



Lattice of compactifications of a topological group

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Abstract. We show that the lattice of compactifications of a topological group G is a complete lattice which is isomorphic to the lattice of all closed normal subgroups of the Bohr compactification bG of G . The correspondence defines a contravariant functor from the category of topological groups to the category of complete lattices. Some properties of the compactification lattice of a topological group are obtained.

1 Introduction

Let X be a Tychonoff space. It is well known that the collection $K(X)$ of all Hausdorff compactifications of X forms a complete upper semi-lattice under the order relation defined by $c_1X \leq c_2X$ if and only if there is a continuous map $f : c_2X \rightarrow c_1X$, which leaves X pointwise fixed. In general $K(X)$ is not a complete lattice and it is a complete lattice if and only if X is locally compact. There are many results studying the relationship between topological properties of X and the order structure of $K(X)$ (see [4], [9]-[13], [16]).

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Here we consider the Hausdorff compactifications of a Hausdorff topological group G .

Let G be a topological group. Recall that a compactification of G is a pair (K, φ) , where K is a compact Hausdorff group and $\varphi : G \rightarrow K$ is a dense continuous homomorphism. For two compactifications (K, φ) and (H, ψ) of G , we say that (K, φ) and (H, ψ) are equivalent if there is a topological isomorphism $h : K \rightarrow H$ such that $\psi = h \circ \varphi$. This defines an equivalence relation on the collection of all Hausdorff compactifications of G . We use the same symbol $K(G)$ (like in the case of topological spaces) to denote the set of all equivalence classes of the Hausdorff compactifications of G . There exists a natural order relation on $K(G)$ defined as follows:

$(K, \varphi) \leq (H, \psi)$ if there exists a continuous homomorphism $\mu : H \rightarrow K$ such that $\varphi = \mu \circ \psi$.

The following result is well known. For convenience of the readers we give a proof here.

Lemma 1.1. *Let (K, φ) and (H, ψ) be two compactifications of G . Then (K, φ) and (H, ψ) are equivalent if and only if $(K, \varphi) \leq (H, \psi)$ and $(H, \psi) \leq (K, \varphi)$ both hold.*

Proof. The necessity is clear. We only need to show the sufficiency. Suppose that $(K, \varphi) \leq (H, \psi)$ and $(H, \psi) \leq (K, \varphi)$ both hold. We have continuous homomorphisms $f : H \rightarrow K$ and $g : K \rightarrow H$ such that $\varphi = f \circ \psi$, $\psi = g \circ \varphi$. It follows that $g \circ f \circ \varphi = g \circ \psi = \varphi$. This implies that $g \circ f = id_H$ since φ is dense. Similarly we have $f \circ g = id_K$. Hence $f : H \rightarrow K$ is a topological isomorphism. \square

By this lemma, $(K(G), \leq)$ is a partially ordered set. Clearly $K(G)$ has the least element, that is the trivial group and trivial homomorphism, and also has the largest element, the Bohr compactification of G .

In this note we investigate the order algebraic structure of the compactification lattice $K(G)$ of a given topological group G and its relationship with the properties of G .

2 Preliminaries

Throughout this paper we consider the category **TopGrp** of Hausdorff topological groups and continuous homomorphisms. **TopAb** and **Ab** respec-

tively denote the category of topological abelian groups and the category of discrete abelian groups. The categories of complete lattice and sup-preserving maps and the category of complete lattice and inf-preserving maps are denoted by **CLat** and **CLat^{op}**, respectively.

We denote by \mathbb{N} the set of positive natural numbers; by \mathbb{Z} the integers, by \mathbb{Q} the rationals, by \mathbb{R} the reals, and by \mathbb{T} the unit circle group which is identified with \mathbb{R}/\mathbb{Z} . We write $\mathbb{U} = \prod_{n=1}^{\infty} U(n)$, where $U(n)$ is the group of all $n \times n$ unitary matrices in $GL_n(\mathbb{C})$. According to Peter-Weyl's theorem, for every compact group K , the continuous homomorphisms $K \rightarrow \mathbb{U}$ separate the points of K [7]. It follows that every compact group can be topologically embedded into some power of \mathbb{U} . Hence we can take \mathbb{U} in **TopGrp** in the place of the circle group \mathbb{T} in the category **TopAb** of topological Abelian groups. The cyclic group of order $n > 1$ is denoted by $\mathbb{Z}(n)$.

The subgroup generated by a subset X of a group G is denoted by $\langle X \rangle$, and $\langle x \rangle$ is the cyclic subgroup of G generated by an element $x \in G$. The abbreviation $K \leq G$ means that K is a subgroup of G , and $N \triangleleft G$ means that N is a normal subgroup of G .

For an arbitrary topological group G , we denote the family of all continuous homomorphisms from G to \mathbb{U} by the symbol $C^*(G)$. The Bohr compactification of G is denoted by $b : G \rightarrow bG$. The group G endowed with the Bohr topology, that is, the topology induced by the family of all continuous homomorphisms from G to \mathbb{U} , is denoted by G^+ . The von Neumann's kernel of G is denoted by $N(G)$, that is, $N(G) = \ker(b) = \bigcap \{ \ker(f) \mid f \in C^*(G) \}$.

Throughout the paper all topological groups are assumed to be Hausdorff. All unexplained topological terms can be found in [3].

3 Compactification lattice of a topological group

Lemma 3.1. *For every topological group G , the partially ordered set $K(G)$ is a complete lattice.*

Proof. We only need to show that every family of compactifications of G has supremum.

Let $\{c_i : G \rightarrow c_iG \mid i \in I\}$ be a family of compactifications of G . Consider the diagonal map $\langle c_i \rangle : G \rightarrow \prod_{i \in I} c_iG$. Denote by $cG = \overline{\langle c_i \rangle(G)}$ the closure of the image of G under $\langle c_i \rangle$, and $c : G \rightarrow cG$ the restriction of the mapping $\langle c_i \rangle$. Then $c : G \rightarrow cG$ is a compactification of G , and

the equalities $p_i \circ c = c_i, i \in I$, imply that $c : G \rightarrow cG$ is an upper bound of $\{c_i : G \rightarrow c_iG \mid i \in I\}$, where $p_i : cG \rightarrow c_iG$ is the i^{th} projection. We now show that the compactification $c : G \rightarrow cG$ is the supremum of $\{c_i : G \rightarrow c_iG \mid i \in I\}$.

Indeed, suppose that $k : G \rightarrow K$ is a compactification of G such that for each $i \in I$, there exists a continuous homomorphism $h_i : K \rightarrow c_iG$ satisfying $c_i = h_i \circ k$. Then the diagonal map $\langle h_i \rangle : K \rightarrow \prod_{i \in I} c_iG$ satisfies $\langle h_i \rangle \circ k = \langle h_i \circ k \rangle = \langle c_i \rangle$. Also we have $\langle h_i \rangle(K) = \langle h_i \rangle(\overline{k(G)}) \subseteq \overline{\langle h_i \rangle(k(G))} = \overline{\langle c_i \rangle(G)}$. Let $h : K \rightarrow cG$ be the restriction of $\langle h_i \rangle$, then we have $h \circ k = c$. \square

Let G be a topological group. For every compactification $c : G \rightarrow cG$ of G , c can be uniquely factored through $b : G \rightarrow bG$ as $c = c^b \circ b$:

$$\begin{array}{ccc}
 G & \xrightarrow{b} & bG \\
 & \searrow c & \swarrow c^b \\
 & & cG
 \end{array}$$

Clearly, c^b is surjective since c is dense and c^b is closed. Denote by $L(cG) = \ker(c^b)$ the kernel of c^b . Then $L(cG)$ is a closed normal subgroup of bG .

Let also $CN(bG)$ be the set of all closed normal subgroups of bG ordered by inverse inclusion. We have the following result.

Proposition 3.2. $L : K(G) \rightarrow CN(bG)$ is an isomorphism.

Proof. Notice that L has an inverse map which sends each closed normal subgroup $N \triangleleft bG$ to the compactification $q \circ b : G \rightarrow bG/N$, where $q : bG \rightarrow bG/N$ is the quotient map. Also for two compactifications $c_1 : G \rightarrow c_1G$ and $c_2 : G \rightarrow c_2G$ of G , it is clear that $c_1 \leq c_2$ if and only if $\ker(c_2^b) \subseteq \ker(c_1^b)$, that is, $L(c_2G) \subseteq L(c_1G)$. Hence $L : K(G) \rightarrow CN(bG)$ is an isomorphism. \square

Corollary 3.3. Let G and H be topological groups. If bG is topologically isomorphic to bH , then $K(G)$ and $K(H)$ are isomorphic.

A classical result in [11] showed that if X and Y are locally compact spaces, then their lattices of compactifications $K(X)$ and $K(Y)$ are isomorphic if and only if $\beta X \setminus X$ and $\beta Y \setminus Y$ are homeomorphic. For a

topological group G , the remainder $bG \setminus b(G)$ in general does not determine the order structure of $K(G)$. Indeed, if we take $S(X)$ the permutation group of an infinite set X , we know that $S(X)$ is a minimally almost periodic groups, that is, $N(S(X)) = S(X)$. Put $H = \mathbb{T}$ the unit circle group. Then $bS(X)$ is the trivial group and $bH = H$. Hence the remainder $bS(X) \setminus b(S(X)) = bH \setminus b(H) = \emptyset$. But it is clear that $K(bS(X))$ is not isomorphic to $K(H)$.

Corollary 3.4. *Let L be a complete lattice. L is isomorphic to the compactification lattice $K(G)$ of a topological group G if and only if L is isomorphic to the lattice of all closed normal subgroups of a compact group.*

Let G be a topological group and let $H \leq G$ be a dense subgroup of G . Then for every compactification $c : G \rightarrow cG$ of G the restriction of c to H is a compactification of H . Conversely every compactification $c : H \rightarrow cH$ of H admits a unique extension $\tilde{c} : G \rightarrow cH$. Hence the following result is clear.

Proposition 3.5. *If H is a dense subgroup of a topological group G , then $K(G) = K(H)$.*

If G is a topological Abelian group, we can give an alternative description of $K(G)$. Let $C^*(G)$ denote the group of all continuous characters on G , that is, the group of all continuous homomorphisms $f : G \rightarrow \mathbb{T}$. Suppose that $N \leq C^*(G)$. Then we have a compactification $c_N : G \rightarrow c_N G$ corresponding to N such that $c_N G$ is the closure of the image of G in \mathbb{T}^N under the diagonal map c_N . For two subgroups $N_1 \leq C^*(G), N_2 \leq C^*(G)$, we write $N_1 \sim N_2$ if the compactifications $c_{N_1} : G \rightarrow c_{N_1} G$ and $c_{N_2} : G \rightarrow c_{N_2} G$ corresponding to N_1 and N_2 , respectively, are equivalent. Denote by $Subgp(C^*(G))/\sim$ the quotient set of the set $Subgp(C^*(G))$ of all subgroups of $C^*(G)$. Let $[N], [M] \in Subgp(C^*(G))/\sim$. Define $[N] \leq [M]$ if and only if $c_N G \leq c_M G$, where $c_N G$ and $c_M G$ are the compactifications corresponding to N and M , respectively.

Proposition 3.6. *The partially ordered set $(Subgp(C^*(G))/\sim, \leq)$ is isomorphic to $K(G)$.*

Corollary 3.7. *Let G be a topological abelian group. Then $|K(G)| \leq 2^{|C^*(G)|}$.*

Note that the above corollary is true for an arbitrary topological group G , though then $C^*(G)$ is not a group.

Let L be a complete lattice. An element $a \in L$ is called compact if for every family $\{a_i \mid i \in I\} \subset L$ with $a \leq \bigvee a_i$, there exists finite set $J \subset I$ such that $a \leq \bigvee_{i \in J} a_i$. Let $k(L)$ be the set of all compact elements of L . A complete lattice L is said to be an algebraic lattice if $a = \bigvee \{c \in k(L) \mid c \leq a\}$ for every $a \in L$. L is said to be a dual algebraic lattice if the dual lattice L^{op} of L is algebraic.

Theorem 3.8. *If L is a compactification lattice of a topological Abelian group, then there is a dual algebraic lattice S such that L is a retraction of S .*

Proof. Suppose that G is a topological Abelian group and L is order isomorphic to the compactification lattice $K(G)$ of G . Let S be the lattice of all subgroups of bG . We first show that S is a dual algebraic lattice (with respect to the inverse inclusion order).

Indeed, S is a complete lattice and it contains L as a sub-upper semi-lattice. It is clear that the family of all finitely generated subgroups of bG and the family of all compact element of S^{op} coincide, and also every subgroup $H \leq bG$ can be represented as a supremum of a family of finitely generated subgroups of bG .

Let $r : S \rightarrow L$ be such that for every $a \in S$, $r(a)$ is in the closure of a in bG . Then r is a retraction. \square

4 The contravariant lattice-valued functor

Let $f : G \rightarrow H$ be a continuous homomorphism. We define $K(f) : K(G) \rightarrow K(H)$ as the mapping which sends every compactification $c : H \rightarrow cH$ to the compactification $c \circ f : G \rightarrow \overline{c(f(G))}$ of G , where $\overline{c(f(G))}$ means the closure of $c(f(G))$ in cH . Equivalently, if we regard $K(G)$ as $CN(bG)$, and $K(H)$ as $CN(bH)$, then $K(f) : CN(bH) \rightarrow CN(bG)$ sends every closed normal subgroup $N \triangleleft bH$ to the closed normal subgroup $bf^-(N) \triangleleft bG$, where $bf : bG \rightarrow bH$ is the extension of f .

Proposition 4.1. *The assignment $G \mapsto K(G)$ is a contravariant functor from the category \mathbf{TopGp}^{op} to the category \mathbf{CLat} of complete lattices.*

Proof. For two continuous homomorphisms $f : G \rightarrow H$ and $g : H \rightarrow M$ of topological groups, it is clear that $K(g \circ f) = K(g) \circ K(f)$. We only need to show that $K(f) : CN(bH) \rightarrow CN(bG)$ preserves arbitrary joins, for any continuous homomorphism $f : G \rightarrow H$.

$K(f)$ maps H to G , that is, it preserves the bottom element. Let $\{N_i \mid i \in I\}$ be a family of closed normal subgroups of bH . We have $\bigvee_{i \in I} N_i = \bigcap_{i \in I} N_i$, and hence

$$bf^{-}\left(\bigvee_{i \in I} N_i\right) = bf^{-}\left(\bigcap_{i \in I} N_i\right) = \bigcap_{i \in I} bf^{-}(N_i) = \bigvee_{i \in I} bf^{-}(N_i).$$

This shows that $K(f)$ preserves arbitrary joins. □

Lemma 4.2. *Let G and H be topological groups and let $f : G \rightarrow H$ be a dense continuous homomorphism. Then $K(f) : K(H) \rightarrow K(G)$ is injective.*

Proof. The composition $G \rightarrow bG \rightarrow bH = G \rightarrow H \rightarrow bH$ is dense, since $b_H : H \rightarrow bH$ is dense. It follows that $bf : bG \rightarrow bH$ is dense and, hence, it is surjective since it is closed:

$$\begin{array}{ccc} G & \xrightarrow{f} & H \\ \downarrow & & \downarrow \\ bG & \xrightarrow{bf} & bH \end{array}$$

Thus $K(f) : K(H) \rightarrow K(G)$ is injective. □

We know that in the category **TopAb** of topological abelian groups, monomorphisms are precisely one-to-one continuous homomorphisms and epimorphisms are precisely dense continuous homomorphisms. Hence we have the following result.

Proposition 4.3. *The functor $K : \mathbf{TopAb}^{\text{op}} \rightarrow \mathbf{CLat}$ preserves monomorphisms.*

Let G be a topological group. We know that the Bohr compactification bG of G can be obtained in two steps:

first taking $pG = (G/N(G))^+$ the quotient group $G/N(G)$ endowed with the Bohr topology. Let $q : G \rightarrow pG$ be the quotient map, which is in fact

the precompact reflection of G . Next we put $bG = \varrho pG$ to be the Raïkov completion of pG . Then bG is the Bohr compactification of G . Suppose that $f : G \rightarrow H$ is a continuous homomorphism. Then f has a continuous homomorphism extension $pf : pG \rightarrow pH$ such that the following diagram commutes:

$$\begin{array}{ccc} G & \xrightarrow{f} & H \\ \downarrow & & \downarrow \\ pG & \xrightarrow{pf} & pH \end{array}$$

It is clear that $\text{Ker}(pf) = \{e_{pG}\}$ if and only if $N(G) = f^{-}(N(H))$. By the construction of the Raïkov completion, we know that $\text{Ker}(pf) = \{e_{pG}\}$ if and only if $\text{Ker}(bf) = \{e_{bG}\}$:

$$\begin{array}{ccc} pG & \xrightarrow{pf} & pH \\ \downarrow & & \downarrow \\ bG & \xrightarrow{bf} & bH \end{array}$$

Lemma 4.4. *Let $f : G \rightarrow H$ be a continuous homomorphism. Then $K(f) : K(H) \rightarrow K(G)$ is surjective if and only if $N(G) = f^{-}(N(H))$.*

Proof. Suppose that $N(G) = f^{-}(N(H))$. By the above argument, $\text{ker}(bf)$ is trivial, hence bf is one-to-one. It follows that $bf : bG \rightarrow bH$ is a closed embedding, thus $K(f) : K(H) \rightarrow K(G)$ is surjective.

Conversely, if $K(f) : K(H) \rightarrow K(G)$ is surjective, then there exists a closed normal subgroup $N \triangleleft bH$ such that $bf^{-}(N) = \{e_{bG}\}$. It follows that $\text{ker}(bf) = \{e_{bG}\}$. This implies that $N(G) = f^{-}(N(H))$. \square

Let G and H be two discrete abelian groups and let $f : G \rightarrow H$ be a one-to-one homomorphism. Then each homomorphism $g : G \rightarrow \mathbb{T}$ has an extension $\hat{g} : H \rightarrow \mathbb{T}$ such that $g = \hat{g} \circ f$. It follows that $N(G) = f^{-}(N(H))$.

Corollary 4.5. *The functor $K : \mathbf{Ab}^{\text{op}} \rightarrow \mathbf{CLat}$ preserves epimorphisms.*

In general the functor $K : \mathbf{TopAb}^{\text{op}} \rightarrow \mathbf{CLat}$ does not preserve epimorphisms. Indeed, if we take a topological Abelian group H such that H is not maximally almost periodic and denote by G the group H endowed with

discrete topology, then $N(G)$ is the trivial group and $N(H) \neq \{e_H\}$. By Lemma 3.4, $K(id) : K(G) \rightarrow K(H)$ is not surjective.

Proposition 4.6. *If $f : G \rightarrow H$ is a dense continuous homomorphism such that $N(G) = f^{-1}(N(H))$, then bG and bH are topologically isomorphic. So $K(G)$ is order isomorphic to $K(H)$.*

Proof. Clearly we have $\ker(bf) = \{e_G\}$ and hence $bf : bG \rightarrow bH$ is injective, since $N(G) = f^{-1}(N(H))$. Also the composition $G \rightarrow bG \rightarrow bH = G \rightarrow H \rightarrow bH$ is dense, since $f : G \rightarrow H$ is dense. It follows that $bf : bG \rightarrow bH$ is dense and surjective, since it is closed. \square

Furthermore, we have the following result.

Proposition 4.7. *Let $f : G \rightarrow H$ be a continuous homomorphism. Then $K(f) : K(H) \rightarrow K(G)$ is an isomorphism if and only if $bf : bG \rightarrow bH$ is an embedding and for any two closed subgroups N_1 and N_2 of bH , $N_1 \cap bG = N_2 \cap bG$ implies that $N_1 = N_2$.*

Proof. If $K(f) : K(H) \rightarrow K(G)$ is surjective, then there exists a closed normal subgroup $N \triangleleft bH$ such that $bf^{-1}(N) = \{e_{bG}\}$, which implies that $\ker(bf) = \{e_{bG}\}$. Hence $K(f) : K(H) \rightarrow K(G)$ is surjective if and only if $bf : bG \rightarrow bH$ is an embedding. It is clear that $K(f) : K(H) \rightarrow K(G)$ is injective if and only if for any two closed subgroups N_1 and N_2 of bH , $N_1 \cap bG = N_2 \cap bG$ implies that $N_1 = N_2$. \square

The functor $K : \mathbf{TopGp}^{\text{op}} \rightarrow \mathbf{CLat}$ does not preserve coproducts. Indeed, if we take $G = H = \mathbb{Z}(2)$, then $bG = bH = \mathbb{Z}(2)$, and $b(G \times H) = \mathbb{Z}(2) \times \mathbb{Z}(2)$. As $|K(G \times H)| = 5$ and $|K(G) \times K(H)| = 4$, so $K(G \times H)$ is not isomorphic to $K(G) \times K(H)$. Since in the category \mathbf{CLat} of complete lattices, coproducts of objects are equivalent to products of objects, hence $K(G \times H)$ is not the coproduct of $K(G)$ and $K(H)$.

The following result was proved in [6].

Lemma 4.8. *Let $\{G_i \mid i \in I\}$ be a family of topological groups. Then $b\prod_{i \in I} G_i$ is topologically isomorphic to $\prod_{i \in I} bG_i$.*

Proposition 4.9. *Let $\{G_i \mid i \in I\}$ be a family of topological groups. Then $\prod K(G_i)$ is a sub-complete lattice of the complete lattice $K(\prod G_i)$, and there exists an inf-preserving mapping $r : K(\prod G_i) \rightarrow \prod K(G_i)$ which leaves $\prod K(G_i)$ pointwise fixed.*

Proof. First $\prod K(G_i)$ is isomorphic to $\prod CN(bG_i)$. By Lemma 4.8, $K(\prod G_i)$ is isomorphic to $CN(\prod bG_i)$. So we have a natural embedding $f : \prod CN(bG_i) \rightarrow CN(\prod bG_i)$ of complete lattices which sends every element $(N_i) \in \prod CN(bG_i)$ to the element $\prod N_i \in CN(\prod bG_i)$.

Let $r : CN(\prod bG_i) \rightarrow \prod CN(bG_i)$ such that for each closed normal subgroup $N \triangleleft \prod bG_i$, $r(N) = (p_i(N))$, where $p_i : \prod bG_i \rightarrow bG_i$ is the projection to i^{th} coordinate. Then r is a mapping satisfying the required condition. \square

Let $\{G_i \mid i \in I\}$ be a family of topological groups. We write $\otimes G_i$ the coproduct of $\{G_i \mid i \in I\}$. It is clear that $C^*(\otimes G_i)$ is a one-to-one correspondence to $\prod C^*(G_i)$. It is natural to ask the following question.

Open Question 4.10. Does the functor $K : \mathbf{TopGp}^{\text{OP}} \rightarrow \mathbf{CLat}$ preserve products?

When focused on topological abelian groups, we have a functor from the category \mathbf{TopAb} of topological abelian groups to the category $\mathbf{CLat}^{\text{OP}}$ of complete lattices.

Indeed, let G, H be topological abelian groups and let $f : G \rightarrow H$ be a continuous homomorphism. Denote by $\hat{K}(G)$ and $\hat{K}(H)$ the closed subgroup lattice (with respect to the inverse inclusion order) of bG and bH , respectively. Then $\hat{K}(G)$ and $\hat{K}(H)$ are isomorphic to the compactification lattices of G and H , respectively, and $bf : bG \rightarrow bH$ maps each closed subgroup $N \leq bG$ onto a closed subgroup $bf(N) \leq bH$. Suppose that $\{N_j \mid j \in J\}$ is a family of closed subgroups of bG . Then $bf(\bigwedge N_j) = bf(\langle \bigcup N_j \rangle) = \overline{bf(\langle \bigcup N_j \rangle)} = \langle \bigcup bf(N_j) \rangle = \bigwedge bf(N_j)$. This implies that we have a functor $\hat{K} : \mathbf{TopAb} \rightarrow \mathbf{CLat}^{\text{OP}}$.

Proposition 4.11. *Let $f : G \rightarrow H$ be a continuous homomorphism of topological abelian groups. Then $\hat{K}(f) : \hat{K}(G) \rightarrow \hat{K}(H)$ is an order isomorphism if and only if $bf : bG \rightarrow bH$ is a topological isomorphism.*

Proof. Suppose that $\hat{K}(f) : \hat{K}(G) \rightarrow \hat{K}(H)$ is an order isomorphism. Then bf is surjective, since $bf(bG) = bH$, and $bf(\{e_{bG}\}) = bf(\ker(bf)) = \{e_{bH}\}$ implies that $\ker(bf) = \{e_{bG}\}$, since $\hat{K}(f)$ is injective. It follows that bf is a topological isomorphism, since bf is closed. The converse is clear. \square

It is a natural question whether for two topological abelian groups G and H , $K(G)$ isomorphic to $K(H)$ implies that bG and bH are topological isomorphic. But in general this is not true. For example, we take $G = \mathbb{Z}(2)$ and $H = \mathbb{Z}(3)$, then $K(G) = K(H)$ are two-elements lattices, but $bG = G$, $bH = H$.

Open Question 4.12. Let G and H be topological groups. Can one give sufficient and necessary conditions on G and H for $K(G)$ to be isomorphic to $K(H)$?

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مشبکه‌ی فشرده‌سازی‌های یک گروه توپولوژیک

وی‌هی و ژوچانگ شیائو

نشان می‌دهیم که مشبکه‌ی فشرده‌سازی‌های یک گروه توپولوژیک چون G ، مشبکه‌ای کامل است که با مشبکه‌ی زیرگروه‌های نرمال بسته‌ی فشرده‌سازی بوهر G ، یعنی bG ، یکریخت است. این تناظر تابعگونی پادورد از رسته‌ی گروه‌های توپولوژیک به رسته‌ی مشبکه‌های کامل تعریف می‌کند. برخی از ویژگی‌های مشبکه‌ی فشرده‌سازی گروه‌های توپولوژیک را به دست می‌آوریم.