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The ring of real-valued functions on a frame

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Abstract. In this paper, we define and study the notion of the real-valued functions on a frame L. We show that $F(L)$, consisting of all frame homomorphisms from the power set of $\mathbb{R}$ to a frame L, is an f-ring, as a generalization of all functions from a set X into $\mathbb{R}$. Also, we show that $F(L)$ is isomorphic to a sub-f-ring of $\mathcal{R}(L)$, the ring of real-valued continuous functions on L. Furthermore, for every frame L, there exists a Boolean frame B such that $F(L)$ is a sub-f-ring of $F(B)$.

1 Introduction

Pointfree topology focuses on the open sets rather than the points of a space, and deals with abstractly defined "lattice of open sets", called frames, and their homomorphisms. The ring of real continuous functions in pointfree topology has been studied by a number of authors, such as B. Banaschewski (see [2, 4, 5]), R.N. Ball and J. Walters-Wayland (see [1]) and T. Dube (see [6–8]).

In this paper, we are going to turn our viewpoint and regard the power

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set of a set X as a frame, and study all its subsets, rather than the points of X. Our future purpose is to consider a frame L endowed with a topoframe as well as the power set of X endowed with a topology (see [9]).

In section 3, we introduce the concept of real-valued functions $F(L)$ and show that $F(L)$ with the operator $\diamond$ defined at the start of this section, is an f-ring.

In section 4, we show that the f-ring $F(L)$ is a generalization of $\mathbb{R}^X$, the collection of all functions from a set X into the set $\mathbb{R}$.

In section 5, we prove that for every frame L, there exists a Boolean frame B such that $F(L)$ is a sub-f-ring of $F(B)$.

In the last section, we show that $F(L)$ is isomorphic to a sub-f-ring of $\mathcal{R}(L)$, the f-ring of all real continuous functions on L, and demonstrate that the inclusion may be strict.

2 Preliminaries

A *frame* is a complete lattice L in which the infinite distributive law

$$x \wedge \bigvee S = \bigvee \{x \wedge s : s \in S\}$$

holds for all $x \in L$ and $S \subseteq L$. We denote the top element and the bottom element of L by $\top$ and $\bot$, respectively. The frame of all subsets of a set X is denoted by $\mathcal{P}(X)$.

A *frame homomorphism* (or *frame map*) is a map between frames which preserves finite meets, including the top element, and arbitrary joins, including the bottom element.

Here we present some of the background facts concerning f-rings which are used in our manuscript. To begin with, a lattice-ordered ring is a ring A with a lattice structure such that, for all $a, b, c \in A$,

$$(a \wedge b) + c = (a + c) \wedge (b + c)$$

or, equivalently,

$$(a \vee b) + c = (a + c) \vee (b + c)$$

and

$$0 \leq ab \text{ whenever } 0 \leq a \text{ and } 0 \leq b\,.$$

As immediate consequences one has that $-(a \vee b) = (-a) \wedge (-b)$, $-(a \wedge b) = (-a) \vee (-b)$ and $a \leq b$ implies $-b \leq -a$.

Further, with the definitions

$$a^+ = a \vee 0,\ a^- = (-a) \vee 0,\ |a| = a \vee (-a)$$

one has the rules

$$0 \leq |a|,\ |a| = a^+ + a^-,\ a = a^+ - a^-,\ a^+ \wedge a^- = 0,$$

$$|a+b| \leq |a| + |b|,\ |ab| \leq |a||b|.$$

A homomorphism of lattice-ordered rings is, of course, a map between such rings which is both a ring and a lattice homomorphism. We note in passing that in certain cases any ring homomorphism automatically preserves the lattice operations.

An ℓ-ideal in a lattice-ordered ring A is a ring ideal J of A with the added property that $|x| \leq |a|$ and $a \in J$ implies $x \in J$, for any $x, a \in A$.

For any $a \in A$, the ℓ-ideal generated by a is

$$[a] = \{x \in A \,:\, |x| \leq |a|b,\ b \geq 0 \text{ in } A\}\,.$$

Now, an f-ring is a lattice-ordered ring A which satisfies any of the following equivalent conditions:

1. $(a \wedge b)c = (ac) \wedge (bc)$ for any a, $b \in A$ and $c \geq 0$ in A.

2. $|ab| = |a||b|$.

3. $[a \wedge b] = [a] \cap [b]$ for any a, $b \geq 0$ in A.

We call a lattice-ordered ring A with unit *strong* if every $a \geq 1$ is invertible in A, and *bounded* if, for each $a \in A$, $|a| \leq n$, for some natural number n (where we permit notational confusion between the natural number n and the sum in A of n summands equal to the unit 1 of A). Further, A is called Archimedean if, whenever $0 \leq a, b$ and $na \leq b$ for all natural n, then $a = 0$. In the following, A is always an Archimedean, strong, and bounded commutative f-ring with unit. Also, homomorphisms between such rings are understood to be unit preserving.

An f-ring A has a natural topology, its uniform topology, with basic neighbourhoods

$$V_n(a) = \{x \in A : |x - a| < \frac{1}{n}\}, \ \ n = 1, 2, ...,$$

for each $a \in A$.

Note that $\mathcal{L}(A)$, the frame of all ℓ-ideals of A, is a compact frame. Moreover, for any $I, J \in \mathcal{L}(A)$, the following statements hold.

1. $I \vee J = I + J = \{a + b : \ a \in I, \ b \in J\}$.

2. $\overline{I \cap J} = \overline{I} \cap \overline{J}$.

The latter statement expresses that $C\mathcal{L}(A)$, the frame of all closed ℓ-ideals of A, is a sublocal of $\mathcal{L}(A)$ and a frame under finite meets in $\mathcal{L}(A)$ and the closure of arbitrary joins in $\mathcal{L}(A)$; in particular it is a compact completely regular frame.

3 An algebraic structure and an order structure on the Real-valued functions on a frame

The main aim of this section is to show that the collection of all real-valued functions on a frame is an f-ring. If a frame happens to be a Boolean algebra we speak of a Boolean frame.

Definition 3.1. A real-valued function on a frame L is a frame homomorphism $f : \mathcal{P}(\mathbb{R}) \to L$, where one assumes $(\mathcal{P}(\mathbb{R}), \subseteq)$ to be a Boolean frame.

In what follows, the set of all real-valued functions on a frame L is denoted by $F_{\mathcal{P}}(L)$. We abbreviate $F_{\mathcal{P}}(L)$ as $F(L)$.

Definition 3.2. Let $\diamond : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be an operation on $\mathbb{R}$ (in particular $\diamond \in \{+, ., \vee, \wedge\}$). Let f, g be two real-valued functions on L. Define $f \diamond g : \mathcal{P}(\mathbb{R}) \to L$ by

$$(f \diamond g)(X) = \bigvee \{f(Y) \wedge g(Z) : Y \diamond Z \subseteq X\},$$

where $Y \diamond Z = \{y \diamond z : y \in Y, z \in Z\}$.

Lemma 3.3. *Let f, g be two real-valued functions on a frame L. Then $f \diamond g$ is a poset homomorphism.*

Proof. Trivial. $\square$

Hereafter, when a topology is used on a subset of $\mathbb{R}$ it is assumed to be the usual topology. The frame of open subsets of a topological space X is denoted by $\mathfrak{O}X$.

For $p, q \in \mathbb{R}$, let

$$\langle p, q\rangle \coloneqq \{x \in \mathbb{Q} : p < x < q\}$$

and

$$\rrbracket p, q\llbracket \coloneqq \{x \in \mathbb{R} : p < x < q\}.$$

Lemma 3.4. *Let f, g be two real-valued functions on a frame L. Then for every $\diamond \in \{+, ., \vee, \wedge\}$, the following statements hold.*

1. $(f \diamond g)(X) = \bigvee\{f(\{x\}) \wedge g(\{y\}) : x \diamond y \in X\}$, *for every* $X \in \mathcal{P}(\mathbb{R})$.
2. $(f \diamond g)(U) = \bigvee\{f(\rrbracket r, s\llbracket) \wedge g(\rrbracket u, v\llbracket) : \rrbracket r, s\llbracket \diamond \rrbracket u, v\llbracket \subseteq U\}$, *for every* $U \in \mathfrak{O}\mathbb{R}$.
3. $f = g$ *if and only if* $f(\{r\}) = g(\{r\})$, *for every* $r \in \mathbb{R}$.

Proof. 1. Let $\diamond : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be an operation on $\mathbb{R}$ and $X \in \mathcal{P}(\mathbb{R})$. Then

$$\begin{aligned}
(f \diamond g)(X) &= \bigvee\{f(Y) \wedge g(Z) : Y \diamond Z \subseteq X\} \\
&= \bigvee\{f(\textstyle\bigcup_{y \in Y}\{y\}) \wedge g(\bigcup_{z \in Z}\{z\}) : Y \diamond Z \subseteq X\} \\
&= \bigvee\{\textstyle\bigvee_{y \in Y} f(\{y\}) \wedge \bigvee_{z \in Z} g(\{z\}) : Y \diamond Z \subseteq X\} \\
&= \bigvee\{\textstyle\bigvee_{y \in Y} \bigvee_{z \in Z}(f(\{y\}) \wedge g(\{z\})) : Y \diamond Z \subseteq X\} \\
&= \bigvee\{f(\{y\}) \wedge g(\{z\}) : y \diamond z \in X\}.
\end{aligned}$$

2. Suppose that $x \diamond y \in U \in \mathfrak{O}\mathbb{R}$. Since $\diamond : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous map, there are $r, s, u, v \in \mathbb{Q}$ such that $x \in \rrbracket r, s\llbracket$, $y \in \rrbracket u, v\llbracket$, and $x \diamond y \in \rrbracket r, s\llbracket \diamond \rrbracket u, v\llbracket \subseteq U$. Thus

$$\begin{aligned}
(f \diamond g)(U) &= \bigvee\{f(\{x\}) \wedge g(\{y\}) : x \diamond y \in U\} \\
&\leq \bigvee\{f(\rrbracket r, s\llbracket) \wedge g(\rrbracket u, v\llbracket) : \rrbracket r, s\llbracket \diamond \rrbracket u, v\llbracket \subseteq U\} \\
&\leq \bigvee\{f(Y) \wedge g(Z) : Y \diamond Z \subseteq U\} \\
&= (f \diamond g)(U).
\end{aligned}$$

Consequently,

$$(f \diamond g)(U) = \bigvee\{f(\rrbracket r, s\llbracket) \wedge g(\rrbracket u, v\llbracket) : \rrbracket r, s\llbracket \diamond \rrbracket u, v\llbracket \subseteq U\}.$$

3. Trivial.

□

Proposition 3.5. *Let f, g be two real-valued functions on a frame L and $\diamond \in \{+, ., \vee, \wedge\}$. Then $f \diamond g$ is a real-valued function on L.*

Proof. Suppose that $h = f \diamond g$ and $X_1, X_2 \subseteq \mathbb{R}$. Then

$$\begin{aligned} h(X_1) \wedge h(X_2) &= \bigvee\{f(Y_1) \wedge g(Z_1) : Y_1 \diamond Z_1 \subseteq X_1\} \wedge \bigvee\{f(Y_2) \wedge g(Z_2) : Y_2 \diamond Z_2 \subseteq X_2\} \\ &= \bigvee\{f(Y_1) \wedge g(Z_1) \wedge f(Y_2) \wedge g(Z_2) : Y_1 \diamond Z_1 \subseteq X_1, Y_2 \diamond Z_2 \subseteq X_2\} \\ &= \bigvee\{f(Y_1 \cap Y_2) \wedge g(Z_1 \cap Z_2) : Y_1 \diamond Z_1 \subseteq X_1, Y_2 \diamond Z_2 \subseteq X_2\} \\ &\leq \bigvee\{f(Y) \wedge g(Z) : Y \diamond Z \subseteq X_1 \cap X_2\} \\ &= h(X_1 \cap X_2). \end{aligned}$$

Hence, by Lemma 3.3, $h(X_1 \cap X_2) = h(X_1) \wedge h(X_2)$.

Now, let $\{X_i : i \in I\}$ be a family of subsets of $\mathbb{R}$. Suppose that $Y \diamond Z \subseteq \bigcup X_i$. If $y \in Y$ and $z \in Z$, then there exists $i \in I$ such that $y \diamond z \in X_i$. So, $f(\{y\}) \wedge g(\{z\}) \leq h(X_i)$. Hence

$$f(Y) \wedge g(Z) = \bigvee\{f(\{y\}) \wedge g(\{z\}) : y \in Y, z \in Z\} \leq \bigvee_{i \in I} h(X_i).$$

Thus

$$h(\bigcup_{i \in I} X_i) = \bigvee\{f(Y) \wedge g(Z) : Y \diamond Z \subseteq \bigcup_{i \in I} X_i\} \leq \bigvee_{i \in I} h(X_i).$$

By Lemma 3.3, $h(\bigcup_{i \in I} X_i) = \bigvee_{i \in I} h(X_i)$. Also, we have

$$\begin{aligned} h(\emptyset) &= \bigvee\{f(\{x\}) \wedge g(\{y\}) : x \diamond y \in \emptyset\} \\ &= \bigvee \emptyset \\ &= \perp \end{aligned}$$

and

$$\begin{aligned} h(\mathbb{R}) &= \bigvee\{f(\{x\}) \wedge g(\{y\}) : x \diamond y \in \mathbb{R}\} \\ &= f(\mathbb{R}) \wedge g(\mathbb{R}) \\ &= \top \wedge \top \\ &= \top. \end{aligned}$$

□

Lemma 3.6. *Let f, g and h be real-valued functions on a frame L and $\diamond \in \{+, ., \vee, \wedge\}$. Then the following statements hold.*

1. $f \diamond g = g \diamond f$.

2. $f \vee f = f$ *and* $f \wedge f = f$.

3. $f \diamond (g \diamond h) = (f \diamond g) \diamond h$.

4. $f \vee (f \wedge g) = f$ *and* $f \wedge (f \vee g) = f$.

5. $(f + g)h = fh + gh$.

6. *If* $(\diamond, \diamond') \in \{(\vee, \wedge), (\wedge, \vee), (+, \wedge), (+, \vee)\}$, *then* $f \diamond (g \diamond' h) = (f \diamond g) \diamond' (f \diamond h)$.

Proof. 1. Trivial.

2. Trivial.

3. For every $r \in \mathbb{R}$, we have

$$\begin{aligned}
((f \diamond g) \diamond h)(\{r\}) &= \bigvee\{(f \diamond g)(\{x\}) \wedge h(\{y\}) : x \diamond y = r\} \\
&= \bigvee\{\bigvee\{f(\{z\}) \wedge g(\{t\}) : z \diamond t = x\} \wedge h(\{y\}) : x \diamond y = r\} \\
&= \bigvee\{f(\{z\}) \wedge g(\{t\}) \wedge h(\{y\}) : (z \diamond t) \diamond y = r\} \\
&= \bigvee\{f(\{z\}) \wedge g(\{t\}) \wedge h(\{y\}) : z \diamond (t \diamond y) = r\} \\
&= \bigvee\{f(\{z\}) \wedge \bigvee\{g(\{t\}) \wedge h(\{y\}) : t \diamond y = v\} : z \diamond v = r\} \\
&= \bigvee\{f(\{z\}) \wedge (g \diamond h)(\{v\}) : z \diamond v = r\} \\
&= (f \diamond (g \diamond h))(\{r\}).
\end{aligned}$$

Hence, $(f \diamond g) \diamond h = f \diamond (g \diamond h)$.

4. For every $r \in \mathbb{R}$, we have

$$\begin{aligned}
f \vee (f \wedge g)(\{r\}) &= \bigvee\{f(\{x\}) \wedge (f \wedge g)(\{t\}) : x \vee t = r\} \\
&= \bigvee\{f(\{x\}) \wedge \bigvee\{f(\{y\}) \wedge g(\{z\}) : y \wedge z = t\} : x \vee t = r\} \\
&= \bigvee\{f(\{x\}) \wedge (f(\{y\}) \wedge g(\{z\})) : x \vee (y \wedge z) = r\} \\
&= \bigvee\{f(\{x\} \cap \{y\}) \wedge g(\{z\}) : x \vee (y \wedge z) = r\} \\
&= \bigvee\{f(\{x\}) \wedge g(\{z\}) : x \vee (x \wedge z) = r\} \\
&= \bigvee\{f(\{x\}) \wedge g(\{z\}) : x = r\} \\
&= f(\{r\}) \wedge \bigvee\{g(\{z\}) : z \in \mathbb{R}\} \\
&= f(\{r\}) \wedge \top \\
&= f(\{r\}).
\end{aligned}$$

Therefore, $f \vee (f \wedge g) = f$ and a similar proof shows that $f \wedge (f \vee g) = f$.

5. For every $r \in \mathbb{R}$, we have

$$\begin{aligned}
(fh + gh)(\{r\}) &= \bigvee\{(fh)(\{a\}) \wedge gh(\{b\}) : a + b = r\} \\
&= \bigvee\{\bigvee\{f(\{z\}) \wedge h(\{y\}) : zy = a\} \\
&\quad \wedge \bigvee\{g(\{t\}) \wedge h(\{w\}) : tw = b\} : a + b = r\} \\
&= \bigvee\{f(\{z\}) \wedge g(\{t\}) \wedge h(\{y\} \cap \{w\}) : zy + tw = r\} \\
&= \bigvee\{f(\{z\}) \wedge g(\{t\}) \wedge h(\{y\}) : zy + ty = r\} \\
&= \bigvee\{f(\{z\}) \wedge g(\{t\}) \wedge h(\{y\}) : (z + t)y = r\} \\
&= \bigvee\{\bigvee\{f(\{z\}) \wedge g(\{t\}) : z + t = x\} \wedge h(\{y\}) : xy = r\} \\
&= \bigvee\{(f + g)(\{x\}) \wedge h(\{y\}) : xy = r\} \\
&= ((f + g)h)(\{r\}).
\end{aligned}$$

Therefore, $(f + g)h = fh + gh$.

6. For every $r, x, y, z, t \in \mathbb{R}$, we have $x \diamond (y \diamond' t) = (x \diamond y) \diamond' (x \diamond t)$, it

follows that

$$\begin{aligned}
(f \diamond (g \diamond' h))(\{r\}) &= \bigvee\{f(\{x\}) \wedge (g \diamond' h)(\{z\}) : x \diamond z = r\} \\
&= \bigvee\{f(\{x\}) \wedge \bigvee\{g(\{y\}) \wedge h(\{t\}) : y \diamond' t = z\} : x \diamond z = r\} \\
&= \bigvee\{f(\{x\}) \wedge g(\{y\}) \wedge h(\{t\}) : x \diamond (y \diamond' t) = r\} \\
&= \bigvee\{f(\{x\}) \wedge g(\{y\}) \wedge h(\{t\}) : (x \diamond y) \diamond' (x \diamond t) = r\} \\
&= \bigvee\{\bigvee\{f(\{x\}) \wedge g(\{y\}) : x \diamond y = v\} \wedge \\
&\qquad \bigvee\{f(\{x\}) \wedge h(\{t\}) : x \diamond t = w\} : v \diamond' w = r\} \\
&= \bigvee\{(f \diamond g)(\{v\}) \wedge (f \diamond h)(\{w\}) : v \diamond' w = r\} \\
&= (f \diamond g) \diamond' (f \diamond h)(\{r\}).
\end{aligned}$$

Therefore, $f \diamond (g \diamond' h) = (f \diamond g) \diamond' (f \diamond h)$.

□

The set $F(L)$ of all continuous real functions on a frame L will be provided with an algebraic and an order structure. The partial ordering on $F(L)$ is defined by:

$$f \leq g \text{ if and only if } f \wedge g = f \text{ if and only if } f \vee g = g.$$

Also, by Lemma 3.6, $F(L)$ is lattice.

Remark 3.7.

1. **The constant real-valued function on a frame** L. For each $c \in \mathbb{R}$, let $\mathbf{c}$ be defined by

$$\mathbf{c}(X) = \begin{cases} \top_L & \text{if } c \in X, \\ \bot_L & \text{if } c \notin X \end{cases}$$

for every $X \in \mathcal{P}(\mathbb{R})$. It is obvious that $\mathbf{c} \in F(L)$. Also, for every $f \in F(L)$,

$$(f + \mathbf{0})(\{x\}) = \bigvee\{f(\{y\}) \wedge \mathbf{0}(\{z\}) : y + z = x\} = f(\{x\})$$

and

$$(f\mathbf{1})(\{x\}) = \bigvee\{f(\{y\}) \wedge \mathbf{1}(\{z\}) : yz = x\} = f(\{x\}),$$

where $x \in \mathbb{R}$. Therefore, $f + \mathbf{0} = f$ and $f\mathbf{1} = f$.

2. **Additive inverse.** Let $f \in F(L)$. The mapping $-f : \mathcal{P}(\mathbb{R}) \to L$ defined by $(-f)(X) = f(-X)$ clearly belongs to $F(L)$, where $X \in \mathcal{P}(\mathbb{R})$ and $-X = \{-x : x \in X\}$. Also, for any $r \in \mathbb{R}$,

$$\begin{aligned}
(f + (-f))(\{r\}) &= \bigvee\{f(\{y\}) \wedge (-f)(\{z\}) : y + z = r\} \\
&= \bigvee\{f(\{y\}) \wedge f(\{-z\}) : y + z = r\} \\
&= \bigvee\{f(\{y\}) \wedge f(\{y - r\}) : y \in \mathbb{R}\} \\
&= \bigvee\{f(\{y\} \wedge \{y - r\}) : y \in \mathbb{R}\} \\
&= \begin{cases} f(\mathbb{R}) & \text{if } r = 0 \\ \bigvee\{\perp_L\} & \text{if } r \neq 0 \end{cases} \\
&= \begin{cases} \top_L & \text{if } r = 0 \\ \perp_L & \text{if } r \neq 0 \end{cases} \\
&= \mathbf{0}(\{r\}).
\end{aligned}$$

Therefore, $f + (-f) = \mathbf{0}$.

3. **Product with a scalar.** For any $f \in F(L)$ and $r \in \mathbb{R}$, define

$$r.f(X) = \begin{cases} \mathbf{0}(X) & \text{if } r = 0, \\ f(\frac{1}{r}X) & \text{if } r \in \mathbb{R} - \{0\}, \end{cases}$$

where $X \in \mathcal{P}(\mathbb{R})$ and $\frac{1}{r}X = \{\frac{1}{r}x : x \in X\}$; a straightforward calculation gives $r.f = \mathbf{r}f$.

Lemma 3.8. *Let $r \in \mathbb{R}$ and $f, g \in F(L)$. Then the following properties hold.*

1. $(f \wedge g)(\{r\}) = (f(\{r\}) \wedge g[r, +\infty)) \vee (f[r, +\infty) \wedge g(\{r\}))$.

2. $(f \vee g)(\{r\}) = (f(\{r\}) \wedge g(-\infty, r]) \vee (f(-\infty, r] \wedge g(\{r\}))$.

3. $(f \wedge \mathbf{0})(\{r\}) = \begin{cases} \perp & \textit{if } r > 0, \\ f[0, +\infty) & \textit{if } r = 0, \\ f(\{r\}) & \textit{if } r < 0. \end{cases}$

4. $$(f \vee \mathbf{0})(\{r\}) = \begin{cases} f(\{r\}) & \textit{if } \ r > 0, \\ f(-\infty, 0] & \textit{if } \ r = 0, \\ \perp & \textit{if } \ r < 0. \end{cases}$$

Proof. Trivial. □

Theorem 3.9. *$(F(L), +, ., \vee, \wedge)$ is an f-ring.*

Proof. By Lemma 3.6 and Remark 3.7, it suffices to show that if $f, g \in F(L)$ with $f, g \geq \mathbf{0}$, then $fg \geq \mathbf{0}$. By Lemma 3.8, we have

$$\begin{aligned}
(fg \wedge \mathbf{0})(\{r\}) &= \begin{cases} (fg)(\{r\}) & \text{if } \ r < 0 \\ fg[0, +\infty) & \text{if } \ r = 0 \\ \perp_L & \text{if } \ r > 0 \end{cases} \\
&= \begin{cases} \bigvee\{f(\{x\}) \wedge g(\{y\}) : xy = r, \ r < 0\} & \text{if } \ r < 0 \\ (f[0, +\infty) \wedge g[0, +\infty)) \vee (f(-\infty, 0] \wedge g(-\infty, 0]) & \text{if } \ r = 0 \\ \perp_L & \text{if } \ r > 0 \end{cases} \\
&= \begin{cases} \bigvee\{f(\{x\}) \wedge g(\{y\}) : xy = r, x > 0, y < 0\} & \\ \vee \bigvee\{f(\{x\}) \wedge g(\{y\}) : xy = r, x < 0, y > 0\} & \text{if } \ r < 0 \\ (\top \wedge \top) \vee (f(-\infty, 0] \wedge g(-\infty, 0]) & \text{if } \ r = 0 \\ \perp_L & \text{if } \ r > 0 \end{cases} \\
&= \begin{cases} \bigvee\{f(\{x\}) \wedge \perp : xy = r, x > 0, y < 0\} & \\ \vee \bigvee\{\perp \wedge g(\{y\}) : xy = r, x < 0, y > 0\} & \text{if } \ r < 0 \\ \top_L & \text{if } \ r = 0 \\ \perp_L & \text{if } \ r > 0 \end{cases} \\
&= \begin{cases} \top_L & \text{if } \ r = 0 \\ \perp_L & \text{if } \ r \neq 0 \end{cases} \\
&= \mathbf{0}(\{r\}).
\end{aligned}$$

Hence $fg \geq \mathbf{0}$. □

Finally, it is worth mentioning that the association $L \longrightarrow F(L)$ from the category of frames to that of real-valued functions on frames is functorial: for any frame homomorphism $\phi : M \longrightarrow L$, the associated map $F\phi : F(M) \longrightarrow F(L)$ takes any $f \in F(M)$ to $\phi f \in F(L)$, and especially takes $\mathbf{1}_{\mathbf{Frm}}$ to the identity arrow of real-valued functions on frames. Obviously, the resulting functor F is a covariant functor.

4 A generalization of $\mathbb{R}^X$

The set $\mathbb{R}^X$ of all real-valued functions on a set X will be provided with an algebraic and an order structure. If $\diamond \in \{+, ., \wedge, \vee\}$, then for every $f, g \in \mathbb{R}^X$ and $x \in X$, define $f \diamond g$ by

$$(f \diamond g)(x) = f(x) \diamond g(x).$$

Since $(\mathbb{R}, +, ., \wedge, \vee)$ is an f-ring, we infer that $\mathbb{R}^X$ is an f-ring (see [10]).

The following theorem shows that $F(L)$, as an f-ring, is a generalization of $\mathbb{R}^X$.

Theorem 4.1. *The assignment $\theta(f) = f^{-1}$ from $\mathbb{R}^X$ to $F(\mathcal{P}(X))$ is an f-ring isomorphism, where*

$$\begin{array}{rcl} f^{-1} : \mathcal{P}(\mathbb{R}) & \longrightarrow & \mathcal{P}(X) \\ A & \longmapsto & \{x \in X : f(x) \in A\}. \end{array}$$

Proof. (i) Clearly θ is a function.

(ii) Let $f, g \in \mathbb{R}^X$ such that $\theta(f) = \theta(g)$. Then for every $x \in X$,

$$x \in f^{-1}(\{f(x)\}) = g^{-1}(\{f(x)\}),$$

which follows that $f(x) = g(x)$. Hence $f = g$. Therefore, θ is one-one.

(iii) To show that θ is surjective, let $g \in F(\mathcal{P}(X))$. The relation h, define by

$$h(x) = \lambda \text{ iff } x \in g(\{\lambda\})$$

is a function from X to $\mathbb{R}$, since $\bigcup_{\lambda \in \mathbb{R}} g(\{\lambda\}) = g(\mathbb{R}) = X$. Therefore for any $x \in X$, there exists $\lambda \in \mathbb{R}$ such that $x \in g(\{\lambda\})$ and hence $Dom(h) = X$. It immediately follows from the defnition that $\theta(h) = h^{-1} = g$.

(iv) By Lemma 3.4, for any $f, g \in \mathbb{R}^X$, $r \in \mathbb{R}$ and $\diamond \in \{+, ., \wedge, \vee\}$, we have

$$(\theta(f)\diamond\theta(g))(\{r\}) = (f^{-1}\diamond g^{-1})(\{r\}) = \bigcup\{f^{-1}(\{a\})\cap g^{-1}(\{b\}) : a\diamond b = r\}.$$

Furthermore,

$$\theta(f \diamond g)(\{r\}) = (f \diamond g)^{-1}(\{r\}) = \{x \in X : (f \diamond g)(x) = r\}.$$

Let $z \in (\theta(f) \diamond \theta(g))(\{r\})$. Then there exist $a, b \in \mathbb{R}$ with $a \diamond b = r$ such that $z \in f^{-1}(\{a\}) \cap g^{-1}(\{b\})$, and thus

$$(f \diamond g)(z) = f(z) \diamond g(z) = a \diamond b = r,$$

which follows that $z \in \theta(f \diamond g)(\{r\})$. Hence,

$$(\theta(f) \diamond \theta(g))(\{r\}) \subseteq \theta(f \diamond g)(\{r\}).$$

To establish the reverse inclusion, consider $z \in \theta(f \diamond g)(\{r\})$, then

$$f(z) \diamond g(z) = (f \diamond g)(z) = r.$$

Since $z \in f^{-1}(\{f(z)\}) \cap g^{-1}(\{g(z)\})$, we conclude that $z \in (\theta(f) \diamond \theta(g))(\{r\})$. Hence,

$$\theta(f \diamond g)(\{r\}) \subseteq (\theta(f) \diamond \theta(g))(\{r\}).$$

Therefore, $\theta(f \diamond g) = \theta(f) \diamond \theta(g)$. This completes the proof of the theorem.

□

5 Boolean algebra

In this section, we show that for every frame L, there exists a Boolean frame B such that $F(L)$ is a sub-f-ring of $F(B)$.

Remark 5.1. Let $(L, \bigvee, \wedge)$ be a frame. It is well-known that

$$BL \coloneqq \{a \in L : a^{\star\star} = a\},$$

is the Booleanization of the frame L which the underlying set BL has meet $\sqcap$ and join $\bigsqcup$ given by:

(i) $a \sqcap b = a \wedge b$

(ii) $\bigsqcup A = (\bigvee A)^{\star\star}$.

Lemma 5.2. *Let L be a frame and $f : \mathcal{P}(\mathbb{R}) \to L$ be a frame map. Then $f(A)' = f(A')$, for every $A \subseteq \mathbb{R}$, where the complement of $f(A)$ is, by definition, $(f(A))'$, abbreviated $f(A)'$.*

Proof. Since f is a frame map, $f(A') \wedge f(A) = \perp$ and $f(A') \vee f(A) = \top$. It follows immediately that $f(A')$ is the complement of $f(A)$. □

Theorem 5.3. *Let L be a frame. Then the mapping*

$$\begin{aligned} \varphi : F(L) &\longrightarrow F(BL) \\ f &\longmapsto f^{\star\star} \end{aligned}$$

is an f-ring embedding, where

$$\begin{aligned} f^{\star\star} : \mathcal{P}(\mathbb{R}) &\longrightarrow BL \\ A &\longmapsto (f(A))^{\star\star} \end{aligned}$$

Proof. By definition of BL, if $f \in F(L)$ and $A, B \in \mathcal{P}(\mathbb{R})$, then

$$\varphi(f)(A \cap B) = (f(A \cap B))^{\star\star} = (f(A) \wedge f(B))^{\star\star} = f(A)^{\star\star} \wedge f(B)^{\star\star} = \varphi(f)(A) \sqcap \varphi(f)(B).$$

Also, if $\{A_\lambda\}_{\lambda \in \Lambda} \subseteq \mathcal{P}(\mathbb{R})$, then, by Remark 5.1 and Lemma 5.2,

$$\varphi(f)(\bigcup_{\lambda \in \Lambda} A_\lambda) = (f(\bigcup_{\lambda \in \Lambda} A_\lambda))^{\star\star} = (\bigvee_{\lambda \in \Lambda} f(A_\lambda))^{\star\star} = \bigsqcup_{\lambda \in \Lambda} \varphi(f)(A_\lambda).$$

Hence $f^{\star\star} : \mathcal{P}(\mathbb{R}) \to BL$ is a frame map. If $f, g \in F(L)$ and $\varphi(f) = \varphi(g)$, then, by Lemma 5.2,

$$f(A) = f^{\star\star}(A) = \varphi(f)(A) = \varphi(g)(A) = g^{\star\star}(A) = g(A)$$

for every $A \in \mathcal{P}(\mathbb{R})$. So $f = g$ and hence φ is one-one.

If $f,g \in F(L)$ and $A \in \mathcal{P}(\mathbb{R})$, then

$$\begin{array}{rcll}
\varphi(f \diamond g)(A) & = & (f \diamond g)^{\star\star}(A) & \\
& = & ((f \diamond g)(A))^{\star\star} & \\
& = & (\bigvee^{L}\{f(\{x\}) \wedge g(\{y\}) : x \diamond y \in A\})^{\star\star} & \\
& = & \bigsqcup\{f(\{x\}) \wedge g(\{y\}) : x \diamond y \in A\} & \text{by Remark 5.1} \\
& = & \bigsqcup\{f(\{x\}) \sqcap g(\{y\}) : x \diamond y \in A\} & \\
& = & \bigsqcup\{(f(\{x\}))^{\star\star} \sqcap (g(\{y\}))^{\star\star} : x \diamond y \in A\} & \text{by Lemma 5.2} \\
& = & \bigsqcup\{\varphi(f)(\{x\}) \sqcap \varphi(g)(\{y\}) : x \diamond y \in A\} & \\
& = & (\varphi(f) \diamond \varphi(g))(A) &
\end{array}$$

for every $\diamond \in \{+, ., \wedge, \vee\}$. Therefore, φ is an f-ring monomorphism. □

6 The relation between $F(L)$ and $\mathcal{R}(L)$

Now, we are going to prove that $F(L)$ is isomorphic to a sub-f-ring of $\mathcal{R}(L)$.

Theorem 6.1. *For any frame L, the mapping $F(L) \longrightarrow \mathcal{R}(L)$ taking any f to $f \circ j$ is an f-ring monomorphism, where $j : \mathcal{L}(\mathbb{R}) \longrightarrow \mathcal{O}(\mathbb{R})$ taking (p,q) to $]\!]p,q[\![$ is an isomorphism.*

Proof. Let $f,g \in F(L)$ such that $f \circ j = g \circ j$. For every $r \in \mathbb{R}$, we have

$$\begin{array}{rcll}
f(\{r\}) & = & f((\mathbb{R} - \{r\})') & \\
& = & (f(\mathbb{R} - \{r\}))' & \text{by Lemma 5.2} \\
& = & ((f \circ j)((-,r) \vee (r,-)))' & \\
& = & ((g \circ j)((-,r) \vee (r,-)))' & \\
& = & (g(\mathbb{R} - \{r\}))' & \\
& = & g((\mathbb{R} - \{r\})') & \text{by Lemma 5.2} \\
& = & g(\{r\}). &
\end{array}$$

Hence $f = g$. Furthermore, for each operator $\diamond$ in $\{+, ., \vee, \wedge\}$ and for each $p,q \in \mathbb{Q}$, we have

$$\begin{array}{rcl}
((f \diamond g) \circ j)(p,q) & = & (f \diamond g)(]\!]p,q[\![) \\
& = & \bigvee\{(f(]\!]r,s[\![) \wedge g(]\!]u,v[\![) :]\!]r,s[\![\diamond]\!]u,v[\![\subseteq]\!]p,q[\![\} \\
& = & \bigvee\{(f \circ j)(r,s) \wedge (g \circ j)(u,v) : < r,s > \diamond < u,v > \subseteq < p,q >\} \\
& = & ((f \circ j) \diamond (g \circ j))(p,q).
\end{array}$$

Hence, $F(L)$ is isomorphic to a sub-f-ring of $\mathcal{R}(L)$. □

We now present a counterexample to show that the relation of inclusion between $F(L)$ and $R(L)$ in Theorem 6.1 may be strict. For this, let $A = C([0,1])$, the ring of real continuous functions on $[0,1]$, and let $C\mathcal{L}(A)$ be the frame of all closed ℓ-ideals of A. First, we show that $F(C\mathcal{L}(A)) \cong \mathbb{R}$ as f-rings. To see this, let $f : \mathcal{P}(\mathbb{R}) \longrightarrow C\mathcal{L}(A)$ be a frame map and let $I_r := f(\{r\})$ for any $r \in \mathbb{R}$. Since $C\mathcal{L}(A)$ is compact, there exist $r_1, ..., r_n \in \mathbb{R}$ such that

$$I_{r_1} + \cdots + I_{r_n} = \overline{I_{r_1}} \vee \cdots \vee \overline{I_{r_n}} = \overline{I_{r_1} \vee \cdots \vee I_{r_n}} = \overline{f(\{r_1\}) \vee ... \vee f(\{r_n\})} = A\,,$$

where $\vee$ is the join of ℓ-ideals among the ℓ-ideals of A. So there exists $\alpha_i \in I_{r_i}$ such that $\mathbf{1}_A = \alpha_1 + \cdots + \alpha_n$ for every $1 \leq i \leq n$, where we can assume that $\alpha_i \geq \mathbf{0}$, by triangle property in f-rings, with at least one of them being nonzero. Assume that $\alpha_1 \neq \mathbf{0}$, say. We show that $\alpha_1 = \mathbf{1}_A$ and $\alpha_i = \mathbf{0}_A$ for any $i \neq 1$. The case in which $n = 1$ is trivial. If $n = 2$, then $I_{r_1} + I_{r_2} = A$ with $\mathbf{1}_A = \alpha + \beta$ for some positive $\alpha \in I_{r_1}$ and nonnegative $\beta \in I_{r_2}$. Applying the notation $[a]$ for the ℓ-ideal generated by $a \in A$, we can write

$$\alpha \wedge \beta \in \overline{[\alpha \wedge \beta]} = \overline{[\alpha] \cap [\beta]} = \overline{[\alpha]} \cap \overline{[\beta]} \subseteq I_{r_1} \cap I_{r_2} = \{\mathbf{0}_A\}\,.$$

So that $\alpha \wedge \beta = \mathbf{0}_A$. Moreover,

$$\alpha \wedge \beta = \frac{\mathbf{1}}{\mathbf{2}}(\alpha + \beta - |\alpha - \beta|) = \frac{\mathbf{1}}{\mathbf{2}}(\mathbf{1} - |\mathbf{1} - \mathbf{2}\beta|)$$

So $|\mathbf{1} - \mathbf{2}\beta| = \mathbf{1}$, whence $(\mathbf{1} - \mathbf{2}\beta)(r) = \pm 1$ for all r in $[0,1]$ and we conclude, using the continuity of $\mathbf{1} - \mathbf{2}\beta$, that $\mathbf{1} - \mathbf{2}\beta = \mathbf{1}$ or $\mathbf{1} - \mathbf{2}\beta = -\mathbf{1}$. Consequently, $\beta = \mathbf{0}$ or $\beta = \mathbf{1}$, respectively. But $\beta = \mathbf{1}$ is impossible because $\alpha \wedge \beta = \mathbf{0}_A$. If $\beta = \mathbf{0}$, then $\alpha = \mathbf{1}$, as desired.

Next, if $n \geq 3$, then

$$I_{r_1} \cap (I_{r_2} + \cdots + I_{r_n}) = \overline{I_{r_1}} \cap (\overline{I_{r_2} \vee \cdots \vee I_{r_n}}) = \overline{(I_{r_1} \wedge I_{r_2}) \vee \cdots \vee (I_{r_1} \wedge I_{r_n})} = 0\,.$$

So that $\alpha_1 \wedge (\alpha_2 + \cdots + \alpha_n) = \mathbf{0}$. Let $\alpha := \alpha_1$, $\beta := \alpha_2 + \cdots + \alpha_n$, and using an argument similar to the latter case above, we conclude that $\alpha = \alpha_1 = \mathbf{1}$ and $\beta = \alpha_2 + \cdots + \alpha_n = \mathbf{0}$. Hence $\alpha_2 = \cdots = \alpha_n = \mathbf{0}$, Since the α_i's are all nonnegative. Consequently, $f(\{r_1\}) = I_{r_1} = A$, and for every $s \neq r_1$, $f(\{s\}) = f(\{s\}) \cap A = f(\{s\}) \cap f(\{r_1\}) = \{\mathbf{0}_A\}$. So every frame map in $F(C\mathcal{L}(A))$ is constant.

Now, by Proposition 6 of [4] and Proposition 4.1 of [3], since A is an Archimedean, strong, and bounded f-ring over $\mathbb{Q}$, A is isomorphic to a subring of $\mathcal{R}(C\mathcal{L}(A))$; indeed, A is isomorphic to $\mathcal{R}(C\mathcal{L}(A))$ since A is complete in its uniform topology (see [4], p.36). Furthermore, the the image of a constant function in $F(C\mathcal{L}(A))$ under the embedding defined in Theorem 6.1 is a constant function in $\mathcal{R}(C\mathcal{L}(A))$, whence $F(C\mathcal{L}(A)) \neq \mathcal{R}(C\mathcal{L}(A))$.

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