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Realization of locally extended affine Lie algebras of type A_1

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Abstract. Locally extended affine Lie algebras were introduced by Morita and Yoshii as a natural generalization of extended affine Lie algebras. After that, various generalizations of these Lie algebras have been investigated by others. It is known that a locally extended affine Lie algebra can be recovered from its centerless core, i.e., the ideal generated by weight vectors corresponding to nonisotropic roots modulo its centre. In this paper, in order to realize locally extended affine Lie algebras of type A_1, using the notion of Tits-Kantor-Koecher construction, we construct some Lie algebras which are isomorphic to the centerless cores of these algebras.

1 Introduction

Extended affine Lie algebras were first introduced 1990 in [4] and later were systematically studied by Allison et al. in [1]. Then, Morita and Yoshii in [8] introduced locally extended affine Lie algebras as a natural generalization of extended affine Lie algebras. After that, various generalizations of these

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Lie algebras have been investigated (see [2, 9–11]).

The ideal of a locally extended affine Lie algebra $\mathcal{L}$ generated by weight vectors corresponding to nonisotropic roots modulo its center is called the centerless core of $\mathcal{L}$. Centerless cores play an important role in the theory of these algebras, in particular, it is shown that any locally extended affine Lie algebra can be recovered from its centerless core. From this point of view, the authors in [1, §III.1] gave a construction for extended affine Lie algebras. More precisely, they considered a class of Lie algebras satisfying certain 11 conditions and, by starting from a Lie algebra $\mathcal{G}$ of this class, they constructed an extended affine Lie algebra whose centerless core is isomorphic to $\mathcal{G}$. Later, Neher in [11, §6] gave a similar construction for locally extended affine Lie algebras starting from a class of Lie algebras called Lie tori. Recently, Azam et al. in [3] introduced a new class, called the class $\mathcal{T}$, of Lie algebras which is a generalization of the class in [1]. In fact, the elements of this class are considered as the centerless cores of invariant affine reflection algebras [11] and in particular locally extended affine Lie algebras. Moreover, it is shown that each element of $\mathcal{T}$ is the direct union of its subalgebras belonging to this class. This provides a framework for realization of the centerless cores of invariant affine reflection algebras and locally extended affine Lie algebras. The aim of this work is to realize centerless cores of some locally extended affine Lie algebras of type A_1. Roughly speaking, we construct some Lie algebras and show that they are isomorphic to the centerless cores of some locally extended affine Lie algebras of type A_1.

The paper is organized as follows. In Section 2, we provide some preliminaries and definitions which we need in the sequel. In Section 3, we construct some Lie algebras, obtained from Jordan algebras using the Tits-Kantor-Koecher (TKK) construction, which are isomorphic to the centerless cores of some locally extended affine Lie algebras of type A_1. These examples are general versions of the ones in [1, §III.1] in the sense that the isotropic roots live in an arbitrary abelian group instead of a free abelian group of finite rank.

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2 Preliminaries

In this section we recall the preliminaries and definitions needed throughout the paper. In this work all algebras and vector spaces are considered over a field $\mathbb{F}$ of characteristic zero and all groups are written additively. Also for a set S, by $|S|$, we mean the cardinal number of S. For a subset S of a group G, we denote by $\langle S\rangle$, the subgroup of G generated by S. For an associative algebra $(A,\cdot)$ and $a,b\in A$, we mean by $[a,b]$, the *commutator* of a and b. Also, for two indices i and j, by $\delta_{i,j}$ we mean the Kronecker delta.

Definition 2.1. Suppose that G is an abelian group. An algebra $(A,\cdot)$ is called a *G-graded algebra* if there are subspaces A^σ $(\sigma\in G)$ such that

$$A=\oplus_{\sigma\in G}A^\sigma \quad \text{and} \quad A^\sigma\cdot A^\tau \ \subseteq\ A^{\sigma+\tau} \quad (\sigma,\tau\in G).$$

Definition 2.2. Let $(A,\cdot)$ be a commutative algebra. It is called a *Jordan algebra* if $[L_a,L_{a^2}]=0$, for all $a\in A$, where the operator L_a is defined by $L_a(b)=a\cdot b$ for $b\in A$.

Definition 2.3. Let $\mathcal{G}$ be a Lie algebra and $\mathcal{H}$ be a subalgebra of $\mathcal{G}$. We say that $\mathcal{H}$ is a *toral* subalgebra of $\mathcal{G}$ and $\mathcal{G}$ *has a root space decomposition* with respect to $\mathcal{H}$ if $\mathcal{G}=\sum_{\alpha\in\mathcal{H}^*}\mathcal{G}_\alpha(\mathcal{H})$ where

$$\mathcal{G}_\alpha(\mathcal{H}):=\{x\in\mathcal{G}\mid [h,x]=\alpha(h)x \text{ for all } h\in\mathcal{H}\},$$

for each $\alpha\in\mathcal{H}^*$. Also $\alpha\in\mathcal{H}^*$ is called a *root* if $\mathcal{G}_\alpha(\mathcal{H})\neq\{0\}$ and $R:=\{\alpha\in\mathcal{H}^*\mid \mathcal{G}_\alpha(\mathcal{H})\neq\{0\}\}$ is called the *root system* of $\mathcal{G}$ with respect to $\mathcal{H}$. We will usually abbreviate $\mathcal{G}_\alpha(\mathcal{H})$ by $\mathcal{G}_\alpha$. Since any toral subalgebra is abelian, $\mathcal{H}\ \subseteq\ \mathcal{G}_0$ and so $0\in R$ (we discard the case $\mathcal{H}=\{0\}=\mathcal{G}$).

Definition 2.4. [7, Definition 3.3] Let $\mathcal{V}$ be a nontrivial vector space and R be a subset of $\mathcal{V}$. R is said to be a *locally finite root system in $\mathcal{V}$ of rank dim$(\mathcal{V})$* if the following are satisfied:

(i) R is locally finite, contains zero and spans $\mathcal{V}$,

(ii) for every $\alpha\in R\backslash\{0\}$, there exists $\alpha^\vee\in\mathcal{V}^*$ such that $\alpha^\vee(\alpha)=2$ and $s_\alpha(\beta)\in R$ for $\alpha,\beta\in R$ where $s_\alpha:\mathcal{V}\to\mathcal{V}$ maps $u\in\mathcal{V}$ to $u-\alpha^\vee(u)\alpha$. We set by convention $\check{0}$ to be zero,

(iii) $\alpha^\vee(\beta)\in\mathbb{Z}$, for $\alpha,\beta\in R$.

The locally finite root system R is also denoted by $(R,\mathcal{V})$.

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Suppose that R is a locally finite root system in $\mathcal{V}$. The subgroup of automorphisms of $\mathcal{V}$ generated by the set $\{s_\alpha \mid \alpha \in R\}$ is called the *Weyl group of* R denoted by $\mathcal{W}_R$. We say two nonzero roots α, β are *connected* if there exist finitely many roots $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_n = \beta$ such that $\alpha_{i+1}^\vee(\alpha_i) \neq 0, 1 \leq i \leq n-1$. Connectedness defines an equivalence relation on $R \setminus \{0\}$ and so $R\setminus\{0\}$ is the disjoint union of its equivalence classes called *connected components* of R. A nonempty subset X of R is called *irreducible*, if each two nonzero elements $x, y \in X$ are connected.

Two locally finite root systems $(R, \mathcal{V})$ and $(S, \mathcal{U})$ are said to be isomorphic if there is a linear isomorphism $f : \mathcal{V} \to \mathcal{U}$ such that $f(R) = S$. Suppose that I is a nonempty index set and $\mathcal{V} := \oplus_{i \in I}\mathbb{F}\epsilon_i$ is the free $\mathbb{F}$-module over the set I. Define the form

$$(\cdot|\cdot) : \mathcal{V} \times \mathcal{V} \longrightarrow \mathbb{F}$$

$$(\epsilon_i|\epsilon_j) := \delta_{i,j}, \quad \text{for} \quad i, j \in I$$

and set

$$\begin{aligned} \dot{A}_I &:= \{\epsilon_i - \epsilon_j \mid i, j \in I\}, \\ D_I &:= \dot{A}_I \cup \{\pm(\epsilon_i + \epsilon_j) \mid i, j \in I,\ i \neq j\}, \\ B_I &:= D_I \cup \{\pm\epsilon_i \mid i \in I\}, \\ C_I &:= D_I \cup \{2\epsilon_i \mid i \in I\}, \\ BC_I &:= B_I \cup C_I. \end{aligned}$$

One can see that these are irreducible locally finite root systems in their $\mathbb{F}$-span's which we refer to as type A, D, B, C and BC respectively. Moreover, each irreducible locally finite root system is either an irreducible finite root system or a locally finite root system of infinite rank isomorphic to one of these root systems (see [7, §4.14, §8]). Note that D_I, B_I, C_I and BC_I span $\mathcal{V}$ while $\dot{A}_I$ spans a subspace of $\mathcal{V}$ of codimension one. Therefore, following the usual notation in the literature, we use $\dot{A}_I$ instead of A_I.

Let G be any abelian group. We recall from [6] that a subset X of G is called a *symmetric reflection subspace* if $X - 2X \subseteq X$. A symmetric reflection subspace X satisfies $X = -X$. Also a symmetric reflection subspace X of G is called a *pointed reflection subspace* if $0 \in X$ and is called *full* if $G = \langle X \rangle$. The special of $G = \langle X \rangle = \mathbb{Z}^n$ has been treated in [1] in which case X is called a *semilattice* in $\mathbb{Z}^n$.

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Next we recall a class of Lie algebras, refereed to as the class $\mathcal{T}$, introduced in [3] which will be used in our main theorem. A 4-tuple $(\mathcal{G}, (\cdot|\cdot), \mathcal{H}, G)$ is said to be in class $\mathcal{T}$ if it satisfies the following axioms:
$(\mathcal{T}1)$ $\mathcal{G}$ is a Lie algebra and $(\cdot|\cdot)$ is an invariant symmetric bilinear form on $\mathcal{G}$,
$(\mathcal{T}2)$ $\mathcal{H}$ is a nontrivial toral subalgebra of $\mathcal{G}$, the corresponding root system is denoted by R,
$(\mathcal{T}3)$ $(\cdot|\cdot)|_{\mathcal{H}\times\mathcal{H}}$ is nondegenerate,
$(\mathcal{T}4)$ $\mathcal{G}$ is generated, as a Lie algebra, by $\sum_{\alpha\in R\setminus\{0\}} \mathcal{G}_\alpha$,
$(\mathcal{T}5)$ G is an abelian group and $\mathcal{G} = \oplus\sum_{\sigma\in G} \mathcal{G}^\sigma$ is a G-graded Lie algebra,
$(\mathcal{T}6)$ G, as a group, is generated by $\sigma \in G$ for which $\mathcal{G}^\sigma \neq \{0\}$,
$(\mathcal{T}7)$ the form $(\cdot|\cdot)$ is G-graded,
$(\mathcal{T}8)$ the G-grading and the $\mathcal{H}^*$-grading on $\mathcal{G}$ are compatible,
$(\mathcal{T}9)$ if $\mathcal{G}^\sigma_\alpha := \mathcal{G}_\alpha \cap \mathcal{G}^\sigma \neq \{0\}$ for some $\alpha \in R \setminus \{0\}$ and $\sigma \in G$, then $[\mathcal{G}^\sigma_\alpha, \mathcal{G}^{-\sigma}_{-\alpha}] \neq \{0\}$, and $\mathcal{G}^0_\alpha \neq \{0\}$ for each $\alpha \in R$ such that $\frac{1}{2}\alpha \notin R$,
$(\mathcal{T}10)$ $\mathcal{H} = \mathcal{G}^0_0$,
$(\mathcal{T}11)$ R is a locally finite root system in its $\mathbb{F}$-span such that the restriction of the form $(\cdot|\cdot)$ to the subspace of $\mathcal{H}^*$ spanned by R is invariant under the Weyl group $\mathcal{W}_R$ and it is nonzero on each irreducible component.

3 Main results

The authors in [1, § III.2] start from a semilattice of a finitely-generated free abelian group to construct a Lie algebra $\mathcal{G}$ which is isomorphic to the centerless core of an extended affine Lie algebra whose root system is of type A_1. In this section, starting from a pointed reflection subspace of an arbitrary abelian group G, we construct new examples of Lie algebras and show that, if G is torsion free, these Lie algebras are isomorphic to the centerless cores of some locally extended affine Lie algebras of type A_1.

From now on, we assume that G is a non-trivial abelian group and S is a full pointed reflection subspace of G. Then $S = \cup_{i\in I} S_i$, where I is a nonempty index set and S_i's are some distinct cosets of $2G$ in G with $0 \in S_i$ for some i (see [1, §II.1]). For the sake of convenience, for each i, we fix a coset representative $\tau_i \in S_i$ of $2G$, namely $S_i = \tau_i + 2G$ for $i \in I$. We take $\tau_0 = 0$ and set $I^\times := I \setminus \{0\}$. Let $\mathcal{A} := \mathbb{F}[G]$ be the group algebra on G, i.e., $\mathcal{A} = \bigoplus_{\sigma\in G} \mathbb{F}x^\sigma$ with $x^\sigma \cdot x^\tau = x^{\sigma+\tau}$. As one knows, $(\mathcal{A}, \cdot)$ is a

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unital commutative associative algebra over $\mathbb{F}$. Consider the free $\mathcal{A}$-module $\mathcal{W} := \bigoplus_{i\in I^\times} \mathcal{A}w_i$ with basis $\{w_i \mid i \in I^\times\}$ and let $f : \mathcal{W}\times\mathcal{W} \longrightarrow \mathcal{A}$ defined by $f(w_i, w_j) = \delta_{i,j}$, for $i, j \in I^\times$, be a symmetric $\mathcal{A}$-bilinear form. Take, $\mathcal{W} = 0$ and $f = 0$ if $I = \{0\}$. Now set $\mathcal{F} := \mathcal{A} \oplus \mathcal{W}$ with product $(a + v)\cdot(b + w) = ab + f(v, w) + aw + bv$ for $a, b \in \mathcal{A}$, $v, w \in \mathcal{W}$. By [5, pg. 14], $\mathcal{F}$ is a Jordan algebra, called the *Jordan algebra of type* f. If we set $w_0 = 1$, we have $\mathcal{F} = \bigoplus_{i\in I} \mathcal{A}w_i$ and its product as an algebra over $\mathcal{A}$ is determined by $w_0 \cdot w_0 = 1$, $w_i \cdot w_0 = w_i$ and $w_i \cdot w_j = \delta_{i,j}w_0$ for $i, j \in I^\times$.

Next let $\mathcal{J} = \mathcal{J}(S)$ be the $\mathbb{F}$-subspace of $\mathcal{F}$ spanned by $\{x^\sigma w_i \mid \sigma \in S_i, i \in I\}$. We show that $\mathcal{J}$ is a subalgebra of $\mathcal{F}$. If $\sigma \in S_0$ and $\tau \in S_i$, $i \in I$, then $\sigma + \tau \in S_0 + S_i = S_i$ and so

$$x^\sigma w_0 \cdot x^\tau w_i = x^{\sigma+\tau} w_i \in \mathcal{J}.$$

Also if $\sigma \in S_i$, $\tau \in S_j$ for some $i, j \in I^\times$, then

$$x^\sigma w_i \cdot x^\tau w_j = \delta_{i,j} x^{\sigma+\tau} w_0 \in \mathcal{J},$$

as $S_i + S_i = S_0$ for each $i \in I$. Thus $\mathcal{J}$ is a subalgebra of the Jordan algebra $\mathcal{F}$. If we identify $x^\sigma w_i$ with x^σ for $\sigma \in S_i$ and $i \in I$, we may write $\mathcal{J} = \bigoplus_{\sigma\in S} \mathbb{F}x^\sigma$ with Jordan algebra multiplication

$$x^\sigma \cdot x^\tau = \begin{cases} x^{\sigma+\tau} & \text{if } \sigma, \tau \in S_0 \cup S_i, i \in I, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Now, set $L_{\mathcal{J}} := \{L_a \mid a \in \mathcal{J}\}$ which is the space of left multiplications on $\mathcal{J}$ and

$$\mathrm{Inder}(\mathcal{J}) := \{\sum_i [L_{a_i}, L_{b_i}] \mid a_i, b_i \in \mathcal{J}\}$$

which is an ideal of derivation algebra of $\mathcal{J}$. Then, one checks that

$$\mathrm{Instrl}(\mathcal{J}) := \mathrm{Inder}(\mathcal{J}) + L_{\mathcal{J}}$$

is a subalgebra of $\mathfrak{gl}(\mathcal{J})$. Using the well-known Tits-Kantor-Koecher (TKK) construction for producing a Lie algebra out of a Jordan algebra, we set

$$\mathcal{G} := \mathcal{J} + \mathrm{Instrl}(\mathcal{J}) + \bar{\mathcal{J}},$$

where $\bar{\mathcal{J}}$ is a copy of $\mathcal{J}$ in which, by $\bar{j}$ ($j \in \mathcal{J}$), we mean the element of $\bar{\mathcal{J}}$ correspon ding to the element j. The Lie bracket on $\mathcal{G}$ is defined as follows.

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Consider the automorphism $^{-}$ of order two on the Lie algebra $\mathrm{Instrl}(\mathcal{J})$ defined by $\overline{L_a + D} = -L_a + D$ and define the Lie algebra bracket on $\mathcal{G}$ by

$$[a_1 + D_1 + \bar{b}_1, a_2 + D_2 + \bar{b}_2] = D_1(a_2) - D_2(a_1) + \overline{\bar{D}_1(b_2)} \\ - \overline{\bar{D}_2(b_1)} + a_1\Delta b_2 - a_2\Delta b_1 + [D_1, D_2],$$

for $a_i \in \mathcal{J}$, $\bar{b}_i \in \bar{\mathcal{J}}$ and $D_i \in \mathrm{Instrl}(\mathcal{J})$, and $a\Delta b = L_{a\cdot b} + [L_a, L_b]$ for $a, b \in \mathcal{J}$. Note that from the definition of the operation Δ, one checks that

$$\mathcal{J}\Delta\overline{\mathcal{J}} = \mathrm{Instrl}(\mathcal{J}). \tag{2}$$

Next we consider the map $\epsilon : \mathcal{A} \longrightarrow \mathbb{F}$ defined by linear extension of

$$\epsilon(x^\sigma) = \begin{cases} 1 & \text{if } \sigma = 0, \\ 0 & \text{otherwise.} \end{cases}$$

This defines a symmetric $\mathbb{F}$-bilinear form $(a|b) := \epsilon(a \cdot b)$ on $\mathcal{J}$ which turns out to be nondegenerate as $S = -S$. Using ϵ, one defines a symmetric invariant bilinear form $(\cdot|\cdot)$ on $\mathrm{Inder}(\mathcal{J})$ by

$$(D|[L_a, L_b]) = (D(a)|b) \quad \text{for } D \in \mathrm{Inder}(\mathcal{J}), a, b \in \mathcal{J}.$$

Then this form extends to a symmetric invariant bilinear form, denoted again by $(\cdot|\cdot)$, on $\mathcal{G}$ by

$$(a_1 + L_{b_1} + D_1 + \bar{c}_1 | a_2 + L_{b_2} + D_2 + \bar{c}_2) = \\ (a_1|c_2) + (a_2|c_1) + (b_1|b_2) + (D_1|D_2), \tag{3}$$

for $a_i, b_i, c_i \in \mathcal{J}$ and $D_i \in \mathrm{Inder}(\mathcal{J})$. Since the form $(\cdot|\cdot)$ in nondegenerate on $\mathcal{J}$, it follows easily that

(4) *the defined form $(\cdot|\cdot)$ on $\mathcal{G}$ is nondegenerate.*

Set $\mathcal{H} := \mathbb{F}L_1$ and define $\alpha \in \mathcal{H}^*$ by $\alpha(L_1) = 1$. For $\beta \in \mathcal{H}^*$ let $\mathcal{G}_\beta = \{x \in \mathcal{G} \mid [h, x] = \beta(h)x \text{ for all } h \in \mathcal{H}\}$. Then we have,

$$\mathcal{G} = \mathcal{G}_\alpha \oplus \mathcal{G}_0 \oplus \mathcal{G}_{-\alpha}, \tag{5}$$

with

$$\mathcal{G}_\alpha = \mathcal{J}, \quad \mathcal{G}_0 = \mathrm{Instrl}(\mathcal{J}), \quad \text{and} \quad \mathcal{G}_{-\alpha} = \bar{\mathcal{J}}. \tag{6}$$

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Also, we have $(L_1|L_1) = 1$. The form on $\mathcal{H}$ can be naturally transferred to $\mathcal{H}^*$, in which case we have

$$(\alpha|\alpha) = 1. \tag{7}$$

We note that $R = \{0, \pm\alpha\}$ is an irreducible finite root system of type A_1. It follows that $\mathcal{G}$ is $\mathcal{H}^*$-graded with $\mathcal{G}_\alpha$, $\alpha \in R$ as its set of non-zero homogeneous spaces.

Next, we would like to put a G-grading on $\mathcal{G}$. First, we put a G-gradation on the Jordan algebra $\mathcal{J}$ by $\mathcal{J}^\sigma = \mathbb{F}x^\sigma$, if $\sigma \in S$ and $\mathcal{J}^\sigma = \{0\}$, otherwise. For $\sigma \in G$, set $\tilde{\mathfrak{g}\ell}(\mathcal{J})^\sigma = \{A \in \mathfrak{g}\ell(\mathcal{J}) \mid A\mathcal{J}^\tau \subseteq \mathcal{J}^{\sigma+\tau} \text{ for } \tau \in G\}$. Then $\tilde{\mathfrak{g}\ell}(\mathcal{J}) := \sum_{\sigma\in G} \tilde{\mathfrak{g}\ell}(\mathcal{J})^\sigma$ is a G-graded algebra which is a subalgebra of $\mathfrak{g}\ell(\mathcal{J})$. Since $\mathrm{Instrl}(\mathcal{J})$ is generated by homogeneous elements with respect to this grading, we have

$$\mathrm{Instrl}(\mathcal{J}) = \bigoplus_{\sigma\in G} \mathrm{Instrl}(\mathcal{J})^\sigma \tag{8}$$

with

$$\mathrm{Instrl}(\mathcal{J})^\sigma = L_{\mathcal{J}^\sigma} \oplus \sum_{\tau,\zeta\in G, \tau+\zeta=\sigma} [L_{\mathcal{J}^\tau}, L_{\mathcal{J}^\zeta}].$$

Thus $\mathcal{G}$ is a G-graded Lie algebra:

$$\mathcal{G} = \bigoplus_{\sigma\in G} \mathcal{G}^\sigma \quad \text{with} \quad \mathcal{G}^\sigma = \mathcal{J}^\sigma \oplus \mathrm{Instrl}(\mathcal{J})^\sigma \oplus \overline{\mathcal{J}^\sigma}, \quad (\sigma \in G). \tag{9}$$

Since $[L_{x^\sigma}, L_{x^{-\sigma}}] = \{0\}$, then, using (8) and (9), we have

$$\mathcal{G}^0 = \mathbb{F}1 \oplus \mathbb{F}L_1 \oplus \overline{\mathbb{F}1}. \tag{10}$$

Theorem 3.1. *The 4-tuple $(\mathcal{G}, (\cdot|\cdot), \mathcal{H}, G)$, constructed above, is in the class $\mathcal{T}$. Moreover, if G is torsion free, $\mathcal{G}$ is the centerless core of a locally extended affine Lie algebra of type A_1.*

Proof. $(\mathcal{T}1)$, $(\mathcal{T}2)$ and $(\mathcal{T}5)$ follow respectively from (3), (5) and (9). Since $\mathcal{H} = \mathbb{F}L_1$ and $(L_1|L_1) = 1$, then the form $(\cdot|\cdot)|_{\mathcal{H}\times\mathcal{H}}$ is nondegenerate and so $(\mathcal{T}3)$ holds. Using (2) and (6), we see that $\mathcal{G}$ is generated by the root spaces corresponding to the nonzero roots and so $(\mathcal{T}4)$ holds. Since S generates G, then $(\mathcal{T}6)$ holds. Now, note that $(\mathcal{J}^\sigma|\mathcal{J}^\tau) = \{0\}$ if $\sigma + \tau \neq 0$, and from

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the way $(\cdot|\cdot)$ is defined on $\mathcal{G}$ (see (3) and (8)), we see that $(\mathcal{G}^\sigma|\mathcal{G}^\tau) \neq \{0\}$ only if $(\mathcal{J}^\sigma|\mathcal{J}^\tau) \neq \{0\}$. Thus $(\mathcal{T}7)$ holds. Using (6), (8) and (9), one checks that for $\alpha \in R$, $\mathcal{G}_\alpha = \bigoplus_{\sigma\in G}(\mathcal{G}_\alpha \cap \mathcal{G}^\sigma)$ which shows that the $\mathcal{H}^*$-grading and G-grading are compatible and so $(\mathcal{T}8)$ is fulfilled. One can easily check $(\mathcal{T}9)$ and $(\mathcal{T}10)$ using (6), (9), (10) and the definition of the bracket on $\mathcal{G}$. Finally, note that R is an irreducible finite root system in its $\mathbb{Q}$-span, say $\mathcal{V}$, and also the restriction of the form $(\cdot|\cdot)$ on $\mathcal{H}^*$ to $\mathcal{V}$ is a positive definite $\mathbb{Q}$-valued form. Then $(\mathcal{T}11)$ holds and so $(\mathcal{G}, (\cdot|\cdot), \mathcal{H}, G)$ is in the class $\mathcal{T}$ with root system R. Next, assume that G is torsion free. Since the form $(\cdot|\cdot)$ is nondegenerate on $\mathcal{G}$ (see (4)), by [3, Remark 2.3 (ii)] we conclude that $\mathcal{G}$ is the centerless core of a locally extended affine Lie algebra of type A_1. This completes the proof. $\square$

Remark 3.2. Note that if G is a torsion free abelian group, the rank of G is defined by

$$\text{rank}(G) = \dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} G).$$

Now let G be a torsion free abelian group of rank 1, i.e., G is a subgroup of $\mathbb{Q}$. Then, the only full pointed reflection subspace of G is G (see [12, Lemma 12]). In this case, the 4-tuple $(\mathcal{G}, (\cdot|\cdot), \mathcal{H}, G)$, constructed above, with $S = G$ is the only Lie algebra obtained from our construction. Therefore, using the construction of a locally extended affine Lie algebra starting from $(\mathcal{G}, (\cdot|\cdot), \mathcal{H}, G)$ in [11, §6], we obtain a locally extended affine Lie algebra of type A_1 which its centerless core is isomorphic to $\mathcal{G}$.

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