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Properties of products for flatness in the category of S-posets

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Abstract. This paper is devoted to the study of products of classes of right S-posets possessing one of the flatness properties and preservation of such properties under products. Specifically, we characterize a pomonoid S over which its nonempty products as right S-posets satisfy some known flatness properties. Generalizing this results, we investigate products of right S-posets satisfying Condition (PWP). Finally, we investigate pomonoids over which products of right S-posets transfer an arbitrary flatness property, projectivity, freeness, and regularity to their components.

1 Introduction

Over the past several decades, a chunk of literatures has been allocated to the flatness properties of acts over monoids. After switching these properties to their conterparts in the ordered algebraic structures by Fakhruddin in 1986 ([6, 7]), a great deal of investigation was devoted to the cognition of these notions in the context of ordered structures and to the results derived from the former known ones. Following this pattern, in this paper, we investigate preservation and reflection of some flatness properties such as

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weak flatness, GP-(po-)flatness, Conditions (WP) and $(WP)_w$.

For a monoid S a right S-act is a nonempty set A together with a map $A \times S \to A$, $(a, s) \rightsquigarrow as$, such that $a1 = a$ and $a(st) = (as)t$. A monoid S endowed with a partial order, compatible with the binary operation, is called a pomonoid. For a pomonoid S, a right S-poset is a poset A which is also a right S-act whose action $A \times S \to A$ is monotone in both arguments. Left S-posets are defined analogously. In the sequel, the term S-poset is used simply to indicate a right S-poset. An S-subposet of an S-poset A_S is a (nonempty) subset of A which is closed under the action of S and denoted by $B \leq A$. Moreover, S-poset morphisms or simply S-morphisms are monotone maps between S-posets which preserve actions. The classes of S-posets and S-morphisms form a category, denoted by $S - POS$, which comprises the main background of this work. For an account on this category and categorical notions used in this paper, the reader is referred to [5].

The subkernel of an S-poset morphism $f : A_S \longrightarrow B_S$ is defined by $\overrightarrow{ker} f := \{(a, a') \in A \times A : f(a) \leq f(a')\}$. An S-poset A_S is called flat (po-flat) if for every left S-poset ${}_SB$ and for all pairs $(a, b), (a', b')$ in $A \times B$, the equality (inequality) $a \otimes b = a' \otimes b'$ ($a \otimes b \leq a' \otimes b'$) in $A_S \otimes_S B$ implies the same equality (inequality) in $A_S \otimes_S (Sb \cup Sb')$. An S-poset A_S is called weakly flat (po-flat) if for $a, b \in A_S, s, t \in S$ the equality (inequality) $as = bt$ ($as \leq bt$) in A_S implies $a \otimes s = b \otimes t$ ($a \otimes s \leq b \otimes t$) in $A_S \otimes {}_S(Ss \cup St)$. Putting $s = t$ in the foregoing definition yields principally weakly flat (po-flat) notion.

Remark 1.1. It is crucial to notice that for S-posets A_S and ${}_SB$ and $(a, b), (a', b') \in A \times B$, $a \otimes b \leq a' \otimes b'$ in $A_S \otimes {}_SB$ if and only if there exists a *scheme* of the form

$$\begin{array}{cc} a \leq a_1 u_1 & \\ a_1 v_1 \leq a_2 u_2 & u_1 b \leq v_1 b_2 \\ \vdots & \vdots \\ a_n v_n \leq a' & u_n b_n \leq v_n b' \end{array}$$

where for $1 \leq i \leq n$, $a_i \in A$, $b_i \in B, u_i, v_i \in S$. In this case we shall call n the length of the scheme and it should be mention that, by adding iterating inequalities, the length of the scheme can be increased to m for each natural number $m \geq n$.

Obviously, $a\otimes b$ and $a'\otimes b'$ are equal in $A_S\otimes_S B$ if and only if $a\otimes b \leq a'\otimes b'$ and $a'\otimes b' \leq a\otimes b$.

An S-poset A_S satisfies Condition (P_w) if, for all $a, b \in A$ and $s, t \in S$, $as \leq bt$ implies $a \leq a'u,\ a'v \leq b, us \leq vt$ for some $a' \in A,\ u, v \in S$. An S-poset A_S satisfies Condition (P) if, for all $a, b \in A$ and $s, t \in S$, $as \leq bt$ implies $a = a'u, b = a'v, us \leq vt$ for some $a' \in A,\ u, v \in S$, and it satisfies Condition (E) if, for all $a \in A$ and $s, t \in S,\ as \leq at$ implies $a = a'u, us \leq ut$ for some $a' \in A,\ u \in S$. An S-poset is called strongly flat if it satisfies both Conditions (P) and (E). Projectivity is defined in the standard categorical manner. For a detailed account of the ingredients needed in this paper we refer the reader to [1, 6, 7, 17].

A pioneering work in the background of this paper goes back to [3], therein Bulman-Fleming investigated coherent and weakly coherent monoids in special cases. In [14] Sedaghatjoo et al. characterized principally weakly and weakly coherent monoids in general case. In [4], Bulman-Fleming characterized monoids over which products of projective acts are projective and in [2], the authors investigated flatness properties of $S \times S$ for a monoid S. Then in [12] some properties of products of S-acts are discussed. In light of S-posets as a generalization of S-acts, a large portion of literatures in the theory of semigroups and their actions has been accumulated to the flatness properties of S-posets, for instance [1, 5, 11, 15–17].

Meanwhile, pursuing the investigations, in [9] Khosravi studied pomonoids over which flatness properties such as strong flatness, Condition (P), Condition (E), Condition (P_w), weak po-flatness and principal weak po-flatness of S-posets are preserved under products. In [10] products of S-posets satisfying Condition $(PWP)_w$ are discussed.

Hereby, continuing these researches, in this paper we investigate products of GP-po-flat, GP-flat, and weakly flat S-posets. Besides, preservation of Conditions (PWP), (WP), and $(WP)_w$ under products are investigated. Ultimately, we reply to the question of when products of S-posets transfer flatness properties, projectivity, freeness, and regularity to their components.

If S is a pomonoid, the cartesian product S^Γ is an S-poset equipped with the componentwise order and action, where Γ is a nonempty set. Moreover, $(s_\gamma)_{\gamma\in\Gamma} \in S^\Gamma$ is denoted simply by (s_γ), and the S-poset $S \times S$ will be denoted by $D(S)$.

2 GP-po-flat, GP-flat, weakly flat

We recall, from [13], that an S-poset A_S is called GP-po-flat, if for every $s \in S$, and $a, a' \in A_S$, $a \otimes s \leq a' \otimes s$ in $A_S \otimes {}_S S$ implies the existence of a natural number m such that $a \otimes s^m \leq a' \otimes s^m$ in $A_S \otimes {}_S S s^m$. Similarly, GP-flat can be defined by replacing $\leq$ by $=$ in the foregoing definition. It is obvious that every principally weakly po-flat S-poset is GP-po-flat, but not the converse. In this section we first concentrate on products of GP-po-flat and GP-flat S-posets over left PSF pomonoids. Then we give equivalent conditions for S^Γ, for each nonempty set Γ, to be GP-(po-)flat or weakly flat. The following is needed to characterize GP-po-flatness.

Lemma 2.1. ([13]) *An S-poset A_S is GP-po-flat if and only if for every $s \in S$, and $a, a' \in A_S$, $a \otimes s \leq a' \otimes s$ in $A_S \otimes {}_S S$ implies that there exist $m, n \in \mathbb{N}, a_1, ..., a_n \in A, u_1, ..., u_n, v_1, ..., v_n \in S$ such that*

$$\begin{array}{cc} a \leq a_1 u_1 & \\ a_1 v_1 \leq a_2 u_2 & u_1 s^m \leq v_1 s^m \\ \vdots & \vdots \\ a_n v_n \leq a' & u_n s^m \leq v_n s^m. \end{array}$$

Recall that a pomonoid S is called left PSF if all principal left ideals of S are strongly flat. For a pomonoid S an element $u \in S$ is called *right semi-po-cancellable* if for $s, t \in S, su \leq tu$ implies that there exists $r \in S$ such that $ru = u,\ sr \leq tr$. It can be readily checked that a pomonoid S is left PSF if and only if every element of S is right semi-po-cancellable.

Lemma 2.2. *Over a left PSF pomonoid S an S-poset A_S is GP-po-flat if and only if for any $a, a' \in A_S, s \in S$, if $as \leq a's$, then there exist $r \in S$ and $m \in \mathbb{N}$ such that $rs^m = s^m$ and $ar \leq a'r$.*

Proof. Let A_S be GP-po-flat and $as \leq a's$ for $s \in S, a, a' \in A_S$. So $a \otimes s \leq a' \otimes s$ in $A_S \otimes {}_S S$ and Lemma 2.1 implies that there exist $m, n \in \mathbb{N}, a_1, ..., a_n \in$

$A, u_1, ..., u_n, v_1, ..., v_n \in S$ such that

$$\begin{array}{rl} a \leq a_1u_1 & \\ a_1v_1 \leq a_2u_2 & u_1s^m \leq v_1s^m \\ \vdots & \vdots \\ a_nv_n \leq a' & u_ns^m \leq v_ns^m. \end{array}$$

Regarding the inequality $u_1s^m \leq v_1s^m$, there exists $r_1 \in S$ such that $r_1s^m = s^m$ and $u_1r_1 \leq v_1r_1$. So $u_2r_1s^m \leq v_2r_1s^m$ implies that there exists $r_2 \in S$ such that $r_2s^m = s^m$ and $u_2r_1r_2 \leq v_2r_1r_2$. Carrying this process on we reach to $r = r_1...r_n \in S$ such that $rs^m = s^m$ and $u_ir \leq v_ir$ for $1 \leq i \leq n$. Therefore,

$$ar \leq a_1u_1r \leq a_1v_1r \leq a_2u_2r \leq ... \leq a'r,$$

as desired. The converse is obvious. □

Lemma 2.3. *Over a left PSF pomonoid S an S-poset A_S is GP-flat if and only if for any $a, a' \in A_S, s \in S$, if $as = a's$, there exist $r \in S$ and $m \in \mathbb{N}$ such that $rs^m = s^m$ and $ar = a'r$.*

Proof. Let $as = a's$ for $s \in S, a, a' \in A_S$. Since A_S is GP-flat, $a \otimes s^m = a' \otimes s^m$ in $A_S \otimes_S Ss^m$. So there exist $k, n \in \mathbb{N}, a_i, a'_j \in A$ and $u_i, u'_j, v_i, v'_j \in S$, for $1 \leq i \leq n,\ 1 \leq j \leq k$ such that

$$\begin{array}{rl} a \leq a_1u_1 & \\ a_1v_1 \leq a_2u_2 & u_1s^m \leq v_1s^m \\ \vdots & \vdots \\ a_nv_n \leq a' & u_ns^m \leq v_ns^m \end{array}$$

$$\begin{array}{rl} a' \leq a'_1u'_1 & \\ a'_1v'_1 \leq a'_2u'_2 & u'_1s^m \leq v'_1s^m \\ \vdots & \vdots \\ a'_kv'_k \leq a & u'_ks^m \leq v'_ks^m. \end{array}$$

Applying the argument used in the proof of the foregoing lemma for the right column of the scheme we get $r \in S$ such that $rs^m = s^m$, $u_is^m \leq v_is^m$

and $u'_j s^m \leq v'_j s^m$ for $1 \leq i \leq n,\ 1 \leq j \leq k$. Therefore,

$$ar \leq a_1u_1r \leq a_1v_1r \leq a_2u_2r \leq ... \leq a'r \leq a'_1u'_1r \leq a'_1v'_1r \leq ... \leq ar,$$

and so $ar = a'r$. The converse is clear. □

As a result of the above lemma we deduce the following corollary.

Corollary 2.4. *Let S be a left PSF pomonoid and $A_i, 1 \leq i \leq n$ be S-posets. Then $\prod_{i=1}^n A_i$ is GP-po-flat if and only if the inequality $(a_1, ..., a_n)s \leq (a'_1, ..., a'_n)s$, for $a_i, a'_i \in A_i,\ 1 \leq i \leq n, s \in S$, implies the existence of $r \in S$ and $m \in \mathbb{N}$ such that $rs^m = s^m$ and $(a_1, ..., a_n)r \leq (a'_1, ..., a'_n)r$.*

The next theorem states that, for left PSF pomonoids, flatness properties such as GP-flatness, GP-po-flatness, principal weak flatness and principal weak po-flatness are preserved under finite products.

Theorem 2.5. *Let S be a left PSF pomonoid and $A_i, 1 \leq i \leq n$ be S-posets. Then we have the following assertions.*

(i) *If A_i is GP-po-flat for $1 \leq i \leq n$, then $\prod_{i=1}^n A_i$ is GP-po-flat.*

(ii) *If A_i is GP-flat for $1 \leq i \leq n$, then $\prod_{i=1}^n A_i$ is GP-flat.*

(iii) *If A_i is principally weakly po-flat for $1 \leq i \leq n$, then $\prod_{i=1}^n A_i$ is principally weakly po-flat.*

(iv) *If A_i is principally weakly flat for $1 \leq i \leq n$, then $\prod_{i=1}^n A_i$ is principally weakly flat.*

Proof. (i): Suppose that $(a_1, ..., a_n)s \leq (a'_1, ..., a'_n)s$, for $s \in S, a_i, a'_i \in A_i,\ 1 \leq i \leq n$. Then $a_1s \leq a'_1s$, and since S is a left PSF pomonoid, there exist $r_1 \in S$ and $m_1 \in \mathbb{N}$ such that $r_1s^{m_1} = s^{m_1}$ and $a_1r_1 \leq a'_1r_1$. The inequality $a_2r_1s^{m_1} \leq a'_2r_1s^{m_1}$ gives $r_2 \in S$ and $m_2 \in \mathbb{N}$ such that $r_2(s^{m_1})^{m_2} = (s^{m_1})^{m_2}$ and $a_2r_1r_2 \leq a'_2r_1r_2$. Continuing this process, we obtain $r_1, \ldots, r_n \in S, m_1, ..., m_n \in \mathbb{N}$ with $r_is^{m_1 \ldots m_i} = s^{m_1 \ldots m_i}$ and $a_ir_1 \ldots r_i \leq a'_ir_1 \ldots r_i$ for each $1 \leq i \leq n$. Put $r = r_1 \ldots r_n$ and $m = m_1...m_n$. Thus $(a_1, \ldots, a_n)r \leq (a'_1, \ldots, a'_n)r$ and $rs^m = s^m$.

Applying Lemma 2.3(ii) is proved analogously.

Putting $m_i = 1, 1 \leq i \leq n$, (iii) and (iv) are proved in the same manners used for (i) and (ii). □

Now, Theorem 2.5 provides another approach to Proposition 2.3 in [9].

Corollary 2.6. *If S is a left PSF pomonoid, then the S-poset S^n is principally weakly flat (GP-po-flat) for each $n \in \mathbb{N}$.*

Now, we engage in GP-po-flatness, GP-flatness and weak flatness of S^Γ.

Proposition 2.7. *Let S be a pomonoid. Then S^Γ is GP-po-flat for each nonempty set Γ if and only if for any $s \in S$ there exist $(s_1, t_1), \dots, (s_n, t_n) \in D(S)$ and $m \in \mathbb{N}$ such that $s_i s^m \leq t_i s^m$ for all $1 \leq i \leq n$, and for $(u, v) \in D(S)$, $us \leq vs$ implies the existence of $u_1, ..., u_n \in S$ such that*

$$\begin{aligned} u &\leq u_1 s_1 \\ u_1 t_1 &\leq u_2 s_2 \\ &\vdots \\ u_n t_n &\leq v. \end{aligned}$$

Proof. **Necessity.** Let $L = \{(u, v) \in D(S) |\ us \leq vs\}$, and index it by a set Γ as $L = \{(u_\gamma, v_\gamma) |\ \gamma \in \Gamma\}$. Since $(u_\gamma)s \leq (v_\gamma)s$ in S^Γ, by assumption $(u_\gamma) \otimes s^m \leq (v_\gamma) \otimes s^m$ in $S^\Gamma \otimes Ss^m$ for some $m \in \mathbb{N}$, which implies that there exist $s_1, \dots, s_n, t_1, \dots, t_n \in S$, $(u^1_\gamma), \dots, (u^n_\gamma) \in S^\Gamma$ such that

$$\begin{array}{rl} (u_\gamma) \leq (u^1_\gamma)s_1 & \\ (u^1_\gamma)t_1 \leq (u^2_\gamma)s_2 & s_1 s^m \leq t_1 s^m \\ \vdots & \vdots \\ (u^n_\gamma)t_n \leq (v_\gamma) & s_n s^m \leq t_n s^m. \end{array}$$

Now the result follows immediately.

Sufficiency. Let $\Gamma \neq \emptyset$ and $(u_\gamma)s \leq (v_\gamma)s$ for $(u_\gamma), (v_\gamma) \in S^\Gamma$. Our assumption implies the existence of $m \in \mathbb{N}$ and (s_1, t_1), $..., (s_n, t_n) \in D(S)$ such that for $1 \leq i \leq n$, $s_i s^m \leq t_i s^m$ and for each $\gamma \in \Gamma$ there exist $u^1_\gamma, ..., u^n_\gamma \in S$ such that

$$\begin{aligned} u_\gamma &\leq u^1_\gamma s_1 \\ u^1_\gamma t_1 &\leq u^2_\gamma s_2 \\ &\vdots \\ u^n_\gamma t_n &\leq v_\gamma. \end{aligned}$$

Thus $(u_\gamma)\otimes s^m \le (u^1_\gamma)s_1 \otimes s^m \le (u^1_\gamma)\otimes s_1 s^m \le (u^1_\gamma)\otimes t_1 s^m \le (u^1_\gamma)t_1 \otimes s^m \le (u^2_\gamma)s_2 \otimes s^m \le ... \le (v_\gamma)\otimes s^m$ in $S^\Gamma \otimes Ss^m$, as required. □

Similar to the proof of the previous proposition and in light of the Remark 1.1, the following result is obtained.

Proposition 2.8. *Let S be a pomonoid. Then S^Γ is GP-flat for each nonempty set Γ if and only if for any $s \in S$ there exist $(s_i, t_i), (s'_i, t'_i) \in D(S)$, $1 \le i \le n$ and $m \in \mathbb{N}$ such that $s_i s^m \le t_i s^m, s'_i s^m \le t'_i s^m$ for all $1 \le i \le n$ and for $(u, v) \in D(S)$, $us = vs$ implies the existence of $u_1, ..., u_n, v_1, \ldots, v_n \in S$ such that*

$$\begin{aligned} u &\le u_1 s_1 \\ u_1 t_1 &\le u_2 s_2 \\ &\vdots \\ u_n t_n \le v &\le v_1 s'_1 \\ v_1 t'_1 &\le v_2 s'_2 \\ &\vdots \\ v_n t'_n &\le u. \end{aligned}$$

If $Ss \cap (St] \neq \emptyset$, $\{(as, a't) |\ as \le a't\}$ is denoted by $H(s, t)$. Recall that finitely generated left S-poset ${}_SB$ is called finitely definable (FD) if the S-morphism $S^\Gamma \otimes B \longrightarrow B^\Gamma$, given by $(s_\gamma)_\Gamma \otimes b \mapsto (s_\gamma b)_\Gamma$, is order-embedding for all nonempty sets Γ. Theorem 2.7 of [9], using finitely definable left ideals, gives the equivalent conditions for which S^Γ is weakly po-flat S-poset for each $\Gamma \neq \emptyset$. Similar considerations can be applied to weak flatness.

Definition 2.9. Let S be a pomonoid. A finitely generated left S-poset ${}_SB$ is called *weakly finitely definable (WFD)* if the S-morphism $S^\Gamma \otimes B \longrightarrow B^\Gamma$ is a monomorphism for each nonempty set Γ.

The next theorem gives characterization of pomonoids over which S^Γ is weakly flat for each nonempty set Γ.

Theorem 2.10. *For a pomonoid S, the following are equivalent:*

(i) *S^Γ is a weakly flat S-poset for each $\Gamma \neq \emptyset$.*

(ii) *Every finitely generated left ideal of S is WFD.*

(iii) *Ss is WFD for each $s \in S$, and*

for every $s, t \in S$, if $Ss \cap St \neq \emptyset$, then $\Delta_{Ss\cap St} \subseteq S(p,q) \cap S(q',p')$ for some $(p,q) \in H(s,t)$ and $(q',p') \in H(t,s)$.

Proof. The equivalence of (i) and (ii) is clear.

(i)$\Rightarrow$(iii): The first part is obvious. Let $s, t \in S$ such that $Ss \cap St \neq \emptyset$. Index the set $Ss \cap St$ by a set Γ as $\{u_\gamma s (= v_\gamma t) | \ \gamma \in \Gamma\}$. Since $S^\Gamma \otimes (Ss \cup St) \to (Ss \cup St)^\Gamma$ is a monomorphism and $(u_\gamma)s = (v_\gamma)t$, then $(u_\gamma) \otimes s = (v_\gamma) \otimes t$ in $S^\Gamma \otimes (Ss \cup St)$. So there exist $s_i, t_i, s'_j, t'_j \in S$, $(u^i_\gamma), (v^j_\gamma) \in S^\Gamma, 1 \leq i \leq n, 1 \leq j \leq m, b_2, \ldots, b_n, c_2, ..., c_m \in Ss \cup St$ such that

$$\begin{array}{rl} (u_\gamma) \leq (u^1_\gamma)s_1 & \\ (u^1_\gamma)t_1 \leq (u^2_\gamma)s_2 & s_1 s \leq t_1 b_2 \\ \vdots & \vdots \\ (u^n_\gamma)t_n \leq (v_\gamma) & s_n b_n \leq t_n t \end{array}$$

$$\begin{array}{rl} (v_\gamma) \leq (v^1_\gamma)s'_1 & \\ (v^1_\gamma)t'_1 \leq (v^2_\gamma)s'_2 & s'_1 t \leq t'_1 c_2 \\ \vdots & \vdots \\ (v^m_\gamma)t'_m \leq (u_\gamma) & s'_m c_m \leq t'_m s. \end{array}$$

Let k and r be the smallest integers such that $b_k \in St$ and $c_r \in Ss$. So $b_{k-1} \in Ss$ and $c_{r-1} \in St$. Take $p = s_{k-1}b_{k-1}$, $q = t_{k-1}b_k$, $q' = s'_{r-1}c_{r-1}$ and $p' = t'_{r-1}c_r$. Thus

$$\begin{gathered} (u_\gamma)s \leq (u^1_\gamma)s_1 s \leq (u^1_\gamma)t_1 b_2 \leq (u^2_\gamma)s_2 b_2 \leq \ldots \leq \\ (u^{k-1}_\gamma)s_{k-1}b_{k-1} \leq (u^{k-1}_\gamma)t_{k-1}b_k \leq \ldots \leq (v_\gamma)t \leq (v^1_\gamma)s'_1 s \\ \leq \ldots \leq (v^{r-1}_\gamma)s'_{r-1}c_{r-1} \leq (v^{r-1}_\gamma)t'_{r-1}c_r \leq \ldots \leq (u_\gamma)s. \end{gathered}$$

Then $(u_\gamma)s = (u^{k-1}_\gamma)p = (u^{k-1}_\gamma)q = (v^{r-1}_\gamma)p' = (v^{r-1}_\gamma)q' = (v_\gamma)t$. Now it can be easily checked that $\Delta_{Ss\cap St} \subseteq S(p,q) \cap S(q',p')$ for $(p,q) \in H(s,t)$ and $(q',p') \in H(t,s)$.

(iii)$\Rightarrow$(i): Let I be a left ideal of S and $(u_\gamma)s = (v_\gamma)t$ for some $(u_\gamma), (v_\gamma) \in S^\Gamma$, $s, t \in I$. Since $\Delta_{Ss\cap St} \subseteq S(p,q) \cap S(q',p')$ for some $(p,q) \in H(s,t)$ and $(q',p') \in H(t,s)$, for each $\gamma \in \Gamma$ there exist $w_\gamma, w'_\gamma \in S$ such that $u_\gamma s = w_\gamma p = w_\gamma q = w'_\gamma p' = w'_\gamma q' = v_\gamma t$. Take $p = cs, q = dt$, $p' = c's$ and

$q' = d't$ for some $c, d, c', d' \in S$. Since Ss and St are WFD, the equalities $(u_\gamma)s = (w_\gamma c)s = (w'_\gamma)p'$ and $(w_\gamma d)t = (w'_\gamma)q' = (v_\gamma)t$ imply the equalities $(u_\gamma) \otimes s = (w_\gamma c) \otimes s = (w'_\gamma c') \otimes s$ and $(w_\gamma d) \otimes t = (w'_\gamma d') \otimes t = (v_\gamma) \otimes t$ in $S^\Gamma \otimes Ss$ and $S^\Gamma \otimes St$, respectively. Therefore

$$\begin{aligned}(u_\gamma) \otimes s = (w_\gamma c) \otimes s = (w_\gamma) \otimes cs \leq (w_\gamma) \otimes dt = (w_\gamma d) \otimes t = (v_\gamma) \otimes t \\ = (w'_\gamma d') \otimes t = (w'_\gamma) \otimes d't \leq (w'_\gamma) \otimes c's = (w'_\gamma c') \otimes s = (u_\gamma) \otimes s\end{aligned}$$

in $S^\Gamma \otimes (Ss \cup St)$. Thus $(u_\gamma) \otimes s = (v_\gamma) \otimes t$ in $S^\Gamma \otimes (Ss \cup St)$. □

3 Conditions (PWP), (WP), $(WP)_w$

Conditions (PWP), (WP), and $(WP)_w$ were introduced in [8] which we need to recall them here. An S-poset A_S satisfies Condition (PWP) if for all $a, a' \in A, t \in S$, the inequality $at \leq a't$ implies the existence of $a'' \in A$, $u, v \in S$ such that $a = a''u,\ a' = a''v, ut \leq vt$. An S-poset A_S satisfies Condition (WP) if for all $s, t \in S, a, a' \in A_S$ and any homomorphism $f : {}_SSs \cup St \longrightarrow {}_SS$, the inequality $af(s) \leq a'f(t)$ implies the existence of $a'' \in A_S$, $p, q \in Ss \cup St$ such that $f(p) \leq f(q)$, $a \otimes s = a'' \otimes p$, and $a' \otimes t = a'' \otimes q$ in $A_S \otimes (Ss \cup St)$. Moreover, A_S satisfies Condition $(WP)_w$ if for all $s, t \in S, a, a' \in A_S$ and any homomorphism $f : {}_SSs \cup St \longrightarrow {}_SS$, the inequality $af(s) \leq a'f(t)$ implies the existence of $a'' \in A_S$, $p, q \in Ss \cup St$ such that $f(p) \leq f(q)$, $a \otimes s \leq a'' \otimes p$, and $a'' \otimes q \leq a' \otimes t$ in $A_S \otimes (Ss \cup St)$. In this section we focus our attention on products of S-posets satisfying Conditions (PWP), (WP), and $(WP)_w$.

The ordered version of a locally cyclic act is called a *weakly locally cyclic* S-poset for which every finitely generated S-subposet is contained in a cyclic S-subposet. Moreover, a left ideal of S which is also weakly locally cyclic is called *weakly locally principal left ideal.* By virtue of the terminology used in [9], the set $L(a, a) = \{(u, v) \in D(S) |\ ua \leq va\}$ is a left S-subposet of $D(S)$.

Proposition 3.1. *For any pomonoid S, the following are equivalent:*

(i) *Any finite product of S-posets satisfying Condition (PWP) satisfies Condition (PWP).*

(ii) *The diagonal S-poset $D(S)$ satisfies Condition (PWP).*

(iii) *For every $a \in S$ the set $L(a, a)$ is a weakly locally cyclic left S-poset.*

Proof. (i)$\Rightarrow$(ii) is clear.

(ii)$\Rightarrow$(iii): Let $(u,v),(u',v') \in L(a,a)$, for $a \in S$. Since $ua \leq va$ and $u'a \leq v'a$, we have $(u,u')a \leq (v,v')a$, and our assumption implies that there exist $(w,w') \in D(S), p,q \in S$ such that $(w,w')p = (u,u')$, $(w,w')q = (v,v')$ and $pa \leq qa$. So $(u,v),(u',v') \in S(p,q) \subseteq L(a,a)$, and it follows that $L(a,a)$ is weakly locally cyclic.

(iii)$\Rightarrow$(i): Suppose that $A_1,...,A_n$ are S-posets each satisfying Condition (PWP). Let $(a_1,...,a_n)u \leq (a'_1,...,a'_n)u$ for $(a_1,...,a_n),(a'_1,...,a'_n) \in \prod_{i=1}^n A_i, u \in S$. For each $1 \leq i \leq n$, $a_i u \leq a'_i u$ implies the existence $a''_i \in A_i, p_i, q_i \in S$ such that $a''_i p_i = a_i$, $a''_i q_i = a'_i$, and $p_i u \leq q_i u$. Then $(p_i,q_i) \in L(u,u)$ for each $1 \leq i \leq n$. Now, by assumption, there exists $(p,q) \in L(u,u)$ such that $(p_i,q_i) \in S(p,q)$. Suppose that $(p_i,q_i) = w_i(p,q)$ for $w_i \in S$, $1 \leq i \leq n$. Then $(a_1,...,a_n) = (a''_1 w_1,...,a''_n w_n)p$, $(a'_1,...,a'_n) = (a''_1 w_1,...,a''_n w_n)q$, and $pu \leq qu$, proving that $\prod_{i=1}^n A_i$ satisfies Condition (PWP). □

The next theorem presents equivalent conditions on a pomonoid S for products of nonempty families of S-posets to satisfy Condition (PWP).

Theorem 3.2. *For a pomonoid S, the following are equivalent:*

(i) *Products of nonempty families of S-posets satisfying Condition (PWP) satisfy Condition (PWP).*

(ii) *S^Γ satisfies Condition (PWP) for each nonempty set Γ.*

(iii) *For every $a \in S$ the set $L(a,a)$ is a cyclic left S-poset.*

Proof. (i)$\Rightarrow$(ii) is clear. (ii)$\Rightarrow$(iii): Suppose that $a \in S$ and index the set $L(a,a)$ by $\{(u_\gamma,v_\gamma) | \gamma \in \Gamma\}$. Since $(u_\gamma)a \leq (v_\gamma)a$ and S^Γ satisfies Condition (PWP), there exist $p,q \in S, (z_\gamma) \in S^\Gamma$ such that $pa \leq qa$, $(u_\gamma) = (z_\gamma)p$ and $(v_\gamma) = (z_\gamma)q$. Thus $(p,q) \in L(a,a)$ and for each $\gamma \in \Gamma$, $(u_\gamma,v_\gamma) = z_\gamma(p,q)$, which prove that $L(a,a)$ is cyclic.

(iii)$\Rightarrow$(i): Let $\{A_i \mid i \in I\}$ be a family of S-posets satisfying Condition (PWP) and $A = \prod_{i \in I} A_i$. Suppose that $(x_i)a \leq (y_i)a$ where $a \in S$ and $(x_i),(y_i) \in A$. For each $i \in I$, since A_i satisfies Condition (PWP), the inequality $x_i a \leq y_i a$ implies the existence of $u_i, v_i \in S, z_i \in A_i$ providing $u_i a \leq v_i a$, $x_i = z_i u_i$, $y_i = z_i v_i$. So $(u_i,v_i) \in L(a,a) = S(p,q)$. Thus for each $i \in I$ there exists $r_i \in S$ such that $(u_i,v_i) = r_i(p,q)$ and hence $x_i = z_i r_i p$, $y_i = z_i r_i q$. Therefore $(x_i) = (z_i r_i)p$, $(y_i) = (z_i r_i)q$ and $pa \leq qa$. □

In what follows we present the equivalent conditions for S^Γ to satisfy Condition (WP) or $(WP)_w$.

Proposition 3.3. *For a pomonoid S, the following are equivalent:*

(i) S^Γ *satisfies Condition* (WP) *for each* $\Gamma \neq \emptyset$.

(ii) *Every finitely generated left ideal of S is WFD, and for any $s,t \in S$ and homomorphism $f : {}_S Ss \cup St \longrightarrow {}_S S$, if*

$$L_f(s,t) = \{(us,vt)|\ (us,vt) \in \overrightarrow{ker} f\} \neq \emptyset,$$

then $L_f(s,t) \subseteq S(p,q)$ for some $(p,q) \in \overrightarrow{ker} f$.

Proof. (i)$\Rightarrow$(ii): By Theorem 2.10, the first part is immediate. Let $f : {}_S Ss \cup St \longrightarrow {}_S S$ be a homomorphism for $s,t \in S$. Suppose that $L_f(s,t) \neq \emptyset$ and index it by the set $\{(u_\gamma s, v_\gamma t)|\ \gamma \in \Gamma\}$. Since S^Γ satisfies Condition (WP), the inequality $(u_\gamma)f(s) \leq (v_\gamma)f(t)$ implies that there exist $(z_\gamma) \in S^\Gamma, p, q \in Ss \cup St$ such that $f(p) \leq f(q)$, $(u_\gamma) \otimes s = (z_\gamma) \otimes p$ and $(v_\gamma) \otimes t = (z_\gamma) \otimes q$ in $S^\Gamma \otimes (Ss \cup St)$. Clearly $(u_\gamma)s = (z_\gamma)p$ and $(v_\gamma)t = (z_\gamma)q$ in S^Γ, which imply that $L_f(s,t) \subseteq S(p,q)$.

(ii)$\Rightarrow$(i): Let $s,t \in S$ and $f : {}_S Ss \cup St \longrightarrow {}_S S$ be a homomorphism. Suppose that $(u_\gamma)f(s) \leq (v_\gamma)f(t)$ in S^Γ. So $(u_\gamma s, v_\gamma t) \in L_f(s,t) \neq \emptyset$ and, by assumption, $L_f(s,t) \subseteq S(p,q)$ for some $(p,q) \in \overrightarrow{ker} f$. Clearly $f(p) \leq f(q)$, and $(u_\gamma s, v_\gamma t) = z_\gamma(p,q)$ for each $\gamma \in \Gamma$. Thus, $(u_\gamma)s = (z_\gamma)p$ and $(v_\gamma)t = (z_\gamma)q$ in $(Ss \cup St)^\Gamma$. Since $(Ss \cup St)$ is WFD we deduce that $(u_\gamma) \otimes s = (z_\gamma) \otimes p$ and $(v_\gamma) \otimes t = (z_\gamma) \otimes q$ in $S^\Gamma \otimes (Ss \cup St)$, as required. $\square$

An adaptation of Proposition 3.3 in the category of S-acts gives the following proposition.

Proposition 3.4. *For a monoid S, the following are equivalent:*

(i) *The S-act S^Γ satisfies Condition (WP) for each $\Gamma \neq \emptyset$.*

(ii) *Every finitely generated left ideal of S is FD, and for any $s,t \in S$ and homomorphism $f : {}_S Ss \cup St \longrightarrow {}_S S$, if $L_f(s,t) = (Ss \times St) \cap ker f \neq \emptyset$, then $L_f(s,t) \subseteq S(p,q)$ for some $(p,q) \in ker f$.*

Herein, we need to use the term $\widehat{S(p,q)}$ for a pair (p,q) in $D(S)$, introduced in [9], indicating the left S-poset $\{(u,v) \in D(S) | \ \exists w \in S, u \leq wp, wq \leq v\}$ containing the cyclic S-poset $S(p,q)$.

Proposition 3.5. *For a pomonoid S, the following are equivalent:*

(i) S^{Γ} *satisfies Condition* $(WP)_w$ *for each* $\Gamma \neq \emptyset$.

(ii) *Every finitely generated left ideal of S is FD, and for any $s,t \in S$ and homomorphism $f : {}_S Ss \cup St \longrightarrow {}_S S$, if $L_f(s,t) = \{(us,vt) | \ (us,vt) \in \overrightarrow{ker} f\} \neq \emptyset$, then $L_f(s,t) \subseteq \widehat{S(p,q)}$ for some $(p,q) \in \overrightarrow{ker} f$.*

Proof. The proof is similar to the proof of Proposition 3.3. □

4 Transferring flatness properties from products to their components

This section is allocated to reply the question of when products of S-posets transfer flatness properties such as projectivity, freeness, and regularity to their components. The following lemma is an updated version of Remark 3.1 in [14] for S-posets, needed in the sequel.

Lemma 4.1. *Let $\{A_i \mid i \in I\}$ be a family of S-posets and ${}_S B$ be a left S-poset. Suppose that $(a_i) \otimes b \leq (a'_i) \otimes b'$ for $(a_i), (a'_i) \in \prod_I A_i, b, b' \in {}_S B$. Then $a_i \otimes b \leq a'_i \otimes b'$ for each $i \in I$.*

We begin our investigation with (po-)torsion freeness. An element c of a pomonoid S is called right po-cancellable if for any $s,t \in S$, $sc \leq tc$ implies $s \leq t$. An S-poset A_S is called (po-)torsion free if for any $a, a' \in A$ and right (po-)cancellable element c of S, from $(ac \leq a'c)$ $ac = a'c$ it follows that $(a \leq a')$ $a = a'$. The proof of the next lemma is straightforward.

Lemma 4.2. *Let S be a pomonoid and $\{A_i \mid i \in I\}$ be a family of S-posets. Then $\prod_I A_i$ is (po-)torsion free if and only if A_i is (po-)torsion free for each $i \in I$.*

For GP-po-flatness we have the following result.

Lemma 4.3. *Let S be a pomonoid and $\{A_i \mid i \in I\}$ be a family of S-posets such that $\prod_I A_i$ is GP-po-flat. Then A_i is GP-po-flat for each $i \in I$.*

Proof. Suppose that $j \in I$ and $as \leq a's$ for $s \in S$, $a, a' \in A_j$. For each $i \neq j$ in I, choose $a_i \in A_i$ and define

$$c_i = \begin{cases} a_i & i \neq j \\ a & i = j \end{cases}$$

and

$$c'_i = \begin{cases} a_i & i \neq j \\ a' & i = j \end{cases}$$

Thus $(c_i)s \leq (c'_i)s$ and, by assumption, $(c_i) \otimes s^m \leq (c'_i) \otimes s^m$ in $\prod_I A_i \otimes_S Ss^m$ for some $m \in \mathbb{N}$. The result now follows by Lemma 4.1. $\square$

The following result could be proved by letting $m = 1$ in the proof of the previous lemma.

Corollary 4.4. *Let S be a pomonoid and $\{A_i \mid i \in I\}$ be a family of S-posets such that $\prod_I A_i$ is principally weakly po-flat. Then A_i is principally weakly po-flat for each $i \in I$.*

Substituting $\leq$ by $=$ in the proofs of Lemma 4.3 and Corollary 4.4, leads us to the following results respectively.

Lemma 4.5. *Let S be a pomonoid and $\{A_i \mid i \in I\}$ be a family of S-posets such that $\prod_I A_i$ is GP-flat. Then A_i is GP-flat for each $i \in I$.*

Corollary 4.6. *Let S be a pomonoid and $\{A_i \mid i \in I\}$ be a family of S-posets such that $\prod_I A_i$ is principally weakly flat. Then A_i is principally weakly flat for each $i \in I$.*

The following arguments are about Conditions (PWP) and $(PWP)_w$.

Proposition 4.7. *Let S be a pomonoid and $\{A_i \mid i \in I\}$ be a family of S-posets. The following statements are verified.*

(i) *If $\prod_I A_i$ satisfies Condition (PWP), then A_i satisfies Condition (PWP) for each $i \in I$.*

(ii) *If $\prod_I A_i$ satisfies Condition $(PWP)_w$, then A_i satisfies Condition $(PWP)_w$ for each $i \in I$.*

Proof. (i): Let $\{A_i \mid i \in I\}$ be a family of S-posets such that $\prod_I A_i$ satisfies Condition (PWP). Suppose that $at \leq a't$ for $a, a' \in A_j, t \in S$. Fix $a_i \in A_i$ for each $i \neq j$ in I and define

$$c_i = \begin{cases} a & i = j \\ a_i & i \neq j \end{cases}$$

and

$$d_i = \begin{cases} a' & i = j \\ a_i & i \neq j \end{cases} .$$

Thus, $(c_i)t \leq (d_i)t$ and, by assumption, there exist $(a''_i) \in \prod_I A_i$ and $u, v \in S$ such that $(c_i) = (a''_i)u,\ (d_i) = (a''_i)v$ and $ut \leq vt$. So $a = a''_j u,\ a' = a''_j v$, and the result follows. By a similar argument, part (ii) is verified. □

Recall from [1] that a pomonoid S is called *weakly right reversible* in case $Ss \cap (St] \neq \emptyset$ for each $s, t \in S$. In what follows, we investigate flatness properties for which transferring from products to their components meets additional conditions on the pomonoid S.

Theorem 4.8. *For a pomonoid S, the following conditions are equivalent:*

(i) *Po-flatness transfers from products to their components.*

(ii) *Weak po-flatness transfers from products to their components.*

(iii) *Flatness transfers from products to their components.*

(iv) *Weak flatness transfers from products to their components.*

(v) *The one-element S-poset Θ_S meets one of the Conditions (P), (P_w), po-flatness, flatness, weak po-flatness or weak flatness.*

(vi) *S is weakly right reversible.*

Proof. The equivalence of conditions (v) and (vi) is shown in [1, Theorem 1].

(i),(ii),(iii),(iv)⇒(v): Since $S_S \cong S_S \times \Theta_S$, all implications are verified.

(vi)⇒(i): Suppose that $\prod_I A_i$ is po-flat for a family $\{A_i \mid i \in I\}$ of S-posets. Let ${}_SB$ be a left S-poset and $a \otimes b \leq a' \otimes b'$ in $A_j \otimes B$ for some $j \in I,\ a, a' \in A_j,\ b, b' \in {}_SB$. Thus, there exists a scheme such as:

$$\begin{array}{rr} a \leq a_1u_1 & \\ a_1v_1 \leq a_2u_2 & u_1b \leq v_1b_2 \\ \vdots & \vdots \\ a_nv_n \leq a' & u_nb_n \leq v_nb' \end{array}$$

where $a_i \in A_j,\ b_i \in B, u_i, v_i \in S$ for $1 \leq i \leq n$. Putting $v_0 = u_{n+1} = 1$, our assumption implies the existence of $c_0, d_0 \in S$ with $c_0v_0 \leq d_0u_1$. Proceeding inductively, we get $c_1, \dots, c_n,\ d_1, \dots, d_n \in S$ such that $c_id_{i-1}v_i \leq d_iu_{i+1}$ for each $1 \leq i \leq n+1$. Fix $a'_i \in A_i$ for each $j \neq i \in I$. Define

$$\alpha_i = \begin{cases} a'_ic_n \dots c_1c_0 & i \neq j \\ a & i = j \end{cases}, \qquad \alpha'_i = \begin{cases} a'_id_n & i \neq j \\ a' & i = j \end{cases}$$

and

$$\beta_{l_i} = \begin{cases} a'_ic_n \dots c_ld_{l-1} & i \neq j \\ a_l & i = j \end{cases},$$

for each $i \in I, 1 \leq l \leq n-1$. Thus $(\alpha_i) \otimes b \leq (\alpha'_i) \otimes b'$ in $\prod_I A_i \otimes B$ by the scheme:

$$\begin{array}{rr} (\alpha_i) \leq (\beta_{1_i})u_1 & \\ (\beta_{1_i})_Iv_1 \leq (\beta_{2_i})u_2 & u_1b \leq v_1b_2 \\ \vdots & \vdots \\ (\beta_{n_i})v_n \leq (\alpha'_i) & u_nb_n \leq v_nb'. \end{array}$$

Therefore, by our assumption, $(\alpha_i) \otimes b \leq (\alpha'_i) \otimes b'$ in $\prod_I A_i \otimes (Sb \cup Sb')$ which gives $a \otimes b \leq a' \otimes b'$ in $A_j \otimes (Sb \cup Sb')$, using Lemma 4.1.

The implications (vi)⇒(ii), (vi)⇒(iii) and (vi)⇒(iv) follow analogously. □

Proposition 4.9. *For a pomonoid S, the following are equivalent:*

(i) *Condition (P) transfers from products to their components.*

(ii) *Condition (P_w) transfers from products to their components.*

(iii) *Θ_S satisfies Condition (P) or Condition (P_w).*

(iv) *S is weakly right reversible.*

Proof. According to the proof of Theorem 4.8, it is enough to prove the implication (iv)⇒(i). Let $\{A_i \mid i \in I\}$ be a family of S-posets such that $\prod_I A_i$ satisfies Condition (P). Let $as \leq a't$ for $a, a' \in A_j,\ s, t \in S$. Fix $a_i \in A_i$ for each $i \neq j$ in I. Since S is weakly right reversible, there exist $u_1, v_1 \in S$ such that $u_1 s \leq v_1 t$. Define

$$c_i = \begin{cases} a & i = j \\ a_i u_1 & i \neq j \end{cases}$$

and

$$d_i = \begin{cases} a' & i = j \\ a_i v_1 & i \neq j \end{cases} .$$

So $(c_i)s \leq (d_i)t$ and by assumption there exist $(a''_i) \in \prod_I A_i, u, v \in S$ such that $(c_i) = (a''_i)u, (d_i) = (a''_i)v$, and $us \leq vt$. Hence $a = a''_j u,\ a' = a''_j v$, and the result follows. □

Theorem 4.10. *For a pomonoid S, the following statements are equivalent:*

(i) *Condition (WP) transfers from products to their components.*

(ii) *Condition $(WP)_w$ transfers from products to their components.*

(iii) *Θ_S satisfies Condition (WP) or Condition $(WP)_w$.*

(iv) *S is weakly right reversible.*

Proof. Since Conditions (WP) and $(WP)_w$ both imply weak po-flatness, the implications (iii)⇒(iv) is valid. Besides, according to the proof of Theorem 4.8 we have the implications (i)⇒(iii) and (ii)⇒(iii).

(iv)⇒(i): Let $\{A_i \mid i \in I\}$ be a family of S-posets such that $\prod_I A_i$ satisfies Condition (WP). Let $s, t \in S$ and $f : (Ss \cup St) \longrightarrow S$ be a homomorphism such that $af(s) \leq a'f(t)$ for $a, a' \in A_j,\ j \in I$. Since S is weakly right reversible, there exist $u_1, v_1 \in S$ such that $u_1 f(s) \leq v_1 f(t)$. Fix $a_i \in A_i$ for $j \neq i \in I$. Let

$$c_i = \begin{cases} a & i = j \\ a_i u_1 & i \neq j \end{cases}$$

and

$$d_i = \begin{cases} a' & i = j \\ a_i v_1 & i \neq j \end{cases}$$

Then $(c_i)f(s) \leq (d_i)f(t)$. By assumption, there exist $(a''_i) \in \prod_I A_i$, $p, q \in Ss \cup St$ such that $(c_i) \otimes s = (a''_i) \otimes p$, $(d_i) \otimes t = (a''_i) \otimes q$ in $\prod_I A_S \otimes_S (Ss \cup St)$ and $f(p) \leq f(q)$. Thus, thanks to Lemma 4.1, $a \otimes s = a''_j \otimes p$, $a' \otimes t = a''_j \otimes q$ in $A_j \otimes_S (Ss \cup St)$ and hence A_j satisfies Condition (WP).

The implication (iv)⇒(ii) is followed analogously. □

Theorem 4.11. *For a pomonoid S, the following assertions are equivalent:*

(i) *Condition (E) transfers from products to their components.*

(ii) *Strong flatness transfers from products to their components.*

(iii) *Θ_S is strongly flat or satisfies Condition (E).*

(iv) *S is left collapsible.*

Proof. The equivalence of (iii) and (iv) is shown in [1, Theorem 1]. The implications (i)⇒(iii) and (ii)⇒(iii) are similar to their peer in the foregoing theorem. (iv)⇒(i): It is similar to the proof of (iii)⇒(i) in Proposition 4.9. (iv)⇒(ii) follows by (iv)⇒(i) and Proposition 4.9. □

Concerning properties projectivity, freeness and regularity, the results are proved similar to the act case ([14]), so their proofs are omitted.

Proposition 4.12. *For a pomonoid S, the following are equivalent:*

(i) *Projectivity transfers from products to their components.*

(ii) *Θ_S is projective.*

(iii) *S contains a left zero.*

Proposition 4.13. *Let S be a pomonoid on which there exists a regular S-poset. The following are equivalent:*

(i) *Regularity transfers from products to their components.*

(ii) *Θ_S is regular.*

(iii) *S contains a left zero.*

Proposition 4.14. *For a pomonoid S, the following are equivalent:*

(i) *Freeness transfers from products to their components.*

(ii) Θ_S *is free.*

(iii) $S = \{1\}$.

Concluding this section, we summarize the results in the following table.

Property	**The necessary and sufficient condition on S for transferring a flatness property from products to their components**
Torsion freeness GP-po-flatness GP-flatness Principal weak po-flatness Principal weak flatness Condition (PWP) Condition $(PWP)_w$	S needs no condition.
Weak flatness Weak po-flatness Flatness Po-flatness Condition (P) Condition (P_w) Condition (WP) Condition $(WP)_w$	S is weakly right reversible.
Condition (E) Strong flatness	S is left collapsible.
Projectivity	S contains a left zero.
Regularity (if there exists a regular S-poset)	S contains a left zero.
Freeness	$S = \{1\}$.

Table 1: Classification of pomonoids by transferring a flatness property from products of S-posets to their components.

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