otimes m \leq a' \otimes m') \wedge (a \otimes n \leq a' \otimes n'),$$

where $a \otimes m \leq a' \otimes m'$ in $A_i \otimes_S M$ and $a \otimes n \leq a' \otimes n'$ in $A_i \otimes_S N$ for $i \in I$. Because $a, a' \in A_i$, and by Lemma 3.1, we have $a \otimes m \leq a' \otimes m'$ in $A \otimes_S M$ and $a \otimes n \leq a' \otimes n'$ in $A \otimes_S N$. Using order embeddability of $\phi$, we obtain $a \otimes (m, n) \leq a' \otimes (m', n')$ in $A \otimes_S P$. But $a \in A_i$ and so $a \otimes (m, n) \leq a' \otimes (m', n')$ in $A_i \otimes_S P$ by Corollary 3.3. Hence $\phi_i$ is an order embedding.

   **Sufficiency.** Let $\phi_i$ is an order embedding for every $i \in I$ and suppose that

$$f(m) \leq g(n) \wedge f(m') \leq g(n') \wedge (a \otimes m \leq a' \otimes m') \wedge (a \otimes n \leq a' \otimes n'),$$

where $a \otimes m \leq a' \otimes m'$ and $a \otimes n \leq a' \otimes n'$ in $A \otimes_S M$ and $A \otimes_S N$, respectively. Because $a \in A_S$, there exists $i \in I$ such that $a \in A_i$. By Corollary 3.3, $a \otimes m \leq a' \otimes m'$ in $A \otimes_S M$ and $a \otimes n \leq a' \otimes n'$ in $A \otimes_S N$ imply $a \otimes m \leq a' \otimes m'$ in $A_i \otimes_S M$ and $a \otimes n \leq a' \otimes n'$ in $A_i \otimes_S N$, respectively. Using order embeddability of $\phi_i$, we obtain $a \otimes (m, n) \leq a' \otimes (m', n')$ in $A_i \otimes_S P$. By Lemma 3.1, we have $a \otimes (m, n) \leq a' \otimes (m', n')$ in $A \otimes_S P$, and so $\phi$ is an order embedding. The proof is complete. □

For every subpullback diagram, the corresponding $\phi$ is a surjection or an order embedding if and only if the corresponding $\phi_i$ is a surjection or an order embedding, and surjectivity or order isomorphism of $\phi$ for a special subpullback diagram is equivalent to certain kind of flatness property. It follows from Theorems 2.1, 2.2, 2.3, 2.4, 3.2, 4.1, 5.3 of [9] and Definition 2.2 that

**Proposition 3.9.** *Let $S$ be a pomonoid and $A = \bigcup\limits_{i \in I}^{\cdot} A_i$, where $A_i$, $i \in I$, are right strongly convex $S$-subposets of $A_S$. Then $A_S$ is torsion free, principally weakly flat, weakly flat, flat, pullback flat, subpullback flat, principally weakly kernel po-flat, weakly kernel po-flat, translation kernel po-flat, and satisfies conditions $(P)$, $(WP)$, $(PWP)$ if and only if $A_i$ has these properties for every $i \in I$.*

From Proposition 3.9 and Theorem 2.3 of [12], we have

**Corollary 3.10.** *Let $S$ be a pomonoid. Then a right $S$-poset $A_S$ is torsion free, principally weakly flat, weakly flat, flat, pullback flat, subpullback flat, principally weakly kernel po-flat, weakly kernel po-flat, translation kernel po-flat, and satisfies conditions $(P)$, $(WP)$, $(PWP)$ if and only if its strongly convex indecomposable components have these properties.*

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# زیر عقب‌برها و همضرب‌های $S$-مجموعه‌های مرتب

## ژینگلیانگ لیانگ و یانفنگ لوو

در سال ۲۰۰۱، سیدنی بولمن – فلمینگ و همکاران مطالعه‌ی سه نوع ویژگی تخت بودن (هسته ضعیف، هسته ضعیف اصلی، و هسته انتقالی تخت بودن) کنش‌های (راست) $A_S$ تکواره‌ی $S$ را، که می‌توانند به وسیله‌ی حافظ عقب‌بر بودن تابعگون $A_S \otimes -$ معرفی شوند، آغاز کردند. در این مقاله، این نتایج را به $S$-مجموعه‌های مرتب توسعه داده و توصیف‌های معادلی برای $S$-مجموعه‌های مرتب تخت ترتیبی هسته‌ای ضعیف، تخت ترتیبی هسته‌ای ضعیف اصلی، و تخت ترتیبی هسته‌ای انتقالی ارائه می‌دهیم. به علاوه، نشان می‌دهیم که اغلب ویژگی‌های تخت بودن $S$-مجموعه‌های مرتب را می‌توان به همضرب آن‌ها منتقل کرد و برعکس.

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