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A cottage industry of lax extensions

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Abstract. In this work, we describe an adjunction between the comma category of Set-based monads under the V-powerset monad and the category of associative lax extensions of Set-based monads to the category of V-relations. In the process, we give a general construction of the Kleisli extension of a monad to the category of V-relations.

1 Introduction

In [1], Barr introduced the concept of a lax extension of a monad by lifting the ultrafilter monad β on Set to a lax monad $\bar{\beta}$ on the category Rel of relations. The category of lax algebras for $\bar{\beta}$ were shown to form the category Top of topological spaces and continuous maps. This concept was developed in several directions, two of which provide the basis for the current work: in [3] and [5], Clementino, Hofmann and Tholen allowed for lax extensions to the category V-Rel of relations valued in a quantale V, and in [10], Seal proposed an alternate construction of a lax extension, that, when applied to the filter monad $\mathbb{F}$, returned Top again as category of lax algebras. In particular, contrarily to the one-to-one correspondence between monads and

categories of algebras, different monads — equipped with appropriate lax extensions — can yield isomorphic categories of lax algebras.

The adjunction presented here as Theorem 5.7 sheds some light on this situation by having a family of monads and their lax extensions correspond to an essentially unique representative. A central ingredient of this adjunction is the *discrete presheaf monad* $\mathbb{P}_V$ on Set (also called the *V-powerset monad*) that extends the powerset monad $\mathbb{P} = \mathbb{P}_2$ from the base 2 to a general quantale V. Indeed, the Kleisli category of this monad is the category V-Rel of V-relations — whose opposite category is used to laxly extend Set-based monads via the basic adjunction

$$(-)^{\circ} \dashv V\text{-Rel}(-, 1) : V\text{-Rel}^{\mathrm{op}} \longrightarrow \mathsf{Set}$$

(see Subsections 2.5, 2.6, 3.1 and Example 3.6). The category of "neighborhood monads", that is, of monads that play the role of the filter monad in our general context, is the comma category $(\mathbb{P}_V \downarrow \mathsf{Mnd}_{\mathsf{Set}})$ of monads on Set under $\mathbb{P}_V$. The category V-LaxExt of "convergence monads", that is, of monads that convey notions of convergence, similarly to the ultrafilter monad mentioned above, has as its objects monads $\mathbb{T}$ on Set equipped with an associative lax extension $\hat{\mathbb{T}}$ to V-Rel. The comma category $(\mathbb{P}_V \downarrow \mathsf{Mnd}_{\mathsf{Set}})$ then appears as a full reflective subcategory of V-LaxExt via the adjunction

$$F \dashv G : (\mathbb{P}_V \downarrow \mathsf{Mnd}_{\mathsf{Set}}) \longrightarrow V\text{-LaxExt}$$

of Theorem 5.7. The reflector F takes a pair $(\mathbb{T}, \hat{\mathbb{T}})$ to the monad induced by the adjunction

$$(-)^{\sharp} \dashv (\mathbb{T}, V)\text{-URel}(-, 1) : (\mathbb{T}, V)\text{-URel}^{\mathrm{op}} \longrightarrow \mathsf{Set},$$

where $(\mathbb{T}, V)$-URel is the category of *unitary* $(\mathbb{T}, V)$-*relations* associated to the associative lax extension $\hat{\mathbb{T}}$ of $\mathbb{T}$ to V-Rel (Proposition 3.5). The embedding functor G sends a monad morphism $\mathbb{P}_V \longrightarrow \mathbb{T}$ to the pair $(\mathbb{T}, \check{\mathbb{T}})$, where $\check{\mathbb{T}}$ is the Kleisli extension of $\mathbb{T}$ to V-Rel (Subsection 4.4); this lax extension generalizes the construction described in [10] for the $V = 2$ case.

Let us make two remarks on the context of our article. First, in [7], Lowen and Vroegrijk show that the category App of approach spaces and non-expansive maps can be presented as lax algebras for a "prime functional ideal" monad laxly extended to Rel = 2-Rel, contrasting the result of [3],

where the ultrafilter monad with a lax extension to V-Rel was used to obtain a lax-algebraic description of App (with $V = \mathsf{P}_+$ the extended non-negative real half-line). These results, generalized in [2, Corollary IV.3.2.3], stem from a different point of view as the one taken here, as they allow for a "change-base-base", namely passing from $V = \mathsf{P}_+$ to $V = 2$ by modifying the underlying monad, whereas the current work fixes the quantale V. Second, the study of lax extensions is rooted in topology and order theory, but current trends show that it is in no way limited to these fields, as the cited references show, but also [6] and [8].

In this article, we use the conventions and terminology of [2] to which the reader is referred in case of doubt. In particular, we use the term "order" for what is elsewhere known as "preorder" (that is, a reflexive and transitive relation, not necessarily symmetric). Because our categories may be variously enriched, we use the term "ordered category" and "monotone functor" rather than their "2-" counterparts to designate the categories and functors enriched in Ord, the category of ordered sets and monotone maps.

2 The V-powerset monad

In this section, we recall the basic definitions and notations used in this paper, mainly pertaining to categories enriched in quantales.

2.1 Quantales

A *quantale* $V = (V, \otimes, k)$ (more precisely, a *unital quantale*) is a complete lattice equipped with an associative binary operation $\otimes$, its *tensor*, that preserves suprema in each variable:

$$a \otimes \bigvee_{i \in I} b_i = \bigvee_{i \in I} (a \otimes b_i) \qquad \text{and} \qquad \bigvee_{i \in I} a_i \otimes b = \bigvee_{i \in I} (a_i \otimes b)$$

for all $a, a_i, b, b_i \in V$ ($i \in I$), and has a neutral element k. A *lax homomorphism* of quantales $f : (W, \otimes, l) \to (V, \otimes, k)$ is a monotone map $f : W \to V$ satisfying

$$f(a) \otimes f(b) \leq f(a \otimes b) \qquad \text{and} \qquad k \leq f(l)$$

for all $a, b \in V$. A map $f : W \to V$ is a *homomorphism* of quantales if it is a sup-map and preserves the tensor and neutral elements (that is, $f(a) \otimes f(b) = f(a \otimes b)$ and $k = f(l)$).

Given $a \in V$, the sup-map $(-) \otimes a$ is left adjoint to an inf-map $((-) \bullet\!\!-\, a) : V \to V$ that is uniquely determined by

$$(v \otimes a) \leq b \iff v \leq (b \bullet\!\!-\, a)$$

for all $v, b \in V$; hence, $(b \bullet\!\!-\, a) = \bigvee\{v \in V \mid v \otimes a \leq b\}$. Symmetrically, the sup-map $a \otimes (-)$ is left adjoint to an inf-map $(a \to\!\!\bullet\, (-)) : V \to V$ that is uniquely determined by

$$(a \otimes v) \leq b \iff v \leq (a \to\!\!\bullet\, b).$$

In the case where the tensor of V is commutative, $((-) \bullet\!\!-\, a)$ and $(a \to\!\!\bullet\, (-))$ coincide, and either of the two notations may be used.

Example 2.1.

(1) The two-element chain $2 = \{0, 1\}$ with $\otimes$ the binary infimum, and $k = \top$ is a quantale. Here, $(b \bullet\!\!-\, a)$ is the Boolean truth value of the statement $a \leq b$. For any other quantale $V = (V, \otimes, k)$, there is a canonical homomorphism $2 \to V$ that necessarily sends 0 to the bottom element $\bot$ of V, and 1 to k.

(2) The unit interval $\mathsf{I} = [0, 1]$ with its natural order, $\otimes$ given by multiplication, and neutral element $k = 1$ is a quantale. In this case,

$$(b \bullet\!\!-\, a) = b \oslash a := \bigvee\{v \in [0, 1] \mid v \cdot a \leq b\},$$

so that $b \oslash a = b/a$ if $b < a$, and $b \oslash a = 1$ otherwise. This quantale is isomorphic to the *extended non-negative reals* quantale $\mathsf{P}_+ = ([0, \infty]^{\mathrm{co}}, +, 0)$ via the monotone bijection $f : [0, 1] \to [0, \infty]^{\mathrm{co}}$ that sends 0 to ∞, and $x \in (0, 1]$ to $-\log(x)$ (see for example [4]; here, $[0, \infty]^{\mathrm{co}}$ denotes the set of extended non-negative reals equipped with the order opposite to the natural order). The embedding $2 \to \mathsf{I}$ is a homorphism of quantales (that is, a sup-map that preserves the tensor and neutral element).

2.2   *V*-relations

Given a quantale $V = (V, \otimes, k)$, a *V-relation* $r : X \to\!\!\!\!\to Y$ from a set X to a set Y is a Set-map $r : X \times Y \to V$. A *V*-relation $r : X \to\!\!\!\!\to Y$ can be

composed with a V-relation $s : Y \nrightarrow Z$ via

$$(s \cdot r)(x, z) = \bigvee_{y \in Y} r(x, y) \otimes s(y, z)$$

(for all $x \in X$, $z \in Z$) to yield a V-relation $s \cdot r : X \nrightarrow Z$. The identity on a set X for this composition is the V-relation $1_X : X \nrightarrow X$ that sends (x, y) to k if $x = y$ and to $\bot$ otherwise (where $\bot$ is the least element of V). Sets and V-relations between them form the category

$$V\text{-Rel}.$$

Example 2.2.

(1) The category 2-Rel is isomorphic to the category Rel of sets with ordinary relations as morphisms.

(2) The category I-Rel is isomorphic to the category $\mathsf{NRel} = \mathsf{P}_+\text{-Rel}$ of numerical relations.

2.3   Ordered V-relations

The hom-set $V\text{-Rel}(X, Y)$ inherits the pointwise order induced by V: given $r : X \nrightarrow Y$ and $r' : X \nrightarrow Y$, we have

$$r \leq r' \iff \forall (x, y) \in X \times Y \ (r(x, y) \leq r'(x, y)).$$

Since the order on V is complete, so is the pointwise order on $V\text{-Rel}(X, Y)$, and since the tensor in V distributes over suprema, V-relational composition preserves suprema in each variable:

$$\left(\bigvee_{i \in I} s_i \right) \cdot r = \bigvee_{i \in I} (s_i \cdot r) \qquad \text{and} \qquad t \cdot \left(\bigvee_{i \in I} s_i \right) = \bigvee_{i \in I} (t \cdot s_i)$$

for V-relations $r : X \nrightarrow Y$, $s_i : Y \nrightarrow Z$ $(i \in I)$, and $t : Z \nrightarrow W$. In particular, given a V-relation $r : X \nrightarrow Y$, the sup-map $(-) \cdot r : V\text{-Rel}(Y, Z) \to V\text{-Rel}(X, Z)$ is left adjoint to an inf-map $(-) \bullet\!\!-\ r : V\text{-Rel}(X, Z) \to V\text{-Rel}(Y, Z)$ defined by

$$(s \bullet\!\!-\ r)(y, z) = \bigwedge_{x \in X} (r(x, y) \multimap s(x, z)),$$

for all V-relations $s : X \nrightarrow Z$, and $y \in Y$, $z \in Z$. Diagrammatically, $s \bullet\!\!- r$ is the extension in V-Rel of s along r:

$$
\begin{array}{ccc}
X & \xrightarrow{\;r\;} & Y \\
{\scriptstyle s}\downarrow & {\scriptstyle \geq} \;\;{\times}\;\; & {\scriptstyle s\,\bullet\!\!-\,r} \\
Z & &
\end{array}
$$

Symmetrically, for a V-relation $t : Z \nrightarrow W$, the sup-map $t \cdot (-) : V\text{-Rel}(Y,Z) \to V\text{-Rel}(Y,W)$ is left adjoint to an inf-map $(t \multimap (-)) : V\text{-Rel}(Y,W) \to V\text{-Rel}(Y,Z)$ given by

$$
(t \multimap s)(y,z) = \bigwedge_{w \in W} (s(y,w) \bullet\!\!- t(z,w)),
$$

for all V-relations $s : Y \nrightarrow W$, and $y \in Y$, $z \in Z$. Diagrammatically, $t \multimap s$ is the lifting in V-Rel of s along t:

$$
\begin{array}{ccc}
Y & & \\
{\scriptstyle s}\downarrow & {\scriptstyle \geq}\;{\times}\;{\scriptstyle t\,\multimap\,s} & \\
W & \xleftarrow[\;t\;]{} & Z
\end{array}
$$

2.4   Opposite V-relations

For a V-relation $r : X \nrightarrow Y$, the *opposite* V-relation is $r^\circ : Y \nrightarrow X$ is defined by

$$
r^\circ(y,x) = r(x,y)
$$

for all $x \in X$, $y \in Y$. Involution preserves the order on the hom-sets:

$$
r \leq s \implies r^\circ \leq s^\circ
$$

(for all V-relations $r, s : X \nrightarrow Y$).

2.5   Maps as V-relations

There is a functor from Set to V-Rel that interprets the graph of a Set-map $f : X \to Y$ as the V-relation $f_\circ : X \nrightarrow Y$ given by

$$
f_\circ(x,y) = \begin{cases} k & \text{if } f(x) = y, \\ \bot & \text{otherwise.} \end{cases}
$$

To keep notations simple, we write $f : X \to Y$ instead of $f_\circ : X \nrightarrow Y$ to designate a V-relation induced by a map; in particular, we write f° instead of $(f_\circ)^\circ$. There is therefore a functor $(-)_\circ : \mathsf{Set} \to V\text{-}\mathsf{Rel}$, and a functor $(-)^\circ : \mathsf{Set} \to V\text{-}\mathsf{Rel}^{\mathrm{op}}$ obtained by sending a map f to $f_\circ = (f_\circ)^\circ$.

Note that without any commutativity assumption on V, composition of V-relations is not necessarily compatible with the involution $(-)^\circ$, but whiskering by Set-maps is:

$$(h \cdot s \cdot f)^\circ = f^\circ \cdot s^\circ \cdot h^\circ$$

for all V-relations $s : Y \to Z$, and Set-maps $f : X \to Y$, $h : Z \to W$.

2.6 The V-powerset monad

Any V-relation $r : X \nrightarrow Y$ yields a map $r^{\mathbb{P}_V} : V^Y \to V^X$ that sends $s : Y \to V$ to $r^{\mathbb{P}_V}(s) : X \to V$ defined by

$$r^{\mathbb{P}_V}(s)(x) = \bigvee_{y \in Y} r(x, y) \otimes s(y)$$

for all $x \in X$. This correspondence describes a functor $V\text{-}\mathsf{Rel}^{\mathrm{op}} \to \mathsf{Set}$ that is right adjoint to $(-)^\circ : \mathsf{Set} \to V\text{-}\mathsf{Rel}^{\mathrm{op}}$, and determines the V-*powerset monad* $\mathbb{P}_V = (P_V, \mu, \eta)$ on Set, where

$$P_V X = V^X, \quad P_V f(r)(y) = \bigvee_{x \in f^{-1}(y)} r(x),$$

$$\mu_X(R)(x) = \bigvee_{s \in V^X} s(x) \otimes R(s), \qquad \eta_X(x)(x') = 1_X(x, x'),$$

for all $x, x' \in X$, $y \in Y$, and maps $f : X \to Y$, $r : X \to V$, $R : V^X \to V$. The 2-monad $\mathbb{P}_2$ is easily seen to be isomorphic to the powerset monad $\mathbb{P}$.

The monad associated to the left adjoint functor $(-)_\circ : \mathsf{Set} \to V\text{-}\mathsf{Rel}$ is of course isomorphic to $\mathbb{P}_V$, but our focus in the rest of this article will be on the adjunction with $V\text{-}\mathsf{Rel}^{\mathrm{op}}$ (see 3.6).

2.7 V-categories

When $V = (V, \otimes, k)$ is a quantale, a small V-category (X, a) is a set X with a transitive and reflexive V-relation a, so that

$$a \cdot a \le a \qquad \text{and} \qquad 1_X \le a,$$

or equivalently,

$$a(x,y) \otimes a(y,z) \leq a(x,z) \qquad \text{and} \qquad k \leq a(x,x),$$

for all $x, y, z \in X$. A V-functor $f : (X, a) \longrightarrow (Y, b)$ of V-categories is a map $f : X \longrightarrow Y$ with

$$a \cdot f^{\circ} \leq f^{\circ} \cdot b,$$

or, in pointwise notation,

$$a(x,y) \leq b(f(x), f(y))$$

for all $x, y, z \in X$. We denote by

$$V\text{-}\mathsf{Cat}$$

the category of small V-categories and their V-functors. The induced order on a V-category (X, a) is defined by

$$x \leq y \iff k \leq a(x, y)$$

(for all $x, y \in X$); with the hom-sets $V\text{-}\mathsf{Cat}(X, Y)$ equipped with the induced pointwise order:

$$f \leq g \iff \forall x \in X \ (f(x) \leq g(x))$$

(for all V-functors $f, g : X \longrightarrow Y$), $V\text{-}\mathsf{Cat}$ is an ordered category.

2.8 V-modules

Let V be a quantale, and (X, a), $(Y.b)$ small V-categories. A V-relation $r : X \nrightarrow Y$ is a V-module if

$$r \cdot a \leq r \qquad \text{and} \qquad b \cdot r \leq r.$$

Since the reverse inequalities always hold, these are in fact equalities:

$$r \cdot a = r \qquad \text{and} \qquad b \cdot r = r.$$

Modules compose as V-relations, and $a : (X, a) \nrightarrow (X, a)$ serves as an identity morphism in the category

$$V\text{-}\mathsf{Mod}$$

whose objects are V-categories and morphisms are V-modules. This category is ordered, with the order inherited from V-Rel; in fact, V-Mod is a quantaloid, with suprema in its hom-sets formed as in V-Rel; similarly, adjoints to the composition maps (as described in 2.3). By sending a V-module to its underlying V-relation, one obtains a lax functor

$$V\text{-Mod} \longrightarrow V\text{-Rel}$$

that preserves composition, but in general only preserves the identity laxly since $1_X \leq a$ for any V-category structure $a : X \rightarrow\!\!\!\!\!\rightarrow X$.

2.9 Maps in V-Mod

For a V-functor $f : (X, a) \to (Y, b)$, one defines a V-module $f^* : (Y, b) \rightarrow\!\!\!\!\!\rightarrow (X, a)$ by

$$f^* := f^\circ \cdot b,$$

that is, $f^*(y, x) = b(y, f(x))$ for all $x \in X$, $y \in Y$. This operation defines a monotone functor

$$(-)^* : V\text{-Cat} \longrightarrow V\text{-Mod}^{\mathrm{op}}.$$

In particular, $1_{(X,a)} = a = 1_X^*$.

3 The $(\mathbb{T}, V)$-powerset monad

In this section, we recall how a quantale V and a monad $\mathbb{T}$ on Set can give rise to a monad $\not\succ$ that incorporates both structures.

3.1 Associative lax extensions

A *lax extension* of a monad $\mathbb{T} = (T, m, e)$ on Set to V-Rel is a lax functor $\hat{T} : V\text{-Rel} \to V\text{-Rel}$ that extends T laxly, and such that $m^\circ : \hat{T} \to \hat{T}\hat{T}$ and $e^\circ : 1_{V\text{-Rel}} \to \hat{T}$ are lax natural transformations. Equivalently, $\hat{T} : V\text{-Rel} \to V\text{-Rel}$ is given by functions

$$\hat{T}_{X,Y} : V\text{-Rel}(X, Y) \longrightarrow V\text{-Rel}(TX, TY)$$

for all sets X, Y (with $\hat{T}_{X,Y}$ simply written as $\hat{T}$), such that

1. $r \leq r' \implies \hat{T}r \leq \hat{T}r'$,   4. $\hat{T}s \cdot \hat{T}r \leq \hat{T}(s \cdot r)$,

2. $(Tf)^{\circ} \leq \hat{T}(f^{\circ})$,   5. $\hat{T}\hat{T}r \cdot m_X^{\circ} \leq m_Y^{\circ} \cdot \hat{T}r$,

3. $\hat{T}(f^{\circ} \cdot r) = (Tf)^{\circ} \cdot \hat{T}r$,   6. $r \cdot e_X^{\circ} \leq e_Y^{\circ} \cdot \hat{T}r$,

for all sets X, Y, Z, V-relations $r, r' : X \nrightarrow Y$, $s : Y \nrightarrow Z$, and maps $f : Z \to Y$. The lax extension is *associative*, if the inequalities in conditions (4) and (5) are equalities, that is, if $\hat{T}$ preserves composition of V-relations and $m^{\circ} : \hat{T}\hat{T} \nrightarrow \hat{T}$ is a natural transformation.

Example 3.1.

(1) An associative lax extension $\hat{\mathbb{I}}$ of the identity monad $\mathbb{I}$ on Set to V-Rel is given by the identity monad on V-Rel.

(2) An associative lax extension of the ultrafilter monad $\mathbb{U}$ on Set to V-Rel is given by the lax functor $\hat{U} : V\text{-Rel} \to V\text{-Rel}$ defined by

$$\hat{U}r(\mathfrak{x}, \mathfrak{y}) = \bigwedge_{A \in \mathfrak{x}, B \in \mathfrak{y}} \bigvee_{x \in A, y \in B} r(x, y)$$

for all ultrafilters $\mathfrak{x} \in UX$ and $\mathfrak{y} \in UY$.

Remark 3.2. The definition of a *lax extension* $\hat{\mathbb{T}}$ of $\mathbb{T}$ to V-Rel given above is equivalent to the one given in [2, Section III.1]. In particular, $\hat{T}$ is a lax functor, and $e^{\circ} : 1_{V\text{-Rel}} \to \hat{T}$ and $m^{\circ} : \hat{T} \to \hat{T}\hat{T}$ are lax natural transformations. The lax extension is called *associative* because the equalities in (4) and (5) are equivalent to the Kleisli convolution being associative on unitary relations (see Proposition 3.3 below).

3.2 $(\mathbb{T}, V)$-relations

For a quantale V and an associative lax extension $\hat{\mathbb{T}}$ to V-Rel of a monad $\mathbb{T}$ on Set, a $(\mathbb{T}, V)$-*relation* $r : X \nrightarrow Y$ is a V-relation $r : TX \nrightarrow Y$. The *Kleisli convolution* $s \circ r : X \nrightarrow Z$ of $(\mathbb{T}, V)$-relations $r : X \nrightarrow Y$ and $s : Y \nrightarrow Z$ is the $(\mathbb{T}, V)$-relation defined by

$$s \circ r := s \cdot \hat{T}r \cdot m_X^{\circ}.$$

A $(\mathbb{T}, V)$-relation $r : X \nrightarrow Y$ is *unitary* if

$$r \circ 1_X^\sharp = r \qquad \text{and} \qquad 1_Y^\sharp \circ r = r,$$

where $1_X^\sharp := e_X^\circ \cdot \hat{T}1_X : TX \nrightarrow X$; these conditions are equivalent to

$$r \cdot \hat{T}1_X = r \qquad \text{and} \qquad e_Y^\circ \cdot \hat{T}r \cdot m_X^\circ = r.$$

With conditions 3.1(2) and (6), the $(\mathbb{T}, V)$-relation $r : X \nrightarrow Y$ is unitary precisely if

$$r \cdot \hat{T}1_X \leq r \qquad \text{and} \qquad e_Y^\circ \cdot \hat{T}r \cdot m_X^\circ \leq r.$$

One readily verifies that $1_X^\sharp$ is a unitary $(\mathbb{T}, V)$-relation. Furthermore, Kleisli convolution is associative on unitary $(\mathbb{T}, V)$-relations, as the following result recalls (see [2, Proposition III.1.9.4]).

Proposition 3.3. *Let $\hat{\mathbb{T}}$ be a lax extension to V-Rel of a monad $\mathbb{T} = (T, m, e)$ on Set. The following are equivalent:*

1. *$\hat{\mathbb{T}}$ is associative;*

2. *$\hat{T} : V\text{-Rel} \to V\text{-Rel}$ preserves composition and $m^\circ : \hat{T} \to \hat{T}\hat{T}$ is natural.*

Hence, one can form the category

$$(\mathbb{T}, V)\text{-URel}$$

whose objects are sets and morphisms are unitary $(\mathbb{T}, V)$-relations (with composition given by Kleisli convolution and identity on X by $1_X^\sharp$).

3.3 Ordered $(\mathbb{T}, V)$-relations

Let V be a quantale and $\hat{\mathbb{T}}$ an associative lax extension to V-Rel of a monad $\mathbb{T}$ on Set. The category $(\mathbb{T}, V)$-URel of unitary $(\mathbb{T}, V)$-relations forms an ordered category when the hom-sets $(\mathbb{T}, V)\text{-URel}(X, Y) \subseteq V\text{-Rel}(TX, Y)$ are equipped with the pointwise order inherited from V-Rel, since the Kleisli convolution preserves this order on the left and right (see 3.1(1)).

The infimum $\bigwedge_{i \in I} \varphi_i$ in $(\mathbb{T}, V)$-$\mathsf{Rel}(X, Y)$ of a family of unitary $(\mathbb{T}, V)$-relations $(\varphi_i : X \nrightarrow Y)_{i \in I}$ is again unitary:

$$\big(\textstyle\bigwedge_{i \in I} \varphi_i \big) \cdot \hat{T} 1_X \leq \bigwedge_{i \in I} (\varphi_i \cdot \hat{T} 1_X) = \bigwedge_{i \in I} \varphi_i \qquad \text{and}$$
$$e_Y^\circ \cdot \hat{T} \big(\textstyle\bigwedge_{i \in I} \varphi_i \big) \cdot m_X^\circ \leq \bigwedge_{i \in I} (e_Y^\circ \cdot \hat{T} \varphi_i \cdot m_X^\circ) \;\; = \bigwedge_{i \in I} \varphi_i.$$

Hence, the ordered category $(\mathbb{T}, V)$-URel has complete hom-sets. Moreover, for a unitary $(\mathbb{T}, V)$-relation $\varphi : X \nrightarrow Y$, the map $(-) \circ \varphi : (\mathbb{T}, V)$-$\mathsf{URel}(Y, Z) \to (\mathbb{T}, V)$-$\mathsf{URel}(X, Z)$ has a right adjoint $(-) \multimapinv \varphi : (\mathbb{T}, V)$-$\mathsf{URel}(X, Z) \to (\mathbb{T}, V)$-$\mathsf{URel}(Y, Z)$ given by

$$(\psi \multimapinv \varphi) = (\psi \bullet\!\!\!- (\hat{T} \varphi \cdot m_X^\circ)).$$

Indeed, for all $(\mathbb{T}, V)$-relations $\gamma : Y \nrightarrow Z$ and $\psi : X \nrightarrow Z$,

$$\gamma \circ \varphi \leq \psi \iff \gamma \cdot \hat{T} \varphi \cdot m_X^\circ \leq \psi \iff \gamma \leq (\psi \bullet\!\!\!- (\hat{T} \varphi \cdot m_X^\circ)).$$

If ψ is moreover unitary, then by associativity of the Kleisli convolution,

$$(1_Z^\sharp \circ (\psi \multimapinv \varphi)) \circ \varphi \;\; \leq 1_Z^\sharp \circ \psi = \psi \qquad \text{and}$$
$$((\psi \multimapinv \varphi) \circ 1_Y^\sharp) \circ \varphi \;\; \leq (\psi \multimapinv \varphi) \circ \varphi \leq \psi;$$

therefore, $1_Z^\sharp \circ (\psi \multimapinv \varphi) \leq (\psi \multimapinv \varphi)$ and $(\psi \multimapinv \varphi) \circ 1_Y^\sharp \leq (\psi \multimapinv \varphi)$, that is, $\psi \multimapinv \varphi$ is unitary. Hence, $\psi \multimapinv \varphi$ is the extension in $(\mathbb{T}, V)$-URel of ψ along φ:

Hence, as it is left adjoint, the map $(-) \circ \varphi : (\mathbb{T}, V)$-$\mathsf{URel}(Y, Z) \to (\mathbb{T}, V)$-$\mathsf{URel}(X, Z)$ preserves suprema.

3.4   Maps as $(\mathbb{T}, V)$-relations

Let V be a quantale and $\hat{\mathbb{T}}$ an associative lax extension to V-Rel of a monad $\mathbb{T}$ on Set. A map $f : X \to Y$ gives rise to a unitary $(\mathbb{T}, V)$-relation $f^\sharp : Y \nrightarrow X$ via

$$f^\sharp := e_X^\circ \cdot \hat{T}(f^\circ).$$

This definition is consistent with the notation used for the identity of the Kleisli convolution, and one also has $f^\sharp \circ g^\sharp = (g \cdot f)^\sharp$ for all maps $f : X \to Y$ and $g : Y \to Z$ in Set. This defines a functor

$$(-)^\sharp : \mathsf{Set} \to (\mathbb{T}, V)\text{-}\mathsf{URel}^{\mathrm{op}}$$

that maps objects identically. For a unitary $(\mathbb{T}, V)$-relation $\varphi : X \nrightarrow Y$, one moreover has

$$f^\sharp \circ \varphi = f^\circ \cdot \varphi$$

for all maps $f : Z \to Y$. Indeed,

$$
\begin{aligned}
f^\sharp \circ \varphi \ &= e_X^\circ \cdot \hat{T}(f^\circ \cdot \varphi) \cdot m_X^\circ = e_X^\circ \cdot (Tf)^\circ \cdot \hat{T}\varphi \cdot m_X^\circ \\
&= f^\circ \cdot e_Y^\circ \cdot \hat{T}1_Y \cdot \hat{T}\varphi \cdot m_X^\circ = f^\circ \cdot (1_X^\sharp \circ \varphi) = f^\circ \cdot \varphi.
\end{aligned}
$$

3.5 An adjunction between $(\mathbb{T}, V)$-$\mathsf{URel}^{\mathrm{op}}$ and Set

We now proceed to show that the functor $(-)^\sharp : \mathsf{Set} \to (\mathbb{T}, V)\text{-}\mathsf{URel}^{\mathrm{op}}$ is left adjoint to the contravariant hom-functor

$$(\mathbb{T}, V)\text{-}\mathsf{URel}(-, 1) : (\mathbb{T}, V)\text{-}\mathsf{URel}^{\mathrm{op}} \to \mathsf{Set},$$

where $1 = \{\star\}$ denotes a singleton. We identify elements $x \in X$ with maps $x : 1 \to X$, and to a unitary $(\mathbb{T}, V)$-relation $\psi : X \nrightarrow Y$ we associate the map $\psi^\flat : Y \to (\mathbb{T}, V)\text{-}\mathsf{URel}(X, 1)$ defined by

$$\psi^\flat(y) := y^\sharp \circ \psi = y^\circ \cdot \psi = \psi(-, y)$$

for all $y \in Y$ (the third equality follows by definition of composition in V-Rel); here, $\psi(-, y)(x, \star) := \psi(x, y)$.

Lemma 3.4. *Let V be a quantale, $\hat{\mathbb{T}}$ an associative lax extension to V-Rel of a monad $\mathbb{T}$ on Set, and $\varphi : X \nrightarrow Y$ a $(\mathbb{T}, V)$-relation. Then φ is unitary if and only if $y^\circ \cdot \varphi$ is unitary for all $y \in Y$.*

Proof. If φ is unitary, then $y^\circ \cdot \varphi = y^\sharp \circ \varphi$ is unitary as well. To verify the other implication, suppose that $y^\circ \cdot \varphi$ is unitary for all $y \in Y$. Then one has

$$y^\circ \cdot (\varphi \circ 1_X^\sharp) = (y^\circ \cdot \varphi) \circ 1_X^\sharp = y^\circ \cdot \varphi$$

and

$$y^\circ \cdot (1_Y^\sharp \circ \varphi) = y^\circ \cdot e_Y^\circ \cdot \hat{T}\varphi \cdot m_X^\circ = e_1^\circ \cdot (Ty)^\circ \cdot \hat{T}\varphi \cdot m_X^\circ = e_1^\circ \cdot \hat{T}(y^\circ \cdot \varphi) \cdot m_X^\circ = y^\circ \cdot \varphi$$

for all $y \in Y$. Since for any V-relation $r : Z \nrightarrow Y$, one has $y^\circ \cdot r(z, \star) = r(z, y)$, one can conclude that $\varphi \circ 1_X^\sharp = \varphi$ and $1_Y^\sharp \circ \varphi = \varphi$. $\qquad \square$

Proposition 3.5. *Let V be a quantale and $\hat{\mathbb{T}}$ an associative lax extension to V-Rel of a monad $\mathbb{T}$ on* Set.

1. *For a set X, the product $1^X = \prod_{x \in X} 1_x$ in $(\mathbb{T}, V)$-URel (with $1_x = 1$ for all $x \in X$) exists, and can be chosen as $1^X = X$ with projections $\pi_x = x^\sharp : X \nrightarrow 1$ $(x \in X)$.*

2. *The contravariant hom-functor*

$$(\mathbb{T}, V)\text{-URel}(-, 1) : (\mathbb{T}, V)\text{-URel}^{\mathrm{op}} \longrightarrow \text{Set}$$

has $(-)^\sharp : \text{Set} \longrightarrow (\mathbb{T}, V)\text{-URel}^{\mathrm{op}}$ as left adjoint. The unit and counit of the associated adjunction are given by the Yoneda maps

$$\mathbf{y}_X = (1_X^\sharp)^\flat : X \longrightarrow (\mathbb{T}, V)\text{-URel}(X, 1), \quad x \longmapsto x^\sharp$$

and the evaluation V-relations

$$\varepsilon_X : X \nrightarrow (\mathbb{T}, V)\text{-URel}(X, 1), \quad \varepsilon_X(\chi, \psi) = \psi(\chi, \star)$$

respectively.

Proof.

1. For a family $(\phi_x : Y \nrightarrow 1)_{x \in X}$ of unitary $(\mathbb{T}, V)$-relations, one can define a $(\mathbb{T}, V)$-relation $\phi : Y \nrightarrow X$ by setting $\phi(y, x) = \phi_x(y, \star)$ for all $y \in TY$. Since $x^\circ \cdot \phi = \phi_x$ is unitary for all $x \in X$, then so is ϕ by Lemma 3.4. Unicity of the connecting morphism $\phi : Y \nrightarrow X$ follows from its definition.

2. Since $1^X = \prod_{x \in X} 1_x$ exists in $(\mathbb{T}, V)$-URel for all sets X, the representable functor $H = (\mathbb{T}, V)\text{-URel}^{\mathrm{op}}(1, -) : (\mathbb{T}, V)\text{-URel}^{\mathrm{op}} \to \text{Set}$ has a left adjoint $F = 1^{(-)}$ that sends a set X to its product $1^X = X$, and a map $f : X \to Y$ to the unitary $(\mathbb{T}, V)$-relation $f^\sharp : Y \nrightarrow X$. Similarly, it follows that $\mathbf{y}$ is the unit of the adjunction and ε its counit.

$\qquad \square$

3.6 The $(\mathbb{T}, V)$-powerset monad

Let V be a quantale, $\hat{\mathbb{T}}$ an associative lax extension to V-Rel of a monad $\mathbb{T}$ on Set. The adjunction described in Proposition 3.5 induces a monad

$$\not\geq = (\Pi, \mathbf{m}, \mathbf{y})$$

on Set, the $(\mathbb{T}, V)$-*powerset monad* (or, more generically, the *discrete presheaf monad*) associated to $\hat{\mathbb{T}}$, where

$$
\begin{aligned}
\Pi X &= (\mathbb{T}, V)\text{-}\mathsf{URel}(X, 1), & \Pi f(\psi) &= \psi \circ f^{\sharp}, \\
\mathbf{m}_X(\Psi) &= \Psi \circ \varepsilon_X, & \mathbf{y}_X(x) &= x^{\sharp},
\end{aligned}
$$

for all $x \in X$, $f : X \to Y$, and unitary $(\mathbb{T}, V)$-relations $\psi : X \nrightarrow 1$, $\Psi : \Pi X \nrightarrow 1$. Let us emphasize that $\not\geq$ depends both on a choice of a quantale V, as well as that of an associative lax extension $\hat{\mathbb{T}}$:

$$\not\geq = \not\geq(\mathbb{T}, \hat{\mathbb{T}}) = \not\geq(\mathbb{T}, V, \hat{\mathbb{T}}).$$

Example 3.6. Consider a quantale V, and the identity monad $\mathbb{T} = \mathbb{I} = (1_{\mathsf{Set}}, 1, 1)$ on Set extended to the identity monad $\hat{\mathbb{I}} = (1_{V\text{-}\mathsf{Rel}}, 1, 1)$ on V-Rel. Unitary $(\mathbb{T}, V)$-relations are simply V-relations, so, denoting by $\mathbb{P}_V = (P_V, \mu, \eta)$ the $(\mathbb{I}, V)$-powerset monad associated to $\hat{\mathbb{I}}$, we compute

$$
\begin{aligned}
P_V X &= V\text{-}\mathsf{Rel}(X, 1), & P_V f(r) &= r \cdot f^{\circ}, \\
\mu_X(R) &= R \cdot \mathrm{ev}_X, & \eta_X(x) &= x^{\circ},
\end{aligned}
$$

for all $x \in X$, $f : X \to Y$, and V-relations $r : X \nrightarrow 1$, $R : P_V X \nrightarrow 1$ (where $\mathrm{ev}_X : X \nrightarrow V\text{-}\mathsf{Rel}(X, 1)$ is given by $\mathrm{ev}_X(x, r) = r(x, \star)$). By further identifying V-relations $X \nrightarrow 1$ with maps $X \to V$, we obtain the V-powerset monad of 2.6. However, from now on we will only use the description given here, as it allows for more concise arguments[1].

3.7 The Kleisli category of $\mathbb{P}_V$

The Kleisli category of the V-powerset monad $\mathbb{P}_V$ has sets as objects, and maps $f : Y \to V\text{-}\mathsf{Rel}(X, 1)$ as morphisms from Y to Z. Kleisli composition

---

[1]In view of this example, $\mathbb{P}_{(\mathbb{T}, V)}$ is a more logical notation for the $(\mathbb{T}, V)$-powerset monad $\not\geq$; nevertheless, we favor the latter, simpler, notation whenever $\mathbb{T}$ is not the identity monad.

$f \circ g : Z \to V\text{-}\mathsf{Rel}(X, 1)$ of $g : Z \to V\text{-}\mathsf{Rel}(Y, 1)$ and $f : Y \to V\text{-}\mathsf{Rel}(X, 1)$ is

$$f \circ g(z) = \mu_X \cdot P_V f \cdot g(z) = g(z) \cdot f^\circ \cdot \mathrm{ev}_X.$$

The comparison functor $K : \mathsf{Set}_{\mathbb{P}_V} \to V\text{-}\mathsf{Rel}^{\mathrm{op}}$ sends a map $f : Y \to V\text{-}\mathsf{Rel}(X, 1)$ to the V-relation $f^\circ \cdot \mathrm{ev}_X : X \nrightarrow Y$, and has an inverse $(-)^\flat :$ $V\text{-}\mathsf{Rel}^{\mathrm{op}} \to \mathsf{Set}_{\mathbb{P}_V}$ that sends a V-relation $r : X \nrightarrow Y$ to its mate $r^\flat : Y \to V\text{-}\mathsf{Rel}(X, 1)$ (see 3.5). Indeed, one verifies that $(-)^\flat$ is a functor:

$$r^\flat \circ s^\flat = (s \cdot r)^\flat,$$

and that it is an inverse to K:

$$(r^\flat)^\circ \cdot \mathrm{ev}_X = r \qquad \text{and} \qquad (f^\circ \cdot \mathrm{ev}_X)^\flat = f,$$

for all V-relations $r : X \nrightarrow Y$, $s : Y \nrightarrow Z$ and maps $f : Y \to V\text{-}\mathsf{Rel}(X, 1)$. When hom-sets of $\mathsf{Set}_{\mathbb{P}_V}$ are equipped with the pointwise order:

$$f \leq g \iff \forall x \in X \ (f(x) \leq g(x))$$

(for $f, g : Y \to (V, 1)\text{-}\mathsf{Rel}$), the isomorphism $V\text{-}\mathsf{Rel}^{\mathrm{op}} \cong \mathsf{Set}_{\mathbb{P}_V}$ is an order-isomorphism.

4 The enriched Kleisli extension

In this section, we show that the monad $\not\succ$, constructed in the previous section via a quantale V and an associative lax extension $\hat{\mathbb{T}}$ to $V\text{-}\mathsf{Rel}$ of a monad $\mathbb{T}$, also comes with an associative lax extension $\check{\not\succ}$ to $V\text{-}\mathsf{Rel}$. Such a lax extension stems from a general construction that depends in particular on the existence of a monad morphism from $\mathbb{P}_V$ into the underlying monad.

4.1 The Eilenberg–Moore category of $\mathbb{P}_V$

The V-powerset monad $\mathbb{P}_V = (P_V, \mu, \eta)$ is induced from the adjunction $(-)^\circ \dashv (V\text{-}\mathsf{Rel}(-, 1)) : V\text{-}\mathsf{Rel}^{\mathrm{op}} \to \mathsf{Set}$. By [9] and since $V\text{-}\mathsf{Rel}^{\mathrm{op}} \cong V^{\mathrm{op}}\text{-}\mathsf{Rel}$, the Eilenberg–Moore category of $\mathbb{P}_V$ is isomorphic to the category of left V^{op}-actions in the category Sup of complete lattices and sup-maps; that is, $\mathsf{Set}^{\mathbb{P}_V}$ is the category of right V-actions in Sup. We now describe this correspondence in our setting.

A right V-action $(-) * (-) : X \times V \to X$ defines a $\mathbb{P}_V$-algebra structure $a : P_V X \to X$ via

$$a(\phi) := \bigvee_{x \in X} x * \phi(x, \star)$$

for all $\phi \in P_V X$.

For $v \in V$, we define the V-relation $\eta_X(x) \otimes v : X \nrightarrow 1$ by $(\eta_X(x) \otimes v)(y) := \eta_X(x)(y, \star) \otimes v$ (that is, $\eta_X(x) \otimes v$ sends $(x, \star)$ to v, and all other pairs $(y, \star)$ to $\bot$). A $\mathbb{P}_V$-algebra (X, a) yields a right V-action on X in Sup via

$$x * v := a(\eta_X(x) \otimes v),$$

(for all $x \in X$ and $v \in V$), so that $(-) * (-) : X \times V \to X$ is a map that preserves suprema in each variable and satisfies

$$x * (v \otimes u) = (x * v) * u \qquad \text{and} \qquad x * k = x$$

for all $u, v \in V$, $x \in X$. In particular, for $\phi \in P_V X$ and $v \in V$, the action induced on the free $\mathbb{P}_V$-algebra $(P_V X, \mu_X)$ is given by $\phi * v = \mu_X(d_{P_V X}(\phi) \otimes v) = (\eta_{P_V X}(\phi) \otimes v) \cdot \mathrm{ev}_X$, and one observes

$$\phi * v = \phi \otimes v.$$

From now on, we denote the category $\mathsf{Set}^{\mathbb{P}_V}$ of right V-actions in Sup by

$$\mathsf{Sup}^V$$

as it is isomorphic to the functor category of V into Sup (with $V = (V, \otimes, k)$ considered as a one-object category).

4.2 From $\mathbb{P}_V$ to $\mathbb{T}$

Certain monads $\mathbb{T}$ on Set equipped with a monad morphism $\tau : \mathbb{P}_V \to \mathbb{T}$ will allow for a particular lax extension to V-Rel. With is in mind, we mention the following equivalences.

Proposition 4.1. *For a monad $\mathbb{T} = (T, m, e)$ on Set, there is a one-to-one correspondence between:*

(i) *monad morphisms $\tau : \mathbb{P}_V \to \mathbb{T}$;*

(ii) *extensions E of the left adjoint $F_{\mathbb{T}} : \mathsf{Set} \to \mathsf{Set}_{\mathbb{T}}$ along the functor* $(-)^{\circ} : \mathsf{Set} \to V\text{-}\mathsf{Rel}^{\mathrm{op}}$*:*

$$
\begin{array}{ccc}
V\text{-}\mathsf{Rel}^{\mathrm{op}} & \xrightarrow{\ E\ } & \mathsf{Set}_{\mathbb{T}} \\
{\scriptstyle (-)^{\circ}}\uparrow & \nearrow{\scriptstyle F_{\mathbb{T}}} & \\
& \mathsf{Set} &
\end{array}
$$

(iii) *liftings L of the right adjoint $G_{\mathbb{T}} : \mathsf{Set}_{\mathbb{T}} \to \mathsf{Set}$ along the forgetful functor $\mathsf{Sup}^{V} \to \mathsf{Set}$:*

$$
\begin{array}{ccc}
\mathsf{Set}_{\mathbb{T}} & \xrightarrow{\ L\ } & \mathsf{Sup}^{V} \\
& \searrow{\scriptstyle G_{\mathbb{T}}} \quad \downarrow & \\
& \mathsf{Set} &
\end{array}
$$

iv. *V-actions in Sup on TX such that $Tf : TX \to TY$ and $m_X : TTX \to TX$ are equivariant sup-maps for all maps $f : X \to Y$ and sets X.*

Proof. The equivalence between (i) and (ii) is a particular instance of the one-to-one correspondence between monad morphisms and functors between Kleisli categories of the respective monads. The equivalence between (i) and (iii) follows from the explicit description of $G_{\mathbb{T}}$ on $\mathsf{Set}_{\mathbb{T}}$-morphisms $f : X \to TY$ as $G_{\mathbb{T}}f = m_Y \cdot Tf$. (iv) is just a restatement of (iii) in which the objects and morphisms of the Eilenberg–Moore category of $\mathbb{P}_V$ are described explicitly. $\qquad\square$

4.3 V-power-enriched monads

As mentioned in [9] and [11], there is a monotone functor

$$
\mathsf{Sup}^{V} \to V\text{-}\mathsf{Cat}
$$

that sends a right V-action (X, a) to the V-category $(X, (a^{\dashv})^{\circ} \cdot \mathrm{ev}_X) = (X, K(a^{\dashv}))$ (that is, $K : \mathsf{Set}_{\mathbb{P}_V} \to V\text{-}\mathsf{Rel}^{\mathrm{op}}$ denotes the left adjoint of $(-)^{\flat}$, see 3.7), where $a^{\dashv} : X \to P_V X$ is the right adjoint retract of a:

$$
1_{P_V X} \le a^{\dashv} \cdot a \qquad \text{and} \qquad a \cdot a^{\dashv} = 1_X,
$$

and induces the V-relation $K(a^{\dashv}) : X \nrightarrow X$:

$$K(a^{\dashv})(x,y) = \bigvee \{\phi(x,\star) \mid \phi \in V\text{-}\mathsf{Rel}(X,1) : a(\phi) = y\}$$

for all $x,y \in X$. Writing $a(\phi) = \bigvee_{x \in X} x * \phi(x,\star)$ (see 4.1), for $v \in V$, one observes

$$x * v \leq y \iff v \leq K(a^{\dashv})(x,y).$$

Setting $v = K(a^{\dashv})(x,y)$, one has $x * K(a^{\dashv})(x,y) \leq y$, so that with $k \leq K(a^{\dashv})(x,x)$, one obtains

$$\bigvee_{x \in X} x * K(a^{\dashv})(x,y) = y$$

for all $x,y \in X$. Moreover, a free $\mathbb{P}_V$-algebra $(P_V X, \mu_X)$ has an underlying V-category with internal hom $K(\mu_X^{\dashv}) : P_V X \nrightarrow P_V X$ given by

$$
\begin{aligned}
K(\mu_X^{\dashv})(r,s) &= \bigvee \{R(r,\star) \mid R \in V\text{-}\mathsf{Rel}(P_V X, 1) : \\
&\qquad\qquad t(-,\star) \otimes R(t,\star) \leq s(-,\star) \text{ for all } t \in P_V X\} \\
&= \bigwedge \{(r(x,\star) \multimap s(x,\star)) \mid x \in X\} = (s \bullet\!\!- r)
\end{aligned}
$$

for all $r,s \in V\text{-}\mathsf{Rel}(X,1)$ (see 2.3).

By Proposition 4.1, a morphism $\tau : \mathbb{P}_V \to \mathbb{T}$ of monads on Set equips the underlying set TX of a free $\mathbb{T}$-algebra with the internal hom

$$(y \bullet\!\!- \chi) := K(m_X \cdot \tau_{TX})^{\dashv}(\chi, y)$$

for all $\chi \in TX$, $y \in TY$. The sets TX are therefore V-categories, and are equipped with an order (see 2.7) that is inherited pointwise by the hom-sets $\mathsf{Set}(X, TY)$:

$$f \leq g \iff \forall x \in X \ (f(x) \leq g(x))$$

for all $f, g : X \to TY$.

A *V-power-enriched monad* is a pair $(\mathbb{T}, \tau)$ with $\mathbb{T}$ a monad on Set and $\tau : \mathbb{P}_V \to \mathbb{T}$ a monad morphism such that

$$f \leq g \implies Lf \leq Lg,$$

for all $f, g : X \to TY$, where $L = m \cdot T(-) : \mathsf{Set}_{\mathbb{T}} \to \mathsf{Sup}^V$ is the lifting of $G_{\mathbb{T}}$ described in Proposition 4.1. If $(\mathbb{T}, \tau)$ is V-power-enriched, then, with

the mentioned order, $\mathsf{Set}_{\mathbb{T}}$ becomes an ordered category, and the functors $E :$ $V\text{-}\mathsf{Rel} \to \mathsf{Set}_{\mathbb{T}}$ and $L : \mathsf{Set}_{\mathbb{T}} \to \mathsf{Sup}^V$ of Proposition 4.1 become monotone.

We denote by $\mathsf{Mnd}_{\mathsf{Set}}$ the category of monads on Set with their morphisms. A *morphism of V-power-enriched monads* $\alpha : (\mathbb{S}, \sigma) \to (\mathbb{T}, \tau)$ is a morphism in the comma category $(\mathbb{P}_V \downarrow \mathsf{Mnd}_{\mathsf{Set}})$, that is, the diagram

$$
\begin{array}{ccc}
 & \mathbb{P}_V & \\
 {}^{\sigma}\swarrow & & \searrow{}^{\tau} \\
\mathbb{S} & \xrightarrow{\quad\alpha\quad} & \mathbb{T}
\end{array}
$$

must commute in $\mathsf{Mnd}_{\mathsf{Set}}$.

4.4 Kleisli extensions of V-power-enriched monads

Let $(\mathbb{T}, \tau)$ be a V-power-enriched monad. By composing the functors $(-)^\flat :$ $V\text{-}\mathsf{Rel}^{\mathrm{op}} \to \mathsf{Set}_{\mathbb{P}_V}$, $E = \mathsf{Set}_\tau : \mathsf{Set}_{\mathbb{P}_V} \to \mathsf{Set}_{\mathbb{T}}$ and $L : \mathsf{Set}_{\mathbb{T}} \to \mathsf{Sup}^V$ of 3.7 and Proposition 4.1, one obtains a functor

$$
(-)^\tau : V\text{-}\mathsf{Rel}^{\mathrm{op}} \xrightarrow{\ (-)^\flat\ } \mathsf{Set}_{\mathbb{P}_V} \xrightarrow{\ \mathsf{Set}_\tau\ } \mathsf{Set}_{\mathbb{T}} \xrightarrow{\ L\ } \mathsf{Sup}^V
$$

that sends a set X to TX, and a V-relation $r : X \nrightarrow Y$ to the map $r^\tau : TY \to TX$, with

$$
r^\tau := m_X \cdot T(\tau_X \cdot r^\flat).
$$

The *Kleisli extension* $\check{T}$ of $\mathbb{T}$ to $V\text{-}\mathsf{Rel}$ (with respect to τ) is defined by the functions $\check{T} = \check{T}_{X,Y} : V\text{-}\mathsf{Rel}(X, Y) \to V\text{-}\mathsf{Rel}(TX, TY)$ (indexed by sets X and Y), with

$$
\check{T}r(\mathfrak{x}, \mathfrak{y}) = (r^\tau(\mathfrak{y}) \leftarrowtail \mathfrak{x})
$$

for all V-relations $r : X \nrightarrow Y$, and $\mathfrak{x} \in TX$, $\mathfrak{y} \in TY$.

4.5 Kleisli extensions are lax extensions

To prove that $\check{T}$ is indeed a lax extension of the Set-functor T to $V\text{-}\mathsf{Rel}$, it is convenient to express the former as a composite of lax functors. In view of this, we remark that $\check{T}r$ (for a relation $r : X \nrightarrow Y$) can be written equivalently as

$$
\check{T}r = (r^\tau)^* : TX \nrightarrow TY \qquad \text{or} \qquad (\check{T}r)^\flat = (m_X \cdot \tau_{TX})^{\dashv} \cdot r^\tau
$$

(where the functor $(-)^* : V\text{-Cat} \to V\text{-Mod}^{op}$ is defined in 2.9, and $(m_X \cdot \tau_{TX})^{\dashv}$ is right adjoint to $m_X \cdot \tau_{TX}$, see 4.3). The Kleisli extension is therefore an ordered functor

$$\check{T} : V\text{-Rel}^{op} \xrightarrow{(-)^\tau} \mathsf{Sup}^V \longrightarrow V\text{-Cat} \xrightarrow{(-)^*} V\text{-Mod}^{op}.$$

There is moreover a lax functor $V\text{-Mod}^{op} \to V\text{-Rel}^{op}$ that assigns to a module its underlying relation: composition of V-modules is composition of V-relations, identity V-modules are transitive and reflexive V-relations, and $1_X \le a$ for any V-category (X, a). Hence, the Kleisli extension $\check{T}^{op}$ can be decomposed as the top line of the commutative diagram

$$
\begin{array}{c}
V\text{-Rel}^{op} \xrightarrow{(-)^\flat} \mathsf{Set}_{\mathbb{P}_V} \xrightarrow{\mathsf{Set}_\tau} \mathsf{Set}_{\mathbb{T}} \xrightarrow{L} \mathsf{Sup}^V \to V\text{-Cat} \xrightarrow{(-)^*} V\text{-Mod}^{op} \to V\text{-Rel}^{op} \\[4pt]
\quad\ \nwarrow {\scriptstyle (-)^\circ} \ \uparrow {\scriptstyle F_{\mathbb{P}_V}} \quad {\scriptstyle F_{\mathbb{T}}} \nearrow \qquad {\scriptstyle G_{\mathbb{T}}} \nearrow \qquad\quad \downarrow \qquad\quad \swarrow \\[4pt]
\mathsf{Set} \xrightarrow{\qquad T \qquad} \mathsf{Set}
\end{array}
$$

(1)

in which all arrows except $V\text{-Mod}^{op} \to V\text{-Rel}^{op}$ are functors, and the latter is a lax functor that fails only to preserve identities.

Proposition 4.2. *Given a V-power-enriched monad $(\mathbb{T}, \tau)$, the Kleisli extension $\check{T}$ of T to V-Rel yields a lax extension $\check{\mathbb{T}} = (\check{T}, m, e)$ of $\mathbb{T} = (T, m, e)$ to V-Rel. Moreover, $\check{T}$ preserves composition of V-relations.*

Proof. The fact that $\check{T} : \mathsf{Rel} \to \mathsf{Rel}$ is monotone and preserves composition of V-relations follows from its decomposition as lax functors preserving composition in the first line of diagram (1). The lax extension condition $(Tf)^\circ \le \check{T}(f)^\circ$ can be deduced from the diagram

$$
\begin{array}{c}
V\text{-Rel}^{op} \xrightarrow{(-)^\tau} V\text{-Cat} \xrightarrow{(-)^*} V\text{-Mod}^{op} \longrightarrow V\text{-Rel}^{op} \\[4pt]
{\scriptstyle (-)^\circ} \uparrow \qquad\qquad\qquad \downarrow \qquad\qquad {\scriptstyle \le} \qquad\qquad \downarrow {\scriptstyle 1_{V\text{-Rel}^{op}}} \\[4pt]
\mathsf{Set} \xrightarrow{\ T\ } \mathsf{Set} \xrightarrow{\qquad (-)^\circ \qquad} V\text{-Rel}^{op}
\end{array}
$$

in which the first line is $\check{T}^{op}$. The condition $\check{T}(h^\circ \cdot r) = (Th)^\circ \cdot \hat{T}r$ for all V-relations $r : X \nrightarrow Y$ and maps $h : Z \to Y$ comes from

$$
\begin{aligned}
\check{T}(h^\circ \cdot r)(\chi, z) &= ((r^\tau \cdot (h^\circ)^\tau(z)) \multimap \chi) \\
&= ((r^\tau \cdot Th(z)) \multimap \chi) = (Th)^\circ \cdot \check{T}r(\chi, z)
\end{aligned}
$$

for all $\chi \in TX$, $z \in TZ$. To verify oplaxness of $e : 1_V\text{-Rel} \to \check{T}$, consider a V-relation $r : X \nrightarrow Y$, and $x \in X$, $y \in Y$. Since $\tau_X : P_V X \to TX$ is a $\mathbb{P}_V$-algebra morphism, one has

$$
\begin{aligned}
e_X(x) * r(x,y) &= \tau_X(\eta_X(x) \otimes r(x,y)) \\
&\leq \tau_X\big(\textstyle\bigvee_{x \in X} \eta_X(x) \otimes r(x,y)\big) = \tau_X \cdot r^\flat(y) = r^\tau \cdot e_Y(y),
\end{aligned}
$$

so that $r(x,y) \leq ((r^\tau \cdot e_Y(y)) \bullet\!\!- e_X(x)) = \hat{T}r(e_X(x), e_Y(y))$, as required.

Via the isomorphism $V\text{-Rel}^{\mathrm{op}} \cong \mathsf{Set}_{\mathbb{P}_V}$, naturality of $m^\circ : \check{T} \to \check{T}\check{T}$ is equivalent to $(m_X^\circ)^\flat \circ (\check{T}\check{T}r)^\flat = (\check{T}r)^\flat \circ (m_Y^\circ)^\flat$. As $(m_X^\circ)^\flat = d_{TX} \cdot m_X$ by commutativity of the left-most triangle in diagram (1), naturality of m° is equivalent to

$$
P_V m_X \cdot (m_{TX} \cdot \tau_{TTX})^{\dashv} \cdot (\check{T}r)^\tau = (m_X \cdot \tau_{TX})^{\dashv} \cdot r^\tau \cdot m_Y.
$$

Since

$$
P_V m_X \cdot (m_{TX} \cdot \tau_{TTX})^{\dashv} \leq (m_X \cdot \tau_{TX})^{\dashv} \cdot m_X,
$$

oplaxness follows from

$$
\begin{aligned}
m_X \cdot (\check{T}r)^\tau &= m_X \cdot m_{TX} \cdot T(\tau_{TX} \cdot (m_X \cdot \tau_{TX})^{\dashv} \cdot r^\tau) \\
&= m_X \cdot T(m_X \cdot \tau_{TX} \cdot (m_X \cdot \tau_{TX})^{\dashv} \cdot r^\tau) \\
&= m_X \cdot T(m_X \cdot T(\tau_X \cdot r^\flat)) = r^\tau \cdot m_Y.
\end{aligned}
$$

$\square$

4.6   Discrete presheaf monads of V-power-enriched monads

Let $(\mathbb{T}, \tau)$ be a V-power-enriched monad. The monotone maps

$$
\begin{aligned}
\mathrm{nbhd} = \mathrm{nbhd}_{X,Y} : V\text{-Rel}(TX, Y) &\to \mathsf{Set}(Y, TX) \\
r &\mapsto (m_X \cdot \tau_{TX}) \cdot r^\flat, \\
\mathrm{conv} = \mathrm{conv}_{Y,X} : \mathsf{Set}(Y, TX) &\to V\text{-Rel}(TX, Y) \\
f &\mapsto ((m_X \cdot \tau_{TX})^{\dashv} \cdot f)^\circ \cdot \mathrm{ev}_{TX},
\end{aligned}
$$

form an adjunction $\mathrm{nbhd} \dashv \mathrm{conv}$ for all sets X, Y, such that moreover $\mathrm{nbhd} \cdot \mathrm{conv} = 1_{\mathsf{Set}(Y,TX)}$. With respect to the Kleisli extension of $\mathbb{T}$, one observes that

$$
((m_X \cdot \tau_{TX})^{\dashv})^\circ \cdot \mathrm{ev}_{TX} = \check{T}1_X,
$$

so $\mathrm{conv}(f) =$, and $\mathrm{conv}(f)$ is a unitary $(\mathbb{T}, V)$-relation.

Lemma 4.3. *Let* $(\mathbb{T}, \tau)$ *be a* V*-power-enriched monad equipped with its Kleisli extension* $\check{T}$ *to* V*-*Rel*. If* $r : TX \nrightarrow Y$ *is a unitary* $(\mathbb{T}, V)$*-relation, then*

$$\mathrm{conv} \cdot \mathrm{nbhd}(r) = r.$$

Proof. Since r is unitary,

$$
\begin{aligned}
r^\flat \ &\leq (m_X \cdot \tau_{TX})^\dashv \cdot (m_X \cdot \tau_{TX}) \cdot r^\flat \\
&= \mu_{TX} \cdot P_V((m_X \cdot \tau_{TX})^\dashv \cdot m_X) \cdot \eta_{TX} \cdot \tau_{TX} \cdot r^\flat \\
&\leq \mu_{TX} \cdot P_V((m_X \cdot \tau_{TX})^\dashv \cdot m_X) \cdot (m_X \cdot \tau_{TX})^\dashv \cdot \tau_{TX} \cdot r^\flat \\
&= \mu_{TX} \cdot P_V((m_X \cdot \tau_{TX})^\dashv) \cdot (e_Y^\circ \cdot \check{T}r \cdot m_X^\circ)^\flat \\
&= (e_Y^\circ \cdot \check{T}r \cdot m_X^\circ \cdot \check{T}1_X)^\flat = r^\flat.
\end{aligned}
$$

This implies $r^\flat = (m_X \cdot \tau_{TX})^\dashv \cdot (m_X \cdot \tau_{TX}) \cdot r^\flat$, and thus $\mathrm{conv} \cdot \mathrm{nbhd}(r) = r$, as claimed. $\qquad\square$

Theorem 4.4. *Let* $(\mathbb{T}, \tau)$ *be a* V*-power-enriched monad. The Kleisli extension* $\check{\mathbb{T}}$ *to* V*-*Rel *of* $\mathbb{T}$ *is associative, and the maps* nbhd *and* conv *yield mutually inverse monotone functors*

$$\mathrm{nbhd} : (\mathbb{T}, V)\text{-}\mathsf{URel}^{\mathrm{op}} \longrightarrow \mathsf{Set}_{\mathbb{T}} \qquad \text{and} \qquad \mathrm{conv} : \mathsf{Set}_{\mathbb{T}} \longrightarrow (\mathbb{T}, V)\text{-}\mathsf{URel}^{\mathrm{op}}$$

that commute with the left adjoint functors $(-)^\sharp : \mathsf{Set} \to (\mathbb{T}, V)\text{-}\mathsf{URel}^{\mathrm{op}}$ *and* $F_{\mathbb{T}} : \mathsf{Set} \to \mathsf{Set}_{\mathbb{T}}$.

Proof. By Lemma 4.3 and the preceding discussion, the maps nbhd and conv yield order-isomorphisms between the set of all unitary V-relations $r : TX \nrightarrow Y$ and $\mathsf{Set}(Y, TX)$ for all sets X, Y. To see that these determine functors, we first need to verify the identities

$$
\begin{aligned}
\mathrm{nbhd}(s \circ r) \ &= \mathrm{nbhd}(r) \circ \mathrm{nbhd}(s), & \mathrm{conv}(f) \circ \mathrm{conv}(g) \ &= \mathrm{conv}(g \circ f) \\
\mathrm{nbhd}(1_X^\sharp) \ &= e_X, & \mathrm{conv}(e_X) \ &= 1_X^\sharp
\end{aligned}
$$

for all unitary V-relations $r, s : TX \nrightarrow Y$, and maps $f, g : Y \to TX$. The fact that nbhd preserves composition can be verified by using that τ is a monad morphism, and that m is the multiplication of $\mathbb{T}$. Preservation of composition by conv then follows because nbhd and conv are mutual inverses. Preservation of units is immediately verified by using that $\mathrm{conv}(f) = f^\circ \cdot \check{T}1_{TX}$. Since Kleisli composition is associative, so is Kleisli

convolution of unitary $(\mathbb{T}, V)$-relations, and therefore the Kleisli extension $\check{T}$ is an associative lax extension by Proposition 3.3. Hence, one can indeed form the category $(\mathbb{T}, V)$-URel.

Commutativity of nbhd with $(-)^{\sharp}$ and $F_{\mathbb{T}}$, follows from the fact that for any map $f : X \to Y$,

$$
\begin{aligned}
m_X \cdot \tau_{TX} \cdot (f^{\sharp})^{\flat} &= m_X \cdot \tau_{TX} \cdot (e_Y^{\circ} \cdot \check{T}(f^{\circ}))^{\flat} \\
&= m_X \cdot \tau_{TX} \cdot \mu_Y \cdot P_V((\check{T}(f^{\circ}))^{\flat}) \cdot (e_X^{\circ})^{\flat} \\
&= m_X \cdot \tau_{TX} \cdot (\check{T}(f^{\circ}))^{\flat} \cdot e_X \\
&= m_X \cdot \tau_{TX} \cdot (m_Y \cdot \tau_{TY})^{\dashv} \cdot (f^{\circ})^{\tau} \cdot e_X \\
&= \tau_Y \cdot (f^{\circ})^{\flat} = e_Y \cdot f,
\end{aligned}
$$

by the definitions and the fact that $(g^{\circ})^{\flat} = \eta_Y \cdot g$ for any map $g : X \to Y$ (here, we continue to use the notation $\mathbb{P}_V = (P_V, \mu, \eta)$). Commutativity of conv with $(-)^{\sharp}$ and $F_{\mathbb{T}}$ is then immediate because conv is inverse to nbhd. $\qquad\square$

Corollary 4.5. *Let $(\mathbb{T}, \tau)$ be a V-power-enriched monad equipped with its Kleisli extension $\check{\mathbb{T}}$. The discrete presheaf monad associate to $\check{\mathbb{T}}$ is order-isomorphic to $\mathbb{T}$:*

$$
\not\succ(\mathbb{T}, \check{\mathbb{T}}) \cong \mathbb{T}.
$$

Proof. By Theorem 4.4, the isomorphism $(\mathbb{T}, V)$-$\mathsf{Rel}^{\mathrm{op}} \cong \mathsf{Set}_{\mathbb{T}}$ commutes with the left adjoint functors from Set, so the induced monads are isomorphic, with corresponding orders on the sets ΠX and TX. $\qquad\square$

4.7   Discrete presheaf monads are V-power-enriched

For an associative lax extension $\hat{\mathbb{T}}$ to V-Rel of a monad $\mathbb{T}$ on Set, there is a functor

$$
(-)_{\sharp} : V\text{-}\mathsf{Rel} \to (\mathbb{T}, V)\text{-}\mathsf{URel}
$$

that sends a V-relation $r : X \nrightarrow Y$ to the unitary $(\mathbb{T}, V)$-relation

$$
r_{\sharp} := e_Y^{\circ} \cdot \hat{T}r : X \nrightarrow Y.
$$

Proposition 4.6. *Consider the discrete presheaf monad $\not\succ$ on Set associated to an associative lax extension $\hat{\mathbb{T}}$ to V-Rel of a monad $\mathbb{T}$ on Set. There is*

a functor V-$\mathsf{Rel}^{\mathrm{op}} \to \mathsf{Set}_{\not\geq}$ and an isomorphism $Q : \mathsf{Set}_{\not\geq} \to (\mathbb{T}, V)$-$\mathsf{URel}^{\mathrm{op}}$ that make the following diagram commute

$$
\begin{array}{ccc}
 & V\text{-}\mathsf{Rel}^{\mathrm{op}} & \\
 \swarrow & & \searrow {\scriptstyle (-)_\sharp} \\
\mathsf{Set}_{\not\geq} & \xrightarrow{\quad Q \quad} & (\mathbb{T}, V)\text{-}\mathsf{URel}^{\mathrm{op}}
\end{array}
$$

under Set. *In particular, there is a monad morphism $\pi : \mathbb{P}_V \to {\not\geq}$ whose component at a set X sends $\varphi \in V$-$\mathsf{Rel}(X, 1)$ to the unitary $(\mathbb{T}, V)$-relation $e_1^\circ \cdot \hat{T}\varphi : X \rightarrowtail 1$.*

Proof. Since the monad Π comes from the adjunction of $(\mathbb{T}, V)$-$\mathsf{URel}^{\mathrm{op}}$ over Set, the functor $Q : \mathsf{Set}_{\not\geq} \to (\mathbb{T}, V)$-$\mathsf{URel}^{\mathrm{op}}$ is the fully faithful comparison functor from the Kleisli category. This functor is bijective on objects because the left adjoint $(-)^\sharp : \mathsf{Set} \to (\mathbb{T}, V)$-$\mathsf{URel}^{\mathrm{op}}$ is so. Hence, $QX = X$ for each set X, and Q sends a morphism $f : X \to \Pi Y$ in $\mathsf{Set}_{\not\geq}$ to the unitary $(\mathbb{T}, V)$-relation $f^\sharp \circ \varepsilon_Y : Y \rightarrowtail X$. The inverse of Q sends a unitary $(\mathbb{T}, V)$-relation $\varphi : Y \rightarrowtail X$ to $\varphi^\flat : X \to \Pi Y$ (see 3.5).

The functor V-$\mathsf{Rel}^{\mathrm{op}} \to \mathsf{Set}_{\not\geq}$ is the composite of $(-)_\sharp$ with the inverse of Q. Explicitly, a V-relation $r : X \rightarrowtail Y$ is sent to the Kleisli morphism $(e_Y^\circ \cdot \hat{T}r)^\flat : Y \to \Pi Y$. The last statement follows from the one-to-one correspondence between monad morphisms and functors between Kleisli categories. $\qquad\square$

Theorem 4.7. *Let $\hat{\mathbb{T}}$ be an associative lax extension to V-Rel of a monad $\mathbb{T}$ on Set. The monad morphism $\pi : \mathbb{P}_V \to {\not\geq}$ makes ${\not\geq}$ into a V-power-enriched monad, and the pointwise order induced on the hom-sets makes $Q : \mathsf{Set}_{\not\geq} \to (\mathbb{T}, V)$-$\mathsf{URel}^{\mathrm{op}}$ into an order-isomorphism*

$$\mathsf{Set}_{\not\geq} \cong (\mathbb{T}, V)\text{-}\mathsf{URel}^{\mathrm{op}}$$

(*see Proposition* 4.6).

Proof. Consider the natural transformation $\pi : \mathbb{P}_V \to {\not\geq}$ of Proposition 4.6, and for $\varphi, \psi \in \Pi X$, define $\chi_{\{\varphi, \psi\}} : TX \rightarrowtail 1$ to be the characteristic function of $\{\phi, \psi\}$ (so that $\chi_{\{\varphi, \psi\}}(\chi, \star)$ takes value $k \in V$ if $\chi \in \{\varphi, \psi\}$ and $\perp$

otherwise). Using that $\chi_{\{\varphi,\psi\}} \cdot \varepsilon_X = \varphi \vee \psi$ (in the pointwise order of $\Pi X = (\mathbb{T}, V)\text{-}\mathsf{URel}(X, 1)$), we compute

$$\mathbf{m}_X \cdot \pi_{TX}(\chi_{\{\varphi,\psi\}}) = e_1^\circ \cdot \hat{T}(\varphi \vee \psi) \cdot m_X^\circ,$$

so in particular $\varphi \vee \psi \leq \mathbf{m}_X \cdot \pi_{TX}(\chi_{\{\varphi,\psi\}})$. Hence, $\varphi \leq \psi$ if and only if $\mathbf{m}_X \cdot \pi_{TX}(\chi_{\{\varphi,\psi\}}) = \psi$. Since the order $\prec$ induced on ΠX by τ is given by

$$\varphi \prec \psi \iff \mathbf{m}_X \cdot \pi_{TX}(\chi_{\{\varphi,\psi\}}) = \psi,$$

the relation $\prec$ describes the pointwise order of ΠX. The condition for $\not\succ$ being V-power-enriched is equivalent to the requirement that $\mathsf{Set}_{\not\succ}$ form an ordered category with respect to the order induced by π; but this follows immediately from the fact that $V\text{-}\mathsf{Rel}$ is ordered. $\qquad\square$

Remark 4.8. The previous Theorem shows that the order induced by $\pi : \mathbb{P}_V \to \Pi$ on $\Pi X = (\mathbb{T}, V)\text{-}\mathsf{URel}(X, 1)$ is the pointwise order of V-relations. With respect to this order, ΠX is then a complete lattice (whose infimum operation can easily be checked to be the infimum in $V\text{-}\mathsf{Rel}$). Moreover, the right V-action on ΠX is given by

$$\varphi * v = \mathbf{m}_X \cdot \pi_{TX}(\eta_{\Pi X}(\varphi) \otimes v) = e_1^\circ \cdot \hat{T}(\varphi \otimes v) \cdot m_X^\circ.$$

for all $\varphi \in \Pi X$ and $v \in V$. The internal hom can then be obtained by noticing that for $\psi \in \Pi X$,

$$\varphi * v \leq \psi \iff e_1^\circ \cdot \hat{T}(\varphi \otimes v) \cdot m_X^\circ \leq \psi \iff \varphi \otimes v \leq \psi;$$

hence, proceeding as in 4.3, we obtain

$$K((\mathbf{m}_X \cdot \pi_{TX})^{\dashv})(\varphi, \psi) \ = \psi \bullet\!\!- \varphi$$

for all $\varphi, \psi \in V\text{-}\mathsf{Rel}(TX, 1)$. In other words, the internal hom of the V-category $(\mathbb{T}, V)\text{-}\mathsf{URel}(X, 1)$ is obtained by restriction of the internal hom of $V\text{-}\mathsf{Rel}(TX, 1)$; this also justifies *a posteriori* the notation used for the internal hom of TX defined in 4.3.

4.8 The Yoneda embedding

In our context, the Yoneda lemma takes on the following form.

Lemma 4.9. *Let $\succnsim$ be the discrete presheaf monad of an associative lax extension $\hat{\mathbb{T}}$ of $\mathbb{T}$ to V-Rel. Then*

$$(\psi \bullet\!\!-\, x^\sharp) = \psi(x, \star)$$

for all $\psi \in \Pi X$ and $x \in X$.

Proof. Remark 4.8 with 4.3 yields

$$(\psi \bullet\!\!-\, \varphi) = \bigwedge\{\varphi(y, \star) -\!\!\bullet \psi(y, \star) \mid y \in X\}$$

for all $\psi \in \Pi X$. Setting $\varphi = x^\sharp$ (for $x \in X$), and observing that $k \leq x^\circ \cdot \hat{T}1_X(x, \star) = x^\sharp(x, \star)$ we obtain

$$(\psi \bullet\!\!-\, x^\sharp) \leq (x^\sharp(x, \star) -\!\!\bullet \psi(x, \star)) \leq (k -\!\!\bullet \psi(x, \star)) = \psi(x, \star).$$

Moreover, ψ is unitary, so $\hat{T}1_X(y, x) \otimes \psi(x, \star) \leq \psi(y, \star)$ for all $y \in X$, that is,

$$\psi(x, \star) \leq (\hat{T}1_X(y, x) -\!\!\bullet \psi(y, \star)),$$

for all $y \in X$, and therefore $\psi(x, \star) \leq (\psi \bullet\!\!-\, x^\sharp)$. $\qquad\qquad\square$

The Yoneda embedding then follows.

Proposition 4.10. *Let $\succnsim$ be the discrete presheaf monad of an associative lax extension $\hat{\mathbb{T}}$ of $\mathbb{T}$ to V-Rel. Then*

$$\hat{T}r(\chi, y) = \check{\Pi}r(\chi^\sharp, y^\sharp)$$

for all V-relations $r : X \nrightarrow Y$, and $\chi \in TX$, $y \in TY$.

Proof. A direct computation shows that $r^\pi(\psi) = \psi \cdot \hat{T}r$ for all $\psi \in \Pi X$. We can then use Lemma 4.9 to write

$$\check{\Pi}r(\chi^\sharp, y^\sharp) = (r^\pi(y^\sharp) \bullet\!\!-\, \chi^\sharp) = r^\pi(y^\sharp)(\chi, \star) = y^\sharp \cdot \hat{T}r(\chi, \star) = \hat{T}r(\chi, y)$$

for all $\chi \in TX$, $y \in TY$. $\qquad\qquad\square$

5 Categories of lax extensions

In this section, we define a category V-LaxExt whose objects are monads on Set equipped with an associative lax extension to V-Rel. The comma category $(\mathbb{P}_V \downarrow \mathsf{Mnd}_{\mathsf{Set}})$ of monads on Set *under* the V-powerset monad $\mathbb{P}_V$ is then shown to be a full reflective subcategory of V-LaxExt, as mentioned in the Introduction.

5.1 The category of associative lax extensions

Given a quantale V, an object of the category[2]

$$V\text{-LaxExt}$$

is a pair $(\mathbb{T}, \hat{\mathbb{T}})$ consisting of a monad $\mathbb{T}$ on Set and of an associative lax extension $\hat{\mathbb{T}}$ of $\mathbb{T}$ to V-Rel. A morphism $\alpha : (\mathbb{S}, \hat{\mathbb{S}}) \to (\mathbb{T}, \hat{\mathbb{T}})$ is a monad morphism $\alpha : \mathbb{S} \to \mathbb{T}$ such that $(m \cdot T\alpha)^\circ \cdot \hat{T}1 : \hat{T} \nrightarrow \hat{T}\hat{S}$ is a natural transformation in V-Rel.

Remark 5.1. Since $m^\circ : \hat{T} \nrightarrow \hat{T}\hat{T}$ is a natural transformation, the identity $1_{\mathbb{T}} : (\mathbb{T}, \hat{\mathbb{T}}) \to (\mathbb{T}, \hat{\mathbb{T}})$ is a morphism of V-LaxExt. Moreover, morphisms of V-LaxExt compose. Indeed, consider monad morphisms $\alpha : \mathbb{S} \to \mathbb{T}$ and $\beta : \mathbb{R} \to \mathbb{S}$ such that $(m \cdot T\alpha)^\circ \cdot \hat{T}1 : \hat{T} \nrightarrow \hat{T}\hat{S}$ and $(n \cdot S\beta)^\circ \cdot \hat{S}1 : \hat{S} \nrightarrow \hat{S}\hat{R}$ are natural transformations in V-Rel. Then $\alpha \cdot \beta : \mathbb{R} \to \mathbb{T}$ is a monad morphism, and for a V-relation $r : X \nrightarrow Y$, one obtains that $(m \cdot T(\alpha \cdot \beta))^\circ \cdot \hat{T}1 : \hat{T} \nrightarrow \hat{T}\hat{R}$ is a natural transformation by using the hypotheses on α and β, together with the fact that $m_X^\circ \cdot \hat{T}1_X = \hat{T}\hat{T}1_X \cdot m_X^\circ \cdot \hat{T}1_X = \hat{T}1_{TX} \cdot m_X^\circ \cdot \hat{T}1_X$ and

---

[2]Strictly speaking, V-LaxExt is a metacategory, but we ignore such size issues here for questions of readability.

that $\hat{\mathbb{S}}$ and $\hat{\mathbb{T}}$ are associative lax extensions:

$$
\begin{aligned}
(\quad & m_Y \cdot T(\alpha_Y \cdot \beta_Y))^\circ \cdot \hat{T}1_Y \cdot \hat{T}r \\
&= (T\beta_Y)^\circ \cdot \hat{T}\hat{S}r \cdot (m_X \cdot T\alpha_X)^\circ \cdot \hat{T}1_X \\
&= (Td_{SRY})^\circ \cdot (TS\beta_Y)^\circ \cdot (Tn_Y)^\circ \cdot \hat{T}\hat{S}r \cdot (m_X \cdot T\alpha_X)^\circ \cdot \hat{T}1_X \\
&= (Td_{SRY})^\circ \cdot \hat{T}((S\beta_Y)^\circ \cdot n_Y^\circ \cdot \hat{S}r) \cdot (m_X \cdot T\alpha_X)^\circ \cdot \hat{T}1_X \\
&= (Td_{SRY})^\circ \cdot \hat{T}(\hat{S}\hat{R}r \cdot (S\beta_X)^\circ \cdot n_X^\circ \cdot \hat{S}1_X) \cdot (m_X \cdot T\alpha_X)^\circ \cdot \hat{T}1_X \\
&= (Td_{SRY})^\circ \cdot \hat{T}(\hat{S}\hat{R}r \cdot (S\beta_X)^\circ \cdot n_X^\circ) \cdot (m_X \cdot T\alpha_X)^\circ \cdot \hat{T}1_X \\
&= (Td_{SRY})^\circ \cdot \hat{T}(\hat{S}\hat{R}r \cdot (S\beta_X)^\circ \cdot n_X^\circ) \cdot \hat{T}(\alpha_X^\circ) \cdot m_X^\circ \cdot \hat{T}1_X \\
&= (Td_{SRY})^\circ \cdot \hat{T}(\hat{S}\hat{R}r \cdot (S\beta_X)^\circ \cdot n_X^\circ \cdot \alpha_X^\circ) \cdot m_X^\circ \cdot \hat{T}1_X \\
&= (Td_{SRY})^\circ \cdot \hat{T}(\hat{S}\hat{R}r \cdot \alpha_{RX}^\circ \cdot (T\beta_X)^\circ \cdot (T\alpha_X)^\circ \cdot m_X^\circ) \cdot m_X^\circ \cdot \hat{T}1_X \\
&= (Td_{SRY})^\circ \cdot \hat{T}(\hat{S}\hat{R}r \cdot \alpha_{RX}^\circ \cdot (T\beta_X)^\circ \cdot (T\alpha_X)^\circ \cdot m_X^\circ \cdot \hat{T}1_X) \cdot m_X^\circ \cdot \hat{T}1_X \\
&= (Td_{SRY})^\circ \cdot \hat{T}\hat{S}\hat{R}r \cdot \hat{T}(\alpha_{RX}^\circ) \cdot \hat{T}\hat{T}((\alpha \cdot \beta_X)^\circ) \cdot (Tm_X)^\circ \cdot m_X^\circ \cdot \hat{T}1_X \\
&= (Td_{SRY})^\circ \cdot \hat{T}\hat{S}\hat{R}r \cdot \hat{T}(\alpha_{RX}^\circ) \cdot m_{RX}^\circ \cdot \hat{T}((\alpha \cdot \beta_X)^\circ) \cdot m_X^\circ \cdot \hat{T}1_X \\
&= \hat{T}\hat{R}r \cdot (m_X \cdot T(\alpha_X \cdot \beta_X))^\circ \cdot \hat{T}1_X.
\end{aligned}
$$

5.2 The Kleisli extension as universal object

We now proceed to describe an adjunction

$$
F \dashv G : (\mathbb{P}_V \downarrow \mathsf{Mnd}_{\mathsf{Set}}) \to V\text{-}\mathsf{LaxExt}
$$

that presents the comma category of monads under $\mathbb{P}_V$ as a full reflective subcategory of the category of associative lax extensions to V-Rel (Theorem 5.7). In particular, the discrete presheaf monad of an associative lax extension to V-Rel appears as its free V-cocompletion (Proposition 5.4).

Lemma 5.2. *A morphism* $\alpha : (\mathbb{S}, \hat{\mathbb{S}}) \to (\mathbb{T}, \hat{\mathbb{T}})$ *in* V-LaxExt *induces a functor* $A : (\mathbb{S}, V)\text{-}\mathsf{URel} \to (\mathbb{T}, V)\text{-}\mathsf{URel}$ *that makes the following diagram commute:*

$$
\begin{array}{ccc}
(\mathbb{S}, V)\text{-}\mathsf{URel}^{\mathrm{op}} & \xrightarrow{\ A^{\mathrm{op}}\ } & (\mathbb{T}, V)\text{-}\mathsf{URel}^{\mathrm{op}} \\
& \overset{(-)^\sharp}{\nwarrow} \quad \overset{(-)^\sharp}{\nearrow} & \\
& \mathsf{Set} &
\end{array}
$$

Proof. For a $(\mathbb{S}, V)$-relation $r : SX \nrightarrow Y$, we define the following $(\mathbb{T}, V)$-relation:

$$
Ar := e_Y^\circ \cdot \hat{T}r \cdot (m_X \cdot T\alpha_X)^\circ \cdot \hat{T}1_X : TX \nrightarrow Y.
$$

To verify that A preserves Kleisli convolution, consider $(\mathbb{S}, V)$-relations $r : X \rightarrow Y$ and $s : Y \rightarrow Z$. We first observe that by naturality of $(m \cdot T\alpha)^\circ \cdot \hat{T}1$,

$$
\begin{aligned}
\hat{T}1_{SX} \cdot (T\alpha_X)^\circ \cdot m_X^\circ \cdot \hat{T}1_X &= \hat{T}1_{SX} \cdot (T\alpha_X)^\circ \cdot m_X^\circ \cdot \hat{T}1_X \cdot \hat{T}1_X \\
&= \hat{T}\hat{S}1_X \cdot (T\alpha_X)^\circ \cdot m_X^\circ \cdot \hat{T}1_X \\
&= (T\alpha_X)^\circ \cdot m_X^\circ \cdot \hat{T}1_X
\end{aligned}
$$

and recall that that $m_X^\circ \cdot \hat{T}1_X = \hat{T}1_{TX} \cdot m_X^\circ \cdot \hat{T}1_X$. These remarks allow us to establish the following chain of equalities:

$$
\begin{aligned}
&As \circ Ar \\
&= e_Z^\circ \cdot \hat{T}s \cdot (m_Y \cdot T\alpha_Y)^\circ \cdot \hat{T}(e_Y^\circ \cdot \hat{T}r \cdot (m_X \cdot T\alpha_X)^\circ \cdot \hat{T}1_X) \cdot m_X^\circ \\
&= e_Z^\circ \cdot \hat{T}s \cdot (m_Y \cdot T\alpha_Y)^\circ \cdot (Te_Y)^\circ \cdot \hat{T}\hat{T}r \cdot (TT\alpha_X)^\circ \cdot (Tm_X)^\circ \cdot \hat{T}\hat{T}1_X \cdot m_X^\circ \\
&= e_Z^\circ \cdot \hat{T}s \cdot (m_Y \cdot T\alpha_Y)^\circ \cdot (Te_Y)^\circ \cdot \hat{T}\hat{T}r \cdot (TT\alpha_X)^\circ \cdot (m_{TX})^\circ \cdot m_X^\circ \cdot \hat{T}1_X \\
&= e_Z^\circ \cdot \hat{T}s \cdot (m_Y \cdot T\alpha_Y)^\circ \cdot (Te_Y)^\circ \cdot m_Y^\circ \cdot \hat{T}r \cdot (T\alpha_X)^\circ \cdot m_X^\circ \cdot \hat{T}1_X \\
&= e_Z^\circ \cdot \hat{T}s \cdot (m_Y \cdot T\alpha_Y)^\circ \cdot \hat{T}r \cdot (T\alpha_X)^\circ \cdot m_X^\circ \cdot \hat{T}1_X \\
&= e_Z^\circ \cdot \hat{T}(s \cdot \hat{S}r) \cdot (m_{SX} \cdot T\alpha_{SX})^\circ \cdot \hat{T}1_{SX} \cdot (T\alpha_X)^\circ \cdot m_X^\circ \cdot \hat{T}1_X \\
&= e_Z^\circ \cdot \hat{T}(s \cdot \hat{S}r) \cdot (T\alpha_{SX})^\circ \cdot m_{SX}^\circ \cdot (T\alpha_X)^\circ \cdot m_X^\circ \cdot \hat{T}1_X \\
&= e_Z^\circ \cdot \hat{T}(s \cdot \hat{S}r) \cdot (T\alpha_{SX})^\circ \cdot (TT\alpha_X)^\circ \cdot (Tm_X)^\circ \cdot m_X^\circ \cdot \hat{T}1_X \\
&= e_Z^\circ \cdot \hat{T}(s \cdot \hat{S}r) \cdot (Tn_X)^\circ \cdot (T\alpha_X)^\circ \cdot \hat{T}1_{TX} \cdot m_X^\circ \cdot \hat{T}1_X \\
&= e_Z^\circ \cdot \hat{T}(s \circ r) \cdot \hat{T}(\alpha_X^\circ) \cdot m_X^\circ \cdot \hat{T}1_X \\
&= e_Z^\circ \cdot \hat{T}(s \circ r) \cdot (T\alpha_X)^\circ \cdot m_X^\circ \cdot \hat{T}1_X = A(s \circ r).
\end{aligned}
$$

Using that $\hat{T}(e_Y^\circ) \cdot m_Y^\circ = \hat{T}1_Y$, one then computes for a map $f : Y \to X$,

$$
A(d_Y^\circ \cdot \hat{S}(f^\circ)) = e_Y^\circ \cdot \hat{T}(d_Y^\circ) \cdot \hat{T}(\alpha_Y^\circ) \cdot m_Y^\circ \cdot \hat{T}(f^\circ) = e_Y \cdot \hat{T}(f^\circ),
$$

so that the stated diagram commutes and in particular $A(1_X^\sharp) = 1_X^\sharp$. Finally, since A preserves Kleisli convolution, one has for a unitary $(\mathbb{S}, V)$-relation r that

$$
Ar \circ 1_X^\sharp = Ar \circ A(1_X^\sharp) = A(r \circ 1_X^\sharp) = Ar
$$

and similarly $1_X^\sharp \circ Ar = Ar$, so A is well-defined. $\qquad\square$

Proposition 5.3. *Morphisms* $\alpha : (\mathbb{S}, \hat{\mathbb{S}}) \to (\mathbb{T}, \hat{\mathbb{T}})$ *in* V-LaxExt *functorially determine morphisms* $\Pi(\alpha) : \Pi(\mathbb{S}, \hat{\mathbb{S}}) \to \Pi(\mathbb{T}, \hat{\mathbb{T}})$ *in* $(\mathbb{P}_V \downarrow \mathsf{Mnd_{Set}})$ *via the components*

$$
\Pi(\alpha)_X(r) = e_1^\circ \cdot \hat{T}r \cdot (m_X \cdot T\alpha_X)^\circ \cdot \hat{T}1_X \in (\mathbb{T}, V)\text{-}\mathsf{URel}(X, 1)
$$

for all $r \in (\mathbb{S}, V)\text{-}\mathsf{URel}(X, 1)$. *That is, there is a functor*

$$F = \Pi : V\text{-}\mathsf{LaxExt} \to (\mathbb{P}_V \downarrow \mathsf{Mnd}_{\mathsf{Set}}).$$

Proof. With the isomorphism $(\mathbb{T}, V)\text{-}\mathsf{URel}^{\mathrm{op}} \cong \mathsf{Set}_\Pi$ (Proposition 4.6), the statement follows from Lemma 5.2 together with the classical one-to-one functorial correspondence between monad morphisms and morphisms of Kleisli categories that commute with the respective left adjoint functors. $\square$

Proposition 5.4. *Let* $\hat{\mathbb{T}}$ *be an associative lax extension of* $\mathbb{T}$ *to* V*-*Rel*, and* Π *its associated discrete presheaf monad. Then the natural transformation* $\mathbf{Y} : T \to \Pi$*, defined componentwise by*

$$\mathbf{Y}_X : TX \to \Pi X , \quad \chi \mapsto \chi^\sharp = \chi^\circ \cdot \hat{T} 1_X,$$

yields a morphism $\mathbf{Y} = \mathbf{Y}_{(\mathbb{T}, \hat{\mathbb{T}})} : (\mathbb{T}, \hat{\mathbb{T}}) \to (\breve{\not\geqslant}, \breve{\not\geqslant})$ *in* V*-*LaxExt.

Proof. The left adjoint functor $(-)^\sharp : \mathsf{Set} \to (\mathbb{T}, V)\text{-}\mathsf{URel}^{\mathrm{op}}$ extends to a functor

$$(-)^\sharp : \mathsf{Set}_\mathbb{T} \to (\mathbb{T}, V)\text{-}\mathsf{URel}^{\mathrm{op}}$$

sending $r : X \to TY$ to $r^\sharp := r^\circ \cdot \hat{T} 1_Y : Y \nrightarrow X$, so the diagram

$$\begin{array}{ccc}
\mathsf{Set}_\mathbb{T} & \xrightarrow{\ (-)^\sharp\ } & (\mathbb{T}, V)\text{-}\mathsf{URel}^{\mathrm{op}} \\
& {\scriptstyle F_\mathbb{T}} \nwarrow \quad \nearrow {\scriptstyle (-)^\sharp} & \\
& \mathsf{Set} &
\end{array}$$

commutes. Hence, $(-)^\sharp : \mathsf{Set}_\mathbb{T} \to (\mathbb{T}, V)\text{-}\mathsf{URel}^{\mathrm{op}}$ induces a monad morphism $\mathbf{Y} : \mathbb{T} \to \Pi$ whose component $\mathbf{Y}_X$ is the composite

$$TX \xrightarrow{\ \mathbf{y}_{TX}\ } (\mathbb{T}, V)\text{-}\mathsf{URel}(TX, 1) \xrightarrow{\ (-) \circ 1_{TX}^\sharp\ } (\mathbb{T}, V)\text{-}\mathsf{URel}(X, 1),$$

that is, $\mathbf{Y}_X(\chi) = \chi^\circ \cdot e_{TX}^\circ \cdot \hat{T} 1_{TX} \cdot \hat{T}\hat{T} 1_X \cdot m_X^\circ = \chi^\circ \cdot \hat{T} 1_X$.

We are left to verify that $(\mathbf{m} \cdot \Pi \mathbf{Y})^\circ \cdot \check{\Pi} 1 : \check{\Pi} \nrightarrow \check{\Pi}\hat{T}$ is natural in V-Rel. For this, we denote by $\pi : \mathbb{P}_V \to \Pi$ the $\mathbb{P}_V$-structure of Π, and remark that for $\chi \in \Pi X$, $\mathcal{Y} \in \Pi T Y$ and a V-relation $r : X \nrightarrow Y$,

$$\begin{aligned}
(\mathbf{m}_Y \cdot \Pi \mathbf{Y}_Y)^\circ \cdot \check{\Pi} 1_Y \cdot \check{\Pi} r(\chi, \mathcal{Y}) &= \check{\Pi} r(\chi, \mathbf{m}_Y \cdot \Pi \mathbf{Y}_Y(\mathcal{Y})) \\
&= ((r^\pi \cdot \mathbf{m}_Y \cdot \Pi \mathbf{Y}_Y(\mathcal{Y})) \leftarrow \chi)
\end{aligned}$$

and

$$\check{\Pi}\hat{T}r \quad \cdot (\mathbf{m}_X \cdot \Pi\mathbf{Y}_X)^\circ \cdot \check{\Pi}1_X(\mathcal{X}, \mathcal{Y})$$
$$= \bigvee\nolimits_{\mathcal{Z} \in \Pi TX}(\mathbf{m}_X \cdot \Pi\mathbf{Y}_X(\mathcal{Z}) \leftspoon \mathcal{X}) \otimes ((\hat{T}r)^\pi(\mathcal{Y}) \leftspoon \mathcal{Z}).$$

By setting $\mathcal{Z} = (\hat{T}r)^\pi(\mathcal{Y})$ in $\mathbf{m}_X \cdot \Pi\mathbf{Y}_X(\mathcal{Z})$, we have $k \le ((\hat{T}r)^\pi(\mathcal{Y}) \leftspoon \mathcal{Z})$ and compute for a unitary $(\mathbb{T}, V)$-relation $\Psi \in TY \rightarrowtail 1$,

$$\begin{aligned}
\mathbf{m}_X \cdot \Pi\mathbf{Y}_X \cdot (\hat{T}r)^\pi(\Psi) &= \mathbf{m}_X \cdot \Pi\mathbf{Y}_X \cdot \mathbf{m}_{TX} \cdot \Pi(\pi_{TX} \cdot (\hat{T}r)^\flat)(\Psi) \\
&= \mathbf{m}_X \cdot \Pi\mathbf{Y}_X \cdot \mathbf{m}_{TX} \cdot \Pi(e_{TX}^\circ \cdot \hat{T}\hat{T}r)^\flat)(\Psi) \\
&= \mathbf{m}_X \cdot \Pi\mathbf{Y}_X(\Psi \circ (((e_{TX}^\circ \cdot \hat{T}\hat{T}r)^\flat)^\sharp \circ \varepsilon_{TY})) \\
&= \mathbf{m}_X \cdot \Pi\mathbf{Y}_X(\Psi \cdot \hat{T}\hat{T}r) \\
&= \Psi \cdot m_X^\circ \cdot \hat{T}r = r^\pi \cdot \mathbf{m}_Y \cdot \Pi\mathbf{Y}_Y(\Psi)
\end{aligned}$$

by using twice that $\mathbf{Y}_Y^\sharp \circ \varepsilon_Y = \mathbf{Y}_Y^\circ \cdot \varepsilon_Y = \hat{T}1_Y$; thus, $(\mathbf{m}_Y \cdot \Pi\mathbf{Y}_Y)^\circ \cdot \check{\Pi}1_Y \cdot \check{\Pi}r \le \check{\Pi}\hat{T}r \cdot (\mathbf{m}_X \cdot \Pi\mathbf{Y}_X)^\circ \cdot \check{\Pi}1_X$. To show the other inequality, it suffices to show that $\mathbf{Y}^\circ : \check{\Pi} \to \hat{T}$ is a lax natural transformation in V-Rel; indeed, in this case, we would have

$$\begin{aligned}
\check{\Pi}\hat{T}r \cdot (\mathbf{m}_X \cdot \Pi\mathbf{Y}_X)^\circ \cdot \check{\Pi}1_X &= \check{\Pi}\hat{T}r \cdot \check{\Pi}(\mathbf{Y}_X^\circ) \cdot \mathbf{m}_X^\circ \cdot \check{\Pi}1_X \\
&\le \check{\Pi}(\mathbf{Y}_Y^\circ) \cdot \check{\Pi}\check{\Pi}r \cdot \mathbf{m}_X^\circ \cdot \check{\Pi}1_X \\
&= (\mathbf{m}_Y \cdot \Pi\mathbf{Y}_Y)^\circ \cdot \check{\Pi}1_Y \cdot \check{\Pi}r.
\end{aligned}$$

But lax naturality of $\mathbf{Y}^\circ$, or equivalently, the inequality

$$\hat{T}r(\mathcal{X}, y) \le (r^\pi(\mathbf{Y}_Y(y)) \leftspoon \mathbf{Y}_X(\mathcal{X}))$$

(for all V-relations $r : X \rightarrowtail Y$, and $\mathcal{X} \in TX$, $y \in TY$), follows from Proposition 4.10. $\qquad\square$

Lemma 5.5. *A morphism of V-power-enriched monads $\alpha : (\mathbb{S}, \sigma) \to (\mathbb{T}, \tau)$ satisfies*

$$\check{S}r \cdot \alpha_X^\circ \le \alpha_Y^\circ \cdot \check{T}r$$

for all V-relations $r : X \rightarrowtail Y$.

Proof. We first remark that

$$r^\tau \cdot \alpha_Y = \alpha_X \cdot r^\sigma$$

The pointwise version of the stated condition reads as

$$\check{S}r(x, y) \le \check{T}r(\alpha_X(x), \alpha_Y(y)),$$

that is,

$$(r^\sigma(y) \bullet\!\!-\, x) \le (\alpha_X \cdot r^\sigma(y) \bullet\!\!-\, \alpha_X(x)).$$

This condition follows from the fact that $\alpha_X : SX \to TX$ is a morphisms of the $\mathbb{P}_V$-algebras $(SX, n_X \cdot \sigma_{SX})$ and $(TX, m_X \cdot \tau_{TX})$, and this in turn follows from $\alpha : \mathbb{S} \to \mathbb{T}$ being a morphism in $(\mathbb{P}_V \!\downarrow\! \mathsf{Mnd}_{\mathsf{Set}})$. $\qquad\square$

Proposition 5.6. *Morphisms* $\alpha : (\mathbb{S}, \sigma) \to (\mathbb{T}, \tau)$ *in* $(\mathbb{P}_V \downarrow \mathsf{Mnd}_{\mathsf{Set}})$ *functorially determine morphisms* $\alpha : (\mathbb{S}, \check{\mathbb{S}}) \to (\mathbb{T}, \check{\mathbb{T}})$ *in* V-LaxExt *(where the* V*-power-enriched monads* $\mathbb{S}$ *and* $\mathbb{T}$ *are equipped with their respective Kleisli extensions). More precisely, there is a functor*

$$G : (\mathbb{P}_V \downarrow \mathsf{Mnd}_{\mathsf{Set}}) \to V\text{-}\mathsf{LaxExt}$$

that commutes with the respective forgetful functors to $\mathsf{Mnd}_{\mathsf{Set}}$.

Proof. We only need to verify that a monad morphism $\alpha : \mathbb{S} \to \mathbb{T}$ such that $\tau = \alpha \cdot \sigma$ yields a natural transformation $(m \cdot T\alpha)^\circ \cdot \check{T}1 : \check{T} \to \check{T}\check{S}$ in V-Rel. Here, we use the notations $\mathbb{T} = (T, m, e)$, $\mathbb{S} = (S, n, d)$, and $\mathbb{P}_V = (P_V, \mu, \eta)$.

First, we remark that for $x \in TX$, $y \in TSY$ and a V-relation $r : X \nrightarrow Y$,

$$
\begin{aligned}
(m_Y \cdot T\alpha_Y)^\circ \cdot \check{T}1_Y \cdot \check{T}r(x, y) &= \check{T}r(x, m_Y \cdot T\alpha_Y(y)) \\
&= ((r^\tau \cdot m_Y \cdot T\alpha_Y(y)) \bullet\!\!-\, x)
\end{aligned}
$$

and

$$
\begin{aligned}
\check{T}\check{S}r \cdot (m_X \cdot \, & T\alpha_X)^\circ \cdot \check{T}1_X(x, y) \\
&= \bigvee\nolimits_{z \in TSX}(m_X \cdot T\alpha_X(z) \bullet\!\!-\, x) \otimes ((\check{S}r)^\tau(y) \bullet\!\!-\, z).
\end{aligned}
$$

Since

$$
\begin{aligned}
m_X \cdot T\alpha_X \cdot (\check{S}r)^\tau &= m_X \cdot T\alpha_X \cdot m_{SX} \cdot T(\alpha_{SX} \cdot \sigma_{SX} \cdot (\check{S}r)^\flat) \\
&= m_X \cdot T(m_X \cdot T\alpha_X \cdot \alpha_{SX} \cdot \sigma_{SX} \cdot (n_X \cdot \sigma_{SX})^\dashv \cdot r^\sigma) \\
&= m_X \cdot T(\alpha_X \cdot n_X \cdot \sigma_{SX} \cdot (n_X \cdot \sigma_{SX})^\dashv \cdot r^\sigma) \\
&= m_X \cdot T(\alpha_X \cdot r^\sigma) \\
&= m_X \cdot T(r^\tau \cdot \alpha_Y) = r^\tau \cdot m_Y \cdot T\alpha_Y,
\end{aligned}
$$

we have $(m_Y \cdot T\alpha_Y)^\circ \cdot \check{T}1_Y \cdot \check{T}r \leq \check{T}\check{S}r \cdot (m_X \cdot T\alpha_X)^\circ \cdot \check{T}1_X$. Moreover, by Lemma 5.5, we also have the other inequality, and the sought naturality follows:

$$
\begin{aligned}
\check{T}\check{S}r \cdot (m_X \cdot T\alpha_X)^\circ \cdot \check{T}1_X
&= \check{T}\check{S}r \cdot \check{T}(\alpha_X^\circ) \cdot m_X^\circ \cdot \check{T}1_X \\
&\leq \check{T}(\alpha_Y^\circ) \cdot \check{T}\check{T}r \cdot m_X^\circ \cdot \check{T}1_X \\
&= (m_Y \cdot T\alpha_Y)^\circ \cdot \check{T}1_Y \cdot \check{T}r.
\end{aligned}
$$

$\square$

Theorem 5.7. *The functor* $G : (\mathbb{P}_V \downarrow \mathsf{Mnd}_{\mathsf{Set}}) \to V\text{-}\mathsf{LaxExt}$ *(see Proposition 5.6) is a full and faithful embedding, and has the functor* $F : V\text{-}\mathsf{LaxExt} \to (\mathbb{P}_V \downarrow \mathsf{Mnd}_{\mathsf{Set}})$ *as left adjoint.*

Proof. To prove the statement, we show that F is left adjoint to G and that the counit of this adjunction is an isomorphism. Hence, consider a $\mathbb{P}_V$-structures monad $(\mathbb{T}, \tau)$ (with its Kleisli extension $\hat{\mathbb{T}}$ to $V\text{-}\mathsf{Rel}$). By Proposition 4.6, Theorem 4.4 and the one-to-one correspondence between monad morphisms and functors between Kleisli categories, all the triangles but the lower-right one in the diagram
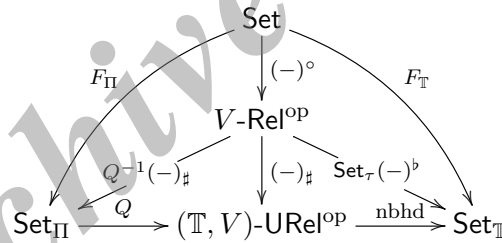
commute. For the last triangle, a computation similar to the last displayed equation in the proof of Theorem 4.4 shows that for any V-relation $r : X \nrightarrow Y$,

$$
\mathrm{nbhd}(r_\sharp) = m_Y \cdot \tau_{TY} \cdot (e_Y^\circ \cdot \check{T}r)^\flat = \tau_X \cdot r^\flat = \mathsf{Set}_\tau(r^\flat).
$$

One deduces that the following diagram commutes
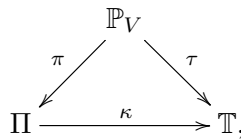
where κ is the monad morphism induced by the composition of nbhd and Q. Since these two functors are isomorphisms, the component $\kappa = \kappa_{(\mathbb{T},\tau)} : FG(\mathbb{T},\tau) \to (\mathbb{T},\tau)$ is itself an isomorphism in $(\mathbb{P}_V \downarrow \mathsf{Mnd_{Set}})$. In fact, one computes $\kappa = m \cdot \tau T$.

We are left to verify that our candidates for the unit and counit of the adjunction, $\mathbf{Y} : 1 \to GF$ and $\kappa : FG \to 1$ respectively, satisfy the triangular identities. For $y \in TX$, we use that $(TX, m_X \cdot \tau_{TX})$ is a $\mathbb{P}_V$-algebra and that $\bigvee_{y \in TX} y * (\chi \bullet\!\!- y) = \chi$ (see 4.3) to write

$$\begin{aligned} \kappa_X \cdot \mathbf{Y}_X(\chi) \ &= m_X \cdot \tau_{TX}(\chi^\circ \cdot \check{T}1_X) \\ &= \bigvee_{y \in TX} y * (\chi^\circ \cdot \check{T}1_X(y,\star)) = \bigvee_{y \in TX} y * (\chi \bullet\!\!- y) = \chi \end{aligned}$$

for all $\chi \in TX$, that is, $G\kappa \cdot \mathbf{Y}G = 1$. For the other identity, we note that for all $r \in (\mathbb{T},V)\text{-URel}(X,1)$,

$$\begin{aligned} \kappa_X \ &\cdot \Pi(\mathbf{Y})_X(r) \\ &= m_X \cdot \tau_{TX}(\mathbf{y}_1^\circ \cdot \check{\Pi}r \cdot (\mathbf{m}_X \cdot \Pi\mathbf{Y}_X)^\circ \cdot \check{\Pi}1_X) \\ &= \bigvee_{\phi \in \Pi X} \phi * (\mathbf{y}_1^\circ \cdot \check{\Pi}r \cdot (\mathbf{m}_X \cdot \Pi\mathbf{Y}_X)^\circ \cdot \check{\Pi}1_X(\phi,\star)) \\ &= \bigvee_{\phi \in \Pi X, \Phi \in \Pi TX} \phi * (\check{\Pi}1_X(\phi, \mathbf{m}_X \cdot \Pi\mathbf{Y}_X(\Phi)) \otimes \check{\Pi}r(\Phi, \mathbf{y}_1(\star))) \\ &= \bigvee_{\phi \in \Pi X, \Phi \in \Pi TX} (\phi * (\mathbf{m}_X \cdot \Pi\mathbf{Y}_X(\Phi) \bullet\!\!- \phi)) * (r^\pi(\mathbf{y}_1(\star)) \bullet\!\!- \Phi) \\ &= \bigvee_{\Phi \in \Pi TX} (\mathbf{m}_X \cdot \Pi\mathbf{Y}_X(\Phi)) * (\pi_{TX}(r) \bullet\!\!- \Phi) \\ &= \bigvee_{\Phi \in \Pi TX} (\Phi \cdot m_X^\circ \cdot \hat{T}1_X) * ((e_1^\circ \cdot \hat{T}r) \bullet\!\!- \Phi). \end{aligned}$$

Since $\Phi = \pi_{TX}(r) = e_1^\circ \cdot \hat{T}r$ is a possible value for Φ running through ΠTX, we obtain

$$r = e_1^\circ \cdot \hat{T}r \cdot m_X^\circ \cdot \hat{T}1_X \le \kappa_X \cdot \Pi(\mathbf{Y})_X(r).$$

Moreover, $\mathbf{m}_X \cdot \Pi\mathbf{Y}_X = (-) \cdot m_X^\circ \cdot \hat{T}1_X : \Pi TX \to \Pi X$ is a $\mathbb{P}_V$-algebra morphism, and therefore also a V-functor of the underling V-categories, so

$$((e_1^\circ \cdot \hat{T}r) \bullet\!\!- \Phi) \le ((e_1^\circ \cdot \hat{T}r \cdot m_X^\circ \cdot \hat{T}1_X) \bullet\!\!- (\Phi \cdot m_X^\circ \cdot \hat{T}1_X)) = (r \bullet\!\!- (\Phi \cdot m_X^\circ \cdot \hat{T}1_X));$$

hence, $(\Phi \cdot m_X^\circ \cdot \hat{T}1_X) * ((e_1^\circ \cdot \hat{T}r) \bullet\!\!- \Phi) \le r$ for all $\Phi \in \Pi TX$, and

$$\kappa_X \cdot \Pi(\mathbf{Y})_X(r) \le r$$

holds. This shows $\kappa F \cdot F\mathbf{Y} = 1$. $\qquad\square$

6 Lax algebras

This section is a small incursion into the realm of lax algebras. Our main Theorem 5.7 above sheds light on categories of lax extensions, but lax extensions were originally motivated by the definition of lax algebras. Here, we show that monad morphisms between power-enriched monads induce isomorphisms between their categories of lax algebras. This proves that, once the defining power-enriched monad $\mathbb{T}$ and the order on $\mathsf{Set}_{\mathbb{T}}$ have been determined, the information pertaining to the quantale V via the enrichment $\tau : \mathbb{P}_V \to \mathbb{T}$ plays little role on the level of lax algebras themselves.

6.1 Categories of monoids

Let $\hat{\mathbb{T}}$ be an associative lax extension of $\mathbb{T} = (T, m, e)$ to V-Rel. A $(\mathbb{T}, V)$-*category* (generically, a *lax algebra*) on a set X is a monoid in $(\mathbb{T}, V)$-$\mathsf{URel}(X, X)$. In other words, a lax algebra (X, a) is a set X with a $(\mathbb{T}, V)$-relation $a : X \nrightarrow X$ satisfying

$$1_X^\sharp \leq a \qquad \text{and} \qquad a \circ a \leq a,$$

or equivalently, $e_X^\circ \leq a$ and $a \cdot \hat{T}a \cdot m_X^\circ \leq a$. These conditions imply in particular that a is unitary and idempotent:

$$1_X^\sharp \circ a = a = a \circ 1_X^\sharp \qquad \text{and} \qquad a \circ a = a$$

(see [2, Section III.1.8] for details). A $(\mathbb{T}, V)$-*functor* between $(\mathbb{T}, V)$ categories (X, a) and (Y, b) is a map $f : X \to Y$ such that

$$a \circ f^\sharp \leq f^\sharp \circ b,$$

or equivalently, $a \leq f^\circ \cdot b \cdot Tf$. The category of $(\mathbb{T}, V)$-categories with $(\mathbb{T}, V)$-functors as morphisms is denoted by

$$(\mathbb{T}, V)\text{-}\mathsf{Cat}.$$

By Theorem 4.7, a monoid in $(\mathbb{T}, V)$-$\mathsf{URel}(X, X)$ is a monoid in $\mathsf{Set}_{\nleq}(X, X)$. More generally, given a 2-enriched monad $\mathbb{T} = (T, m, e)$, a $\mathbb{T}$-*monoid* (or a *Kleisli monoid*) is a set X with a map $\nu : X \to TX$ such that

$$e_X \leq \nu \qquad \text{and} \qquad \nu \circ \nu \leq \nu.$$

A morphism between $\mathbb{T}$-monoids (X, ν) and (Y, ξ) is a map $f : X \longrightarrow Y$ satisfying

$$f_\natural \circ \nu \leq \xi \circ f_\natural,$$

where $f_\natural = e_Y \cdot f$; equivalently, this expression can be written $Tf \cdot \nu \leq \xi \cdot f$. The category of $\mathbb{T}$-monoids and their morphisms is denoted by

$$\mathbb{T}\text{-Mon.}$$

In the general case where $\mathbb{T}$ is a monad on Set (not necessarily 2-enriched) equipped with an an associative lax extension $\hat{\mathbb{T}}$ to V-Rel, the discrete presheaf monad $\not\geq$ is always V-power-enriched—and in particular 2-enriched. Via the order-isomorphism of Theorem 4.7, it is easily verified that the morphisms of monoids correspond, so there is an isomorphism

$$(\mathbb{T}, V)\text{-Cat} \cong \not\geq\text{-Mon.}$$

6.2   The Kleisli extension and monoids

It is shown in Theorem 4.4 that if $\mathbb{T}$ is a V-power-enriched monad, then there is an order-isomorphism

$$\mathsf{Set}_\mathbb{T} \cong (\mathbb{T}, V)\text{-URel}^{\mathrm{op}}$$

when $\mathbb{T}$ comes with its Kleisli extension $\check{\mathbb{T}}$ to V-Rel. By the V-enrichment, there is a monad morphism $\tau : \mathbb{P}_V \longrightarrow \mathbb{T}$, and the order on the hom-sets of $\mathsf{Set}_\mathbb{T}$ is induced by the order on the sets TX, that is, by the semilattice structure. In other words, the 2-enrichment $\tau \cdot \iota : \mathbb{P} \longrightarrow \mathbb{T}$ yields the same ordered category $\mathsf{Set}_\mathbb{T}$ as the V-enrichment $\tau : \mathbb{P}_V \longrightarrow \mathbb{T}$ (here, ι is the monad morphism $\iota : \mathbb{P} \longrightarrow \mathbb{P}_V$ given by the canonical quantale homomorphism $2 \longrightarrow V$ described in Example 2.12.1).

Proposition 6.1. *Let $\mathbb{T}$ a monad on Set, and $\kappa : \mathbb{P}_W \longrightarrow \mathbb{P}_V$, $\tau : \mathbb{P}_V \longrightarrow \mathbb{T}$ monad morphisms such that $(\mathbb{T}, \tau)$ is V-power-enriched. Then $(\mathbb{T}, \tau \cdot \kappa)$ is W-enriched, and there is an isomorphism*

$$(\mathbb{T}, V)\text{-Cat} \cong (\mathbb{T}, W)\text{-Cat},$$

where $(\mathbb{T}, \tau)$ and $(\mathbb{T}, \tau \cdot \kappa)$ are equipped with their respective Kleisli extensions $\check{\mathbb{T}}_V$ to V-Rel, and $\check{\mathbb{T}}_W$ to W-Rel.

Proof. By Theorem 4.4, there are order-isomorphisms

$$(\mathbb{T}, W)\text{-URel}^{\mathrm{op}} \cong \mathsf{Set}_{\mathbb{T}} \cong (\mathbb{T}, V)\text{-URel}^{\mathrm{op}}.$$

Hence, monoids in the corresponding hom-sets are in bijective correspondence, and so are morphisms between these, that is,

$$(\mathbb{T}, V)\text{-Cat} \cong \mathbb{T}\text{-Mon} \cong (\mathbb{T}, W)\text{-Cat},$$

which proves the claim.   □

Acknowledgements

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بازاری از گسترش‌های نرم

دیرک هافمن و گاوین جی. سیل

در این مقاله، یک الحاقی بین رسته‌ی کامایی موناد‌های مجموعه–بنیاد تحت موناد V–مجموعه‌ی توانی و رسته‌ی گسترش‌های سست شرکت‌پذیر موناد‌های مجموعه–بنیاد به رسته‌ی V–رابطه‌ها ارائه می‌دهیم. در این روند، یک روش کلی ساختن گسترش کلیسلی یک موناد به رسته‌ی V–رابطه‌ها را ارائه می‌کنیم.

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