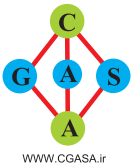
Categories and
General Algebraic Structures
with Applications
Volume 7, Special issue on the Occasion of Banaschewski's 90th Birthday,
July 2017, 89-105.



Adjoint relations for the category of local dcpos

Bin Zhao, Jing Lu, and Kaiyun Wang*

Dedicated to Bernhard Banaschewski on the occasion of his 90th birthday

Abstract. In this paper, we consider the forgetful functor from the category **LDcpo** of local dcpos (respectively, **Dcpo** of dcpos) to the category **Pos** of posets (respectively, **LDcpo** of local dcpos), and study the existence of its left and right adjoints. Moreover, we give the concrete forms of free and cofree S-ldcpos over a local dcpo, where S is a local dcpo monoid. The main results are: (1) The forgetful functor U : **LDcpo** $\longrightarrow$ **Pos** has a left adjoint, but does not have a right adjoint; (2) The inclusion functor I : **Dcpo** $\longrightarrow$ **LDcpo** has a left adjoint, but does not have a right adjoint; (3) The forgetful functor U : **LDcpo**-S $\longrightarrow$ **LDcpo** has both left and right adjoints; (4) If $(S, \cdot, 1)$ is a good ldcpo-monoid, then the forgetful functor U: **LDcpo**-S $\longrightarrow$ **Pos**-S has a left adjoint.

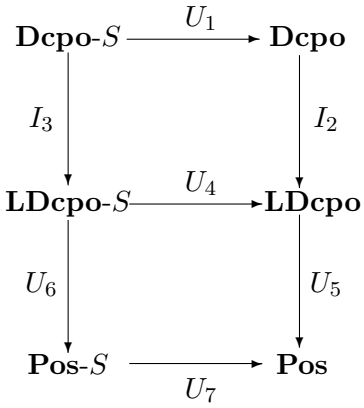
1 Introduction

Domain theory is a branch of mathematics about a special class of partially ordered sets. It has essential applications in theoretical computer science

*Corresponding author
Keywords: Dcpo, local dcpo, S-ldcpo, forgetful functor.
Mathematics Subject Classification [2010]: 06A99, 06F05, 20M99.
Received: 20 October 2016, Accepted: 16 January 2017
ISSN Print: 2345-5853 Online: 2345-5861

as well as other areas of mathematics. A poset is called directed complete (or a dcpo, for short), if every its directed subset has a supremum. In particular, the category **Dcpo** of all dcpos with Scott continuous maps plays an important role in domain Theory (see [5]). Moreover, the free dcpo over a poset has been given in [3]. Unfortunately, the set of real numbers $\mathbb{R}$ and the set of natural numbers $\mathbb{N}$ are not dcpos under their usual orders. This restricts the realm of applications of domain theory. In [8], Mislove introduced the notion of local dcpos (also were called cups by Erné in [4]), that is, each upper bounded directed subset has a supremum, and presented the local dcpo-completion of posets. Using the dcpo-completion of posets, Zhao and Fan (see [10]) construct a new type of local dcpo-completion of posets, which revises the one given by Mislove. It is proved in [10] that the category $\mathbf{LDcpo}_d$ of all local dcpos with Scott continuous maps is the reflective subcategory of the category $\mathbf{Pos}_d$ of all posets with Scott continuous maps. Clearly, the notion of local dcpos is a generalization of the notion of dcpos. It can be easily seen that $\mathbb{R}$ and $\mathbb{N}$ are local dcpos. So the notion of local dcpos extends the research scope of domain theory. In [7], Mahmoudi and Moghbeli considered the free and cofree objects in the category **Dcpo**-S of all S-dcpos; dcpos equipped with a compatible right action of a dcpo-monoid S, with Scott continuous action-preserving maps.

Let **LDcpo** denote the category of all local dcpos with local dcpo maps. In this paper, we consider the forgetful (respectively, inclusion) functor from **LDcpo** (respectively, **Dcpo**) to the category **Pos** of posets (respectively, **LDcpo**), and study the existence of its left and right adjoints. Motivated by the work of Mahmoudi and Moghbeli, we also study the existence of the free and cofree objects in the category **LDcpo**-S of all local dcpos equipped with a compatible right action of a local dcpo monoid S. In fact, we consider the following two squares of functors



and study the existence of the left and right adjoints for these functors. It is proved in [7] that the horizontal forgetful functor U_1 has both left and right adjoints. Following the idea of dcpo-completions in [10], it follows from [11] that the vertical inclusion functor I_2 has a left adjoint. Also, the horizontal forgetful functor U_7 has both left and right adjoints, which has been found in [2].

Here, we show that none of the vertical forgetful functors in the above two squares has a right adjoint, and U_i $(i = 5, 6)$ has a left adjoint (for the existence of U_6-free object, we had to add a condition on S). We also prove that the horizontal forgetful functor U_4 has both left and right adjoints.

2 Preliminaries

We refer to [5] for lattice theory, to [1] for category theory, and to [2] for S-posets.

If X is a subset of a poset P, then $\downarrow X = \{y \in P \mid y \leq x \text{ for some } x \in X\}$ and dually, $\uparrow X = \{y \in P \mid y \geq x \text{ for some } x \in X\}$. The subset X is called a *lower set* (respectively, an *upper set*) if $X =\downarrow X$ (respectively, $X =\uparrow X$). We denote $\downarrow \{x\}$ and $\uparrow \{x\}$ by $\downarrow x$ and $\uparrow x$, respectively.

A nonempty subset D of a poset P is called *directed*, if for every $a, b \in D$ there exists $c \in D$ such that $a, b \leq c$. A subset A of a poset P is called an *ideal* if A is a directed lower set. A poset P is called *directed complete* (or a *dcpo*, for short), if for every directed subset D, the directed join $\bigvee D$ exists in P (or simply, $\bigvee D$ exists).

A map $f : P \longrightarrow Q$ between dcpos is *Scott continuous* if for each directed subset $D \subseteq P$, $\bigvee f(D)$ exists in Q and $f(\bigvee D) = \bigvee f(D)$. Let **Dcpo** denote the category of all dcpos with Scott continuous maps, and let **Pos** denote the category of all posets with order-preserving (monotone) maps. Obviously, **Dcpo** is a subcategory of **Pos**.

Definition 2.1. (see [8]) A poset P is called a *local dcpo* if every directed subset of P with an upper bound in P has a least upper bound.

Remark 2.2. (1) A local dcpo is also called a cup by Erné in [4].

(2) Every dcpo is a local dcpo. But the converse may not be true. For example, the set of natural numbers $\mathbb{N}$ is a local dcpo under the usual order, but not a dcpo.

(3) The cartesian product of a collection of local dcpos under the componentwise order is a local dcpo.

(4) Let P be a local dcpo. Then for every $x \in P$, $\downarrow x$ is a dcpo.

A *po-monoid* S is a monoid $(S, \cdot, 1)$ with a partial order $\leq$ which is compatible with the monoid operation: for all $s, t, \overline{s}, \overline{t} \in S$, $s \leq t$, $\overline{s} \leq \overline{t}$ imply $s \cdot \overline{s} \leq t \cdot \overline{t}$. Similarly, a *dcpo-monoid* is a monoid which is also a dcpo whose binary operation is a Scott continuous map. Throughout this paper, unless otherwise stated, 1 is always to be regard as the identity element, whenever a monoid concept is mentioned in the context.

Definition 2.3. (see [2]) Let $(S, \cdot, 1)$ be a monoid. A (*right*) *S-act* is a set A equipped with an action $* : A \times S \longrightarrow A$ such that $a * 1 = a$ and $a * (s \cdot t) = (a * s) * t$ for all $a \in A$ and $s, t \in S$.

Definition 2.4. (see [2]) A map $f : (A, *) \longrightarrow (B, \star)$ between S-acts is called *action-preserving* if for all $a \in A, s \in S, f(a * s) = f(a) \star s$.

For a po-monoid S, a (*right*) *S-poset* is a poset A which is also an S-act whose action $* : A \times S \longrightarrow A$ is a monotone map, where $A \times S$ is considered as a poset with the componentwise order. Let **Pos**-S denote the category of all S-posets with monotone action-preserving maps.

For a dcpo-monoid S, an *S-dcpo* is a dcpo A which is also an S-act whose action $* : A \times S \longrightarrow A$ is a Scott continuous map. A map $f : A \longrightarrow B$ between S-dcpos is called an *S-dcpo map* if it is both Scott continuous and action-preserving. Let **Dcpo**-S denote the category of all S-dcpos with S-dcpo maps. Clearly, **Dcpo**-S is a subcategory of **Pos**-S.

3 Adjunctions among LDcpo, Dcpo, and Pos

Let P and Q be local dcpos. A map $f : P \longrightarrow Q$ is called a *local dcpo map*, if for every upper bounded directed subset D of P, $\bigvee f(D)$ exists and $f(\bigvee D) = \bigvee f(D)$. Let **LDcpo** denote the category of all local dcpos with local dcpo maps. If $f : P \longrightarrow Q$ is a local dcpo map and $x, y \in P$ with $x \leq y$, then $\downarrow y$ is an upper bounded directed subset of P, and $f(y) = f(\bigvee \downarrow y) = \bigvee f(\downarrow y) \geq f(x)$. Thus every local dcpo map is monotone. Clearly, **LDcpo** is a subcategory of **Pos**. One can show that a Scott continuous map between dcpos is a local dcpo map. Thus **Dcpo** is a full subcategory of **LDcpo**.

Remark 3.1. Let $f : P \longrightarrow Q$ be a monotone map between local dcpos. If D is an upper bounded directed subset of P, then $f(D)$ is an upper bounded directed subset of Q.

Lemma 3.2. *Let P, Q, R be local dcpos, and $f : P \times Q \longrightarrow R$ a map of two variables. Then f is a local dcpo map if and only if f is a local dcpo map in each variable, that is, for all $a \in P$, $b \in Q$, $f_a : Q \longrightarrow R$ $(b \mapsto f(a, b))$ and $f_b : P \longrightarrow R$ $(a \mapsto f(a, b))$ are local dcpo maps.*

Proof. Let D be an upper bounded directed subset of Q. Then $\{a\} \times D$ is an upper bounded directed subset of $P \times Q$ and $\bigvee(\{a\} \times D) = (a, \bigvee D)$ for all $a \in P$. Since f is a local dcpo map, f_a is a monotone map. By Remark 3.1, $f_a(D)$ is an upper bounded directed subset of R, and thus $f_a(\bigvee D) = \bigvee_{d \in D} f_a(d)$. Similarly, for all $b \in Q$, one can prove that f_b is a local dcpo map.

Conversely, let D be an upper bounded directed subset of $P \times Q$. Since $P \times Q$ is a local dcpo, $\bigvee D$ exists and $\bigvee D = (\bigvee D_1, \bigvee D_2)$, where $D_1 = \{p \in P \mid \text{there exists } q \in Q \text{ such that } (p, q) \in D\}$ and $D_2 = \{q \in Q \mid \text{there exists } p \in P \text{ such that } (p, q) \in D\}$. One can check that D_1 and D_2 are upper bounded directed subsets. Since f_a and f_b are local dcpo maps for all $a \in P$ and $b \in Q$, f is a monotone map. By Remark 3.1, $f(D)$ is an upper bounded directed subset. So $f(\bigvee D) = \bigvee_{d \in D} f(d)$. Thus f is a local dcpo map. □

Lemma 3.3. *Let P be a poset. Then the set $L(P)$, of all upper bounded ideals of P, is a local dcpo under the inclusion order.*

Proof. Let $\mathcal{D}$ be an upper bounded directed subset of $L(P)$. Then there exists $F \in L(P)$ such that $D \subseteq F$ for all $D \in \mathcal{D}$. Thus $\bigcup \mathcal{D} \subseteq F$. Since $F \in L(P)$, there exists $x \in P$ such that $F \subseteq{\downarrow} x$. Then $\bigcup \mathcal{D} \subseteq{\downarrow} x$. Obviously, $\bigcup \mathcal{D}$ is an ideal, Thus $\bigcup \mathcal{D} \in L(P)$. □

Proposition 3.4. *Let $f : P \longrightarrow X$ be a monotone map from a poset P to a local dcpo X. Then $\downarrow f(F) \in L(X)$, for all $F \in L(P)$.*

Proof. Let $F \in L(P)$. Then there exists $x \in P$ such that $F \subseteq{\downarrow} x$. Thus $\downarrow f(F) \subseteq{\downarrow} f(\downarrow x) \subseteq{\downarrow} f(x)$, where the last inclusion is true because f is a monotone map. Hence $\downarrow f(F)$ is an upper bounded directed subset of X and so $\downarrow f(F) \in L(X)$, as required. □

Theorem 3.5. *The forgetful functor U_5 : **LDcpo**$\longrightarrow$ **Pos** has a left adjoint.*

Proof. Let P be a poset. By Lemma 3.3, $L(P)$ is a local dcpo. Define a map $\eta : P \longrightarrow L(P)$ as follows:

$$\forall\ a \in P,\ \eta(a) =\downarrow a.$$

Obviously, η is monotone. Next, we shall prove that η is universal. To see this, let X be a local dcpo and $f : P \longrightarrow X$ a monotone map. Define a map $\overline{f} : L(P) \longrightarrow X$ as follows:

$$\forall\ F \in L(P),\ \overline{f}(F) = \bigvee f(F).$$

By Remark 3.1, $f(F)$ is an upper bounded directed subset of X, and so $\bigvee f(F)$ exists in X. Hence $\overline{f}$ is well-defined. Also it is clear to see that $\overline{f}$ is monotone. To prove that $\overline{f}$ is a local dcpo map, let $\mathcal{D}$ be an upper bounded directed subset of $L(P)$. Then $\{\overline{f}(D) \mid D \in \mathcal{D}\}$ is an upper bounded directed subset of X and $\bigvee \mathcal{D} = \bigcup \mathcal{D}$. Thus $\bigvee f(\bigcup \mathcal{D})$ exists. Also, for all $D \in \mathcal{D}$, $\bigvee f(D) \leq \bigvee f(\bigcup \mathcal{D})$. Then $\bigvee f(\bigcup \mathcal{D})$ is an upper bound of $\{\bigvee f(D) \mid D \in \mathcal{D}\}$. Suppose that $b \in X$ is an upper bound of $\{\bigvee f(D) \mid D \in \mathcal{D}\}$. Then for all $D \in \mathcal{D}$, $\bigvee f(D) \leq b$. For all $y \in f(\bigcup \mathcal{D})$, there exists $d \in \bigcup \mathcal{D}$ such that $y = f(d)$. Since $d \in \bigcup \mathcal{D}$, there exists $D_1 \in \mathcal{D}$ such that $d \in D_1$. Then $y = f(d) \in f(D_1)$, and thus $y \leq \bigvee f(D_1) \leq b$. One can conclude that $\bigvee f(\bigcup \mathcal{D}) \leq b$. Then $\bigvee_{D\in\mathcal{D}}(\bigvee f(D)) = \bigvee f(\bigcup \mathcal{D})$, that is, $\overline{f}(\bigcup \mathcal{D}) = \bigvee_{D\in\mathcal{D}} \overline{f}(D)$. So $\overline{f}$ is a local dcpo map. Also, for all $x \in P$, $\overline{f}(\eta(x)) = \overline{f}(\downarrow x) = \bigvee f(\downarrow x) = f(x)$. Thus $\overline{f} \circ \eta = f$. To prove the uniqueness of $\overline{f}$, let $g : L(P) \longrightarrow X$ be a local dcpo map such that $f = g \circ \eta$. For all $I \in L(P)$, $\{\downarrow x \mid x \in I\}$ is an upper bounded directed subset of $L(P)$, and $\bigvee\{\downarrow x \mid x \in I\} = \bigcup_{x\in I} \downarrow x = I$. Since g is a local dcpo map, we have

$$g(I) = g\Big(\bigcup_{x\in I} \downarrow x\Big) = \bigvee_{x\in I} g(\downarrow x) = \bigvee_{x\in I} (g \circ \eta)(x) = \bigvee_{x\in I} f(x) = \overline{f}(I).$$

Thus $g = \overline{f}$. Thus the forgetful functor U_5 : **LDcpo**$\longrightarrow$ **Pos** has a left adjoint. □

In the following, we can see that the right adjoint of the forgetful functor U_5 : **LDcpo** $\longrightarrow$ **Pos** does not necessarily exist.

Proposition 3.6. *Let $P = \{0, 1\}$ be a poset with $0 < 1$. Then the cofree local dcpo over P does not exist.*

Proof. Suppose that $K(P)$ is the cofree local dcpo over P. Take $\xi : K(P) \longrightarrow P$ to be the cofree monotone map. Then we can conclude that ξ is injective. Thus $|K(P)| \leq 2$. Define a map $f : \mathcal{P}(\mathbb{N}) \longrightarrow P$ as follows:

$$f(A) = \begin{cases} 0, & \text{if } A \text{ is finite,} \\ 1, & \text{otherwise.} \end{cases}$$

Then f is monotone. By the universal property of cofree maps, there exists a unique local dcpo map $\overline{f} : \mathcal{P}(\mathbb{N}) \longrightarrow K(P)$ with $\xi \circ \overline{f} = f$. Now, we consider two cases:

Case 1: When $|K(P)| = 1$, without loss of generality, assume that $K(P) = \{a\}$. If $\xi(a) = 0$, then $\xi(\overline{f}(\mathbb{N})) = 0$ and $f(\mathbb{N}) = 1$. Thus $\xi \circ \overline{f} \neq f$. But this is a contradiction. If $\xi(a) = 1$, then $\xi(\overline{f}(\{n\})) = 1$ and $f(\{n\}) = 0$ for all $n \in \mathbb{N}$. Thus $\xi \circ \overline{f} \neq f$, which is a contradiction.

Case 2: When $|K(P)| = 2$, without loss of generality, assume that $K(P) = \{x, y\}$. Then the order on $K(P)$ can be three cases.

(1) The order on $K(P)$ is discrete. Since ξ is injective, we can conclude that $\xi(x) = 0$, $\xi(y) = 1$ or $\xi(x) = 1$, $\xi(y) = 0$. If $\xi(x) = 0$ and $\xi(y) = 1$, whenever $\overline{f}(\mathbb{N}) = x$, then $0 = \xi(\overline{f}(\mathbb{N})) = f(\mathbb{N}) = 1$. But this is a contradiction; whenever $\overline{f}(\mathbb{N}) = y$, then $\overline{f}(\{n\}) = y$ for all $n \in \mathbb{N}$. Thus $1 = \xi(\overline{f}(\{n\})) = f(\{n\}) = 0$, which is a contradiction. Therefore, $\xi(x) = 0$ and $\xi(y) = 1$ do not hold. Similarly, we can prove that $\xi(x) = 1$ and $\xi(y) = 0$ do not hold.

(2) If $x \leq y$, then $\xi(x) = 0$, $\xi(y) = 1$. Whenever $\overline{f}(\mathbb{N}) = x$, then $0 = \xi(\overline{f}(\mathbb{N})) = f(\mathbb{N}) = 1$, which is a contradiction; whenever $\overline{f}(\mathbb{N}) = y$, since $\overline{f}$ is a local dcpo map, we can conclude that $y = \overline{f}(\mathbb{N}) = \overline{f}(\bigcup\{F \mid F \text{ is a finite subset of } \mathbb{N}\}) = \bigvee\{\overline{f}(F) \mid F \text{ is a finite subset of } \mathbb{N}\}$. Then there exists a finite subset F of $\mathbb{N}$ such that $\overline{f}(F) = y$. Thus $1 = \xi(\overline{f}(F)) = f(F) = 0$, which is a contradiction.

(3) The case $y \leq x$ is proved similar to (2).

Therefore, the cofree dcpo over P does not exist. □

Corollary 3.7. *The forgetful functor U_5 :* **LDcpo** $\longrightarrow$ **Pos** *does not have a right adjoint.*

Let P be a poset. A subset A of P is called *D-closed* if for any directed subset $D \subseteq A$, if $\bigvee D$ exists then $\bigvee D \in A$. The set of all D-closed subsets of P form the set of all closed sets of a topology on P, which will be called the D-topology (see [6, 10]). A subset U of P is called *Scott closed* if it is a lower set and D-closed. The set of all Scott closed sets of P is denoted by $\Gamma(P)$.

Now, we consider the left adjoint of the inclusion functor $I_2 : \mathbf{Dcpo} \longrightarrow \mathbf{LDcpo}$. Let P be a local dcpo, and let $\Psi(P) = \{\downarrow x \mid x \in P\}$. Define a map $\eta : P \longrightarrow cl_d(\Psi(P))$ by $\eta(x) = \downarrow x$, where $cl_d(\Psi(P))$ is the closure of $\Psi(P)$ in $\Gamma(P)$ with respect to the D-topology. Then $cl_d(\Psi(P))$ is a dcpo and η is universal (see Theorem 1 in [11]).

Proposition 3.8. Dcpo *is a full reflective subcategory of* **LDcpo**.

In the following, we see that the right adjoint of the inclusion functor $I_2 : \mathbf{Dcpo} \longrightarrow \mathbf{LDcpo}$ does not necessarily exist.

Example 3.9. We consider the local dcpo $\mathbb{N}$. Assume that the inclusion functor $I_2 : \mathbf{Dcpo} \longrightarrow \mathbf{LDcpo}$ has the right adjoint. Then there exist the dcpo $K(\mathbb{N})$ and the universal mapping $\xi : K(\mathbb{N}) \longrightarrow \mathbb{N}$. Obviously, ξ is injective. Let $Q = \mathbb{N} \oplus \{\infty\}$. Then Q is a dcpo. Next, we shall consider the following two cases.

Case 1: If $\xi(K(\mathbb{N})) \neq \mathbb{N}$, then there exists $n_0 \in \mathbb{N}$ such that $n_0 \notin \xi(K(\mathbb{N}))$. Define a map $f : Q \longrightarrow \mathbb{N}$ by $f(x) = n_0$, for all $x \in Q$. Then f is a local dcpo map. Since ξ is universal, there exists a unique Scott continuous map $\overline{f} : Q \longrightarrow K(\mathbb{N})$ with $\xi \circ \overline{f} = f$. Since $n_0 \notin \xi(K(\mathbb{N}))$, we can conclude that $\xi(\overline{f}(x)) \neq n_0$ for all $x \in Q$. Thus $\xi \circ \overline{f} \neq f$. But this is a contradiction.

Case 2: If $\xi(K(\mathbb{N})) = \mathbb{N}$. Assume that $K(\mathbb{N})$ is a directed set. Then $\bigvee K(\mathbb{N})$ exists. Since ξ is a local dcpo map, we have that $\xi(\bigvee K(\mathbb{N})) = \bigvee \xi(K(\mathbb{N})) = \bigvee \mathbb{N}$, but $\bigvee \mathbb{N}$ does not exist, which is a contradiction. So $K(\mathbb{N})$ is not a directed set. Thus there exist x_1, $x_2 \in K(\mathbb{N})$ such that $x_1 \not\leq d$ or $x_2 \not\leq d$ for all $d \in K(\mathbb{N})$. Therefore, x_1 and x_2 are incomparable. Let $\xi(x_1) = n_1$ and $\xi(x_2) = n_2$. Since $\mathbb{N}$ is a chain, without loss of generality, we assume that $n_1 <_{\mathbb{N}} n_2$. Define a map $f : Q \longrightarrow \mathbb{N}$ as follows:

$$f(x) = \begin{cases} x, & x \leq n_2, \\ n_2, & \text{otherwise.} \end{cases}$$

Thus f is a local dcpo map. Since ξ is universal, there exists a unique Scott continuous map $\overline{f} : Q \longrightarrow K(\mathbb{N})$ with $\xi \circ \overline{f} = f$. Therefore, $\xi(\overline{f}(n_1)) = f(n_1) = n_1 = \xi(x_1)$ and $\xi(\overline{f}(n_2)) = f(n_2) = n_2 = \xi(x_2)$. Since ξ is injective, we have $\overline{f}(n_1) = x_1$ and $\overline{f}(n_2) = x_2$. Since $n_1 <_{\mathbb{N}} n_2$, we can conclude that $x_1 \leq x_2$. This contradicts the fact that x_1 and x_2 are incomparable. Therefore, the inclusion functor $I_2 : \mathbf{Dcpo} \longrightarrow \mathbf{LDcpo}$ does not have a right adjoint.

Proposition 3.10. Dcpo *is not a coreflective subcategory of* **LDcpo**.

4 Free S-ldcpo and cofree S-ldcpo over a local dcpo

In this section, we consider the forgetful functor U_4, and prove that U_4 has both a left adjoint and a right adjoint.

Definition 4.1. A monoid $(S, \cdot, 1)$ is called a *local dcpo monoid* (or an *ldcpo-monoid*, for short) if it satisfies the following conditions:

(1) S is a local dcpo;

(2) For each $a \in S$ and each upper bounded directed subset $D \subseteq S$,

$$a \cdot (\bigvee D) = \bigvee_{d \in D} (a \cdot d) \text{ and } (\bigvee D) \cdot a = \bigvee_{d \in D} (d \cdot a).$$

Definition 4.2. Let $(S, \cdot, 1)$ be an ldcpo-monoid. An *S-ldcpo* is a pair $(A, \bullet)$ such that

(1) A is a local dcpo;

(2) $(A, \bullet)$ is an S-act;

(3) The action $\bullet : A \times S \longrightarrow A$ is a local dcpo map, where $A \times S$ is considered with the componentwise order.

Remark 4.3. Every S-dcpo is an S-ldcpo.

A *homomorphism* $f : (A, \bullet_A) \longrightarrow (B, \bullet_B)$ between S-ldcpos is a local dcpo map such that $f(a \bullet_A s) = f(a) \bullet_B s$ for all $a \in A$ and $s \in S$. Let **LDcpo**-S denote the category of all S-ldcpos with homomorphisms. Obviously, **Dcpo**-S is a subcategory of **LDcpo**-S.

Definition 4.4. Let $(S, \cdot, 1)$ be an ldcpo-monoid and P a local dcpo. A *free S-ldcpo* over a local dcpo P is a pair (F, τ), where F is an S-ldcpo and $\tau : P \longrightarrow F$ is a local dcpo map with the universal property that for each S-ldcpo A and a local dcpo map $f : P \longrightarrow A$ there exists a unique homomorphism $\overline{f} : F \longrightarrow A$ such that $\overline{f} \circ \tau = f$.

Theorem 4.5. *Let $(S, \cdot, 1)$ be an ldcpo-monoid and P a local dcpo. Then $(P \times S, \eta)$ is the free S-ldcpo over P, where $\eta : P \longrightarrow P \times S$ is defined as follows:*

$$\forall\ p \in P,\ \eta(p) = (p, 1).$$

Proof. By Remark 2.2(3), $P \times S$ is a local dcpo. Define a map $\star : (P \times S) \times S \longrightarrow P \times S$ as follows:

$$\forall\ (p, s) \in P \times S,\ t \in S,\ (p, s) \star t = (p, s \cdot t).$$

One can prove that $(P \times S, \star)$ is an S-ldcpo. Recalling that $P \times S$ is the free S-poset over the poset P with the universal map $\eta : P \longrightarrow P \times S$, given by $x \mapsto (x, 1)$ (see [2]), we show that η is a local dcpo map. Let D be an upper bounded directed subset of P. Obviously, η is a monotone map, we can conclude that $\eta(D)$ is an upper bounded directed subset of $P \times S$. Thus

$$\eta(\bigvee D) = (\bigvee D, 1) = \bigvee_{d \in D} \eta(d).$$

To prove the universal property of η, let $(B, *)$ be an S-ldcpo and $f : P \longrightarrow B$ a local dcpo map. Then the map $\overline{f} : P \times S \longrightarrow B$ defined by $\overline{f}((p, s)) = f(p) * s$, which is the unique action-preserving map with $\overline{f} \circ \eta = f$ (see [2]), is a local dcpo map. By Lemma 3.2, it suffices to show that for all $s \in S$ and $p \in P$, $\overline{f}_s$ and $\overline{f}_p$ are local dcpo maps. Let D be an upper bounded directed subset of P. Clearly, $\overline{f}_s$ is a monotone map. Thus $\overline{f}_s(D)$ is an upper bounded directed subset, and so one can conclude that

$$\overline{f}_s(\bigvee D) = \overline{f}(\bigvee D, s) = f(\bigvee D) * s = \Big(\bigvee_{d \in D} f(d)\Big) * s = \bigvee_{d \in D} \overline{f}_s(d).$$

Therefore, $\overline{f}_s$ is a local dcpo map. Similarly, we can prove that $\overline{f}_p$ is a local dcpo map. □

Corollary 4.6. *The forgetful functor* U_4 *:* **LDcpo**-$S \longrightarrow$ **LDcpo** *has a left adjoint.*

Definition 4.7. Let $(S, \cdot, 1)$ be an ldcpo-monoid and P a local dcpo. A *cofree S-ldcpo* over a local dcpo P is a pair (K, ξ), where K is an S-ldcpo and $\xi : K \longrightarrow P$ is a local dcpo map with the universal property that for each S-ldcpo A and a local dcpo map $f : A \longrightarrow P$ there exists a unique homomorphism $\overline{f} : A \longrightarrow K$ such that $\xi \circ \overline{f} = f$.

Lemma 4.8. *Let* $(S, \cdot, 1)$ *be an ldcpo-monoid and* P *a local dcpo. Then* $P^{(S)}$ *is a local dcpo under the pointwise order, where* $P^{(S)}$ *denotes the set of all local dcpo maps from* S *to* P*.*

Proof. Let $\{f_i \mid i \in I\}$ be an upper bounded directed subset of $P^{(S)}$. For all $s \in S$, we have that $\{f_i(s) \mid i \in I\}$ is an upper bounded directed subset of P. Since P is a local dcpo, $\bigvee_{i\in I} f_i(s)$ exists. Take $g(s) = \bigvee_{i\in I} f_i(s)$ for all $s \in S$. One can show that $g \in P^{(S)}$ and $\bigvee_{i\in I} f_i = g$. Therefore, $P^{(S)}$ is a local dcpo. □

Lemma 4.9. *Let* $(S, \cdot, 1)$ *be an ldcpo-monoid and* P *a local dcpo. Define an action* $\star : P^{(S)} \times S \longrightarrow P^{(S)}$ *as follows:*

$$\forall\ f \in P^{(S)},\ s,\ t \in S,\ (f \star s)(t) = f(s \cdot t).$$

Then $(P^{(S)}, \star)$ *is an S-ldcpo.*

Proof. For all $f \in P^{(S)}, s \in S$, $f \star s \in P^{(S)}$. In fact, let T be an upper bounded directed subset of S. Then $\bigvee T$ exists, and thus

$$\begin{aligned}(f \star s)(\bigvee T) = f(s \cdot (\bigvee T)) &= f\Big(\bigvee_{a\in T}(s \cdot a)\Big)\\ &= \bigvee_{a\in T} f(s \cdot a) = \bigvee_{a\in T}((f \star s)(a)).\end{aligned}$$

Next, we shall prove that $(P^{(S)}, \star)$ is an S-ldcpo. Since for all $f \in P^{(S)}$ and $t \in S$, $(f \star 1)(t) = f(1 \cdot t) = f(t)$, we have that $f \star 1 = f$. For all $f \in P^{(S)}$ and $s, t \in S$, we shall prove that $(f \star s) \star t = f \star (s \cdot t)$. For all $a \in S$, $((f \star s) \star t)(a) = (f \star s)(t \cdot a) = f(s \cdot (t \cdot a)) = f((s \cdot t) \cdot a) = (f \star (s \cdot t))(a)$. Thus $(f \star s) \star t = f \star (s \cdot t)$. Now, we show that the action is a local dcpo

map. Applying Lemma 3.2, let $\{f_i \mid i \in I\}$ be an upper bounded directed subset of $P^{(S)}$, $s, t \in S$. Then

$$\begin{aligned}\Big(\Big(\bigvee_{i\in I} f_i\Big) \star s\Big)(t) &= \Big(\bigvee_{i\in I} f_i\Big)(s \cdot t)\\ &= \bigvee_{i\in I} f_i(s \cdot t)\\ &= \bigvee_{i\in I} (f_i \star s)(t)\\ &= \Big(\bigvee_{i\in I} (f_i \star s)\Big)(t).\end{aligned}$$

Thus $\Big(\bigvee_{i\in I} f_i\Big) \star s = \bigvee_{i\in I}(f_i \star s)$. Now, let T be an upper bounded directed subset of S and $f \in P^{(S)}$. Moreover, for all $a \in S$, we have that

$$(f \star (\bigvee T))(a) = f\Big(\bigvee_{t\in T} (t \cdot a)\Big) = \bigvee_{t\in T} f(t \cdot a) = \Big(\bigvee_{t\in T} (f \star t)\Big)(a).$$

Thus $f \star (\bigvee T) = \bigvee_{t\in T}(f \star t)$. Therefore, $(P^{(S)}, \star)$ is an S-ldcpo. □

Theorem 4.10. *Let $(S, \cdot, 1)$ be an ldcpo-monoid and P a local dcpo. Then $(P^{(S)}, \xi)$ is a cofree S-ldcpo over P, where $\xi : P^{(S)} \longrightarrow P$ is defined as follows:*

$$\forall\ f \in P^{(S)},\ \xi(f) = f(1).$$

Proof. By Lemma 4.9, we have that $(P^{(S)}, \star)$ is an S-ldcpo. Recalling that $P^{(S)}$ is the cofree S-poset over the poset P with the universal map ξ (see [2]). One can prove that ξ is a local dcpo map. Next, we shall prove the universal property of ξ. Let $(A, \diamond)$ be an S-ldcpo and $f : A \longrightarrow P$ a local dcpo map. Define a map $\overline{f} : A \longrightarrow P^{(S)}$ as follows:

$$\forall\ a \in A, s \in S,\ \overline{f}(a)(s) = f(a \diamond s).$$

Then $\overline{f}$ is the unique action-preserving map with $\xi \circ \overline{f} = f$ (see [2]). Finally, we prove that $\overline{f}$ is a local dcpo map. Let T be an upper bounded directed subset of S. Then for all $a \in A$, $\{a \diamond b \mid b \in T\}$ is an upper bounded directed subset of A, and thus

$$\overline{f}(a)(\bigvee T) = f(a \diamond (\bigvee T)) = f\Big(\bigvee_{b\in T} (a \diamond b)\Big) = \bigvee_{b\in T} f(a \diamond b) = \bigvee_{b\in T} \overline{f}(a)(b).$$

Therefore, $\overline{f}(a) \in P^{(S)}$. Let D be an upper bounded directed subset of A. Then for all $t \in S$,

$$\overline{f}(\bigvee D)(t) = f((\bigvee D) \diamond t) = f\Big(\bigvee_{d\in D}(d \diamond t)\Big) = \bigvee_{d\in D} f(d \diamond t) = \Big(\bigvee_{d\in D} \overline{f}(d)\Big)(t).$$

Thus $\overline{f}(\bigvee D) = \bigvee_{d\in D} \overline{f}(d)$. Therefore, $(P^{(S)}, \xi)$ is a cofree S-ldcpo over P. □

Corollary 4.11. *The forgetful functor* U_4 *:* **LDcpo**-$S\longrightarrow$ **LDcpo** *has a right adjoint.*

5 Adjoint Relations for LDcpo-S

In this section, we consider the forgetful functor U_6, and show that if S satisfied a condition which we call "good", then U_6 has a left adjoint. Also we show that U_6 does not have a right adjoint.

Definition 5.1. Let P be a local dcpo, and D be an upper bounded directed subset of P. If $\bigvee D \in D$, then we say that P is *good*.

Remark 5.2. A local dcpo P is good if and only if every upper bounded directed subset has a top element.

Definition 5.3. (see [9]) Let P be a poset and $x, y \in P$. We write $x \ll_l y$ and say that x *universally approximates* y if for any directed set D and any upper bound z of D with $y \leq \bigvee_z D$ ($\bigvee_z D$ means that D has a supremum in $\downarrow z$), there is $d \in D$ such that $x \leq d$.

Proposition 5.4. *Let* P *be a local dcpo. Then* P *is good if and only if* $x \ll_l x$ *for all* $x \in P$.

Proof. Let B be a directed subset of P, and d be an upper bound of B with $x \leq \bigvee_d B$. Since P is a local dcpo, we have that $x \leq \bigvee B = \bigvee_d B$. Since P is good, $\bigvee_d B \in B$. Thus $x \ll_l x$.

Conversely, let D be an upper bounded directed subset of P. Then there exists $x \in P$ such that $D \subseteq \downarrow x$. Since P is a local dcpo, $\bigvee D$ exists and $\bigvee D \leq x$. Then $\bigvee_x D = \bigvee D$. Since $\bigvee D \ll_l \bigvee D = \bigvee_x D$, there exists $d \in D$ such that $\bigvee D \leq d$. Then $\bigvee D = d$, and thus $\bigvee D \in D$. □

Proposition 5.5. *Let $(S, \cdot, 1)$ be a good ldcpo-monoid and $(P, *)$ an S-poset. Define an action $\odot : L(P) \times S \longrightarrow L(P)$ as follows:*

$$\forall\ F \in L(P),\ s \in S,\ F \odot s = {\downarrow}(F * s),$$

where $F * s = \{a * s \mid a \in F\}$. Then $(L(P), \odot)$ is an S-ldcpo.

Proof. By Lemma 3.3, we have that $L(P)$ is a local dcpo. For all $F \in L(P)$, $s, t \in S$, we have that

$$F \odot 1 = {\downarrow}\{a * 1 \mid a \in F\} = {\downarrow}\{a \mid a \in F\} = {\downarrow}F = F$$

and

$$F \odot (s \cdot t) = {\downarrow}(F * (s \cdot t)) = {\downarrow}((F * s) * t) = {\downarrow}(({\downarrow}(F * s)) * t) = (F \odot s) \odot t.$$

Now, we show that the action is a local dcpo map. Applying Lemma 3.2, let $\mathcal{D}$ be an upper bounded directed subset of $L(P)$. By the given action, we have that

$$\begin{aligned}
(\bigvee \mathcal{D}) \odot s &= (\bigcup \mathcal{D}) \odot s \\
&= {\downarrow}((\bigcup_{D \in \mathcal{D}} D) * s) \\
&= {\downarrow}\Big(\bigcup_{D \in \mathcal{D}} (D * s)\Big) \\
&= \bigcup_{D \in \mathcal{D}} {\downarrow}(D * s) \\
&= \bigvee_{D \in \mathcal{D}} (D \odot s).
\end{aligned}$$

Now, let T be an upper bounded directed subset of S and $F \in L(P)$. Since S is good, $\bigvee T \in T$. Then

$$F \odot (\bigvee T) = {\downarrow}(F * (\bigvee T)) = \bigvee_{t \in T} {\downarrow}(F * t) = \bigvee_{t \in T} (F \odot t).$$

Then $(L(P), \odot)$ is an S-ldcpo. □

Theorem 5.6. *Let $(S, \cdot, 1)$ be a good ldcpo-monoid. For a given S-poset $(P, *)$, the free S-ldcpo over P is $(L(P), \odot)$.*

Proof. By Proposition 5.5, we have that $(L(P), \odot)$ is an S-ldcpo. By the proof of Theorem 3.5, the map η, defined in Theorem 3.5, is an S-poset

map. Finally, we show that η is universal. Let $(B, \bullet)$ be an S-ldcpo and $f : P \longrightarrow B$ be an S-poset map. By the proof of Theorem 3.5, there exists a unique local dcpo map $\overline{f} : L(P) \longrightarrow B$ such that $\overline{f} \circ \downarrow = f$, where $\overline{f}$ is defined by $\overline{f}(F) = \bigvee f(F)$. It suffices to prove that $\overline{f}$ is action-preserving. For all $F \in L(P)$ and $s \in S$, we have that

$$\overline{f}(F \odot s) = \overline{f}(\downarrow (F * s)) = \bigvee f(\downarrow (F * s)) = \bigvee f(F * s) = \overline{f}(F) \bullet s.$$

Then $\overline{f}$ is a homomorphism, and thus η is universal. □

Corollary 5.7. *If $(S, \cdot, 1)$ is a good ldcpo-monoid, then the forgetful functor U_6 : **LDcpo**-S $\longrightarrow$**Pos**-S has a left adjoint.*

Finally, by taking $S = \{1\}$, and applying Corollary 3.7 and Proposition 3.10, we have the following Propositions.

Proposition 5.8. *The forgetful functor U_6 : **LDcpo**-S $\longrightarrow$ **Pos**-S does not have a right adjoint for a general ldcpo-monoid S.*

Proposition 5.9. **Dcpo**-S *is not a coreflective subcategory of* **LDcpo**-S *for a general dcpo-monoid S.*

6 Conclusions

Applying the investigations done in the above sections, we get:

(1) The free local dcpo over a poset P is $L(P)$.
(2) The free S-ldcpo over a local dcpo P is $P \times S$.
(3) If S is a good ldcpo-monoid, the free S-ldcpo over an S-poset P is $L(P)$.
(4) The free S-dcpo over a local dcpo P is $cl_d(\Psi(P)) \times S$.
(5) The free S-ldcpo over a poset P is $L(P) \times S$.
(6) The cofree local dcpo over a poset P does not necessarily exist.
(7) The cofree S-ldcpo over a local dcpo P is $P^{(S)}$.
(8) The cofree S-ldcpo over an S-poset P does not necessarily exist.
(9) The cofree S-ldcpo over a poset P does not necessarily exist.
(10) The cofree S-dcpo over a local dcpo P does not necessarily exist.
(11) **Dcpo** is a full reflective subcategory of **LDcpo**.

(12) **Dcpo** is not a coreflective subcategory of **LDcpo**.

(13) **Dcpo**-S is not a coreflective subcategory of **LDcpo**-S for a general dcpo-monoid S.

However, there are several basic questions to which we possess no answers. For instance, it is not known whether the inclusion functor I_3 : **Dcpo**-$S \longrightarrow$ **LDcpo**-S has a left adjoint. Also, for any ldcpo-monoid S, whether the forgetful functors U_6 : **LDcpo**-$S \longrightarrow$ **Pos**-S has a left adjoint.

Acknowledgements

This work is supported by the National Natural Science Foundation of China (Grant nos. 11531009, 11301316), the Natural Science Program for Basic Research of Shaanxi Province (Grant no. 2015JM1020) and the Fundamental Research Funds for the Central Universities (Grant no. GK201501001).

The authors would like to thank the anonymous referee for his (her) valuable comments and suggestions.

References

[1] Adámek, J., Herrlich, H., and Strecker, G.E., "Abstract and Concrete Categories: The Joy of Cats", John Wiley & Sons, New York, 1990.

[2] Bulman-Fleming, S. and Mahmoudi, M., *The category of S-posets*, Semigroup Forum 71 (2005), 443-461.

[3] Crole, R.L., "Categories for Types", Cambridge University Press, Cambridge, 1994.

[4] Erné, M., *Minimal bases, ideal extensions, and basic dualities*, Topology Proc. 29 (2005), 445-489.

[5] Gierz, G., Hofmann, K.H., Keimel, K., Lawson, J.D., Mislove, M., and Scott, D.S., "Continuous Lattices and Domains", Encyclopedia of Mathematics and its Applications 93, Cambridge University Press, 2003.

[6] Keimel, K. and Lawson, J.D., *D-completions and the d-topology*, Ann. Pure Appl. Logic 159(3) (2009), 292-306.

[7] Mahmoudi, M. and Moghbeli, H., *Free and cofree acts of dcpo-monoids on directed complete posets*, Bull. Malaysian Math. Sci. Soc. 39 (2016), 589-603.

[8] Mislove, M.W., *Local dcpos, local cpos, and local completions*, Electron. Notes Theor. Comput. Sci. 20 (1999), 399-412.

[9] Xu, L. and Mao, X., *Strongly continuous posets and the local Scott topology*, J. Math. Anal. Appl. 345 (2008), 816-824.

[10] Zhao, D. and Fan, T., *Dcpo-completion of posets*, Theoret. Comput. Sci. 411 (2010), 2167-2173.

[11] Zhao, D., *Partial dcpo's and some applications*, Arch. Math. (Brno) 48 (2012), 243-260.

Bin Zhao, *Department of Mathematics, Shaanxi Normal University, Xi'an, 710062, P.R. China.*

Email: zhaobin@snnu.edu.cn

Jing Lu, *Department of Mathematics, Shaanxi Normal University, Xi'an, 710062, P.R. China.*

Email: lujing0926@126.com

Kaiyun Wang, *Department of Mathematics, Shaanxi Normal University, Xi'an, 710062, P.R. China.*

Email: wangkaiyun@snnu.edu.cn

رسته‌ها و ساختارهای کلی جبری با کاربردها
جلد ۷، شماره ۱، تیر ۱۳۹۶ (شماره ویژه برای بزرگداشت ۹۰ سالگی برنارد بناشفسکی)

رابطه‌های الحاقی رسته‌ی dcpoهای موضعی

بین ژائو، ژینگ لو، و کابون وانگ

در این مقاله، تابعگون فراموشکار از رسته‌ی **LDcpo** از dcpoهای موضعی (به ترتیب، رسته‌ی **Dcpo** متشکل از dcpoها) به رسته‌ی **Pos** از مجموعه‌های مرتب جزئی (به ترتیب، به رسته **LDcpo** از dcpoهای موضعی) را در نظر می‌گیریم، و وجود الحاقی‌های چپ و راست آن را مطالعه می‌کنیم. به علاوه، صورت ملموس $S-\mathrm{ldcpo}$ها را روی یک dcpo موضعی ارائه می‌کنیم، که در آن S یک dcpo تکواره‌ی موضعی است. قضیه‌های اصلی عبارت‌اند از:

(۱) تابعگون فراموشکار $U : \mathbf{LDcpo} \to \mathbf{Pos}$ دارای الحاق چپ است، ولی الحاق راست ندارد؛

(۲) تابعگون شمولی $I : \mathbf{Dcpo} \to \mathbf{LDcpo}$ دارای الحاق چپ است، ولی الحاق راست ندارد؛

(۳) تابعگون فراموشکار $U : \mathbf{LDcpo}\text{-}S \to \mathbf{LDcpo}$ دارای الحاق چپ و راست است؛

(۴) اگر $(S, \cdot, 1)$ یک ldcpo تکواره‌ی خوب باشد، آنگاه تابعگون فراموشکار

$$U : \mathbf{LDcpo}\text{-}S \to \mathbf{Pos}\text{-}S$$

دارای الحاق چپ است.

دریافت: ۲۰ اکتبر ۲۰۱۶، پذیرش: ۱۶ ژانویه ۲۰۱۷
شاپا چاپی: ۵۸۵۳-۲۳۴۵ برخط: ۵۸۶۱-۲۳۴۵