



On GPW -flat acts

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Abstract. In this article, we present GPW -flatness property of acts over monoids, which is a generalization of principal weak flatness. We say that a right S -act A_S is GPW -flat if for every $s \in S$, there exists a natural number $n = n_{(s, A_S)} \in \mathbb{N}$ such that the functor $A_S \otimes_{S-}$ preserves the embedding of the principal left ideal ${}_S(Ss^n)$ into ${}_SS$. We show that a right S -act A_S is GPW -flat if and only if for every $s \in S$ there exists a natural number $n = n_{(s, A_S)} \in \mathbb{N}$ such that the corresponding φ is surjective for the pullback diagram $P(Ss^n, Ss^n, \iota, \iota, S)$, where $\iota : {}_S(Ss^n) \rightarrow {}_SS$ is a monomorphism of left S -acts. Also we give some general properties and a characterization of monoids for which this condition of their acts implies some other properties and vice versa.

1 Introduction

In 1970, Kilp [7] initiated a study of flatness of acts. In 1983, further investigation of (principal) weak version of flatness was done by Kilp [8]. In 2001, Laan [10] gave equivalents of different flatness properties according to surjectivity of φ corresponding to some pullback diagram.

In this article, in Section 2, we introduce a generalization of principal weak flatness, called GPW -flatness and will give some general properties.

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In Section 3, we give conditions for a (Rees factor) cyclic act to be *GPW*-flat. In Section 4, we give a characterization of monoids over which all right *S*-acts are *GPW*-flat and also a characterization of monoids *S* for which this condition of their right *S*-acts implies some other properties and vice versa.

In this paper *S* will stand for a monoid and \mathbb{N} the set of natural numbers. A nonempty set *A* is called a *right S-act*, denoted A_S , if there exists a mapping $A \times S \rightarrow A$, $(a, s) \mapsto as$, such that $(as)t = a(st)$ and $a1 = a$, for all $a \in A$ and all $s, t \in S$. An act A_S is called *weakly flat* if the functor $A_S \otimes_{S-}$ preserves all embeddings of left ideals into *S*. An act A_S is called *principally weakly flat* if the functor $A_S \otimes_{S-}$ preserves all embeddings of principal left ideals into *S*. A right *S*-act A_S is called *torsion free* if $ac = a'c$ for any $a, a' \in A_S$ and right cancellable element $c \in S$ implies $a = a'$. A right *S*-act A_S satisfies *Condition (P)* if for every $a, a' \in A_S$, $s, t \in S$, $as = a't$ implies that $a = a''u$, $a' = a''v$ and $us = vt$ for some $a'' \in A_S$, $u, v \in S$. A right *S*-act A_S satisfies *Condition (E)* if for every $a \in A_S$, $s, t \in S$, $as = at$ implies that $a = a'u$ and $us = ut$ for some $a' \in A_S$, $u \in S$. A right *S*-act A_S satisfies *Condition (PWP)* if for every $a, a' \in A_S$, $s \in S$, $as = a's$ implies that $a = a''u$, $a' = a''v$ and $us = vs$ for some $a'' \in A_S$, $u, v \in S$. A right *S*-act A_S satisfies *Condition (P')* if for every $a, a' \in A_S$, $s, t, z \in S$, $as = a't$ and $sz = tz$ imply that $a = a''u$, $a' = a''v$ and $us = vt$ for some $a'' \in A_S$, $u, v \in S$.

Let *K* be a proper right ideal of *S*. If x, y , and z denote elements not belonging to *S*, define $A(K) = (\{x, y\} \times (S \setminus K)) \cup (\{z\} \times K)$, and define a right *S*-action on $A(K)$ by

$$(x, v)_S = \begin{cases} (x, vs) & vs \notin K \\ (z, vs) & vs \in K \end{cases}$$

$$(y, v)_S = \begin{cases} (y, vs) & vs \notin K \\ (z, vs) & vs \in K \end{cases}$$

$$(z, v)_S = (z, vs).$$

Then clearly $A(K)$ is a right *S*-act.

2 General properties

In this section, we introduce GPW-flatness property of acts and will give some general properties.

Definition 2.1. A right S -act A_S is called *GPW-flat* if for every $s \in S$, there exists $n = n_{(s, A_S)} \in \mathbb{N}$, such that the functor $A_S \otimes_S -$ preserves the embedding of the principal left ideal ${}_S(Ss^n)$ into ${}_S S$.

Clearly every principally weakly flat right S -act is GPW-flat, but, by the following example, we see that the converse is not true.

Example 2.2. Suppose $S = \{1, x, 0\}$ with $x^2 = 0$, and let $K_S = \{x, 0\}$. Clearly the right Rees factor S -act S/K is GPW-flat, but it is not principally weakly flat.

Proposition 2.3. For any right S -act A_S , the following statements are equivalent:

- (1) A_S is GPW-flat.
- (2) For every $s \in S$ there exists $n \in \mathbb{N}$ such that for any $a, a' \in A_S$, $a \otimes s^n = a' \otimes s^n$ in $A_S \otimes_S S$ implies $a \otimes s^n = a' \otimes s^n$ in $A_S \otimes_S (Ss^n)$.
- (3) For every $s \in S$ there exists $n \in \mathbb{N}$ such that for any $a, a' \in A_S$, $as^n = a's^n$ implies $a \otimes s^n = a' \otimes s^n$ in $A_S \otimes_S (Ss^n)$.
- (4) For every $s \in S$ there exists $n \in \mathbb{N}$ such that for any $a, a' \in A_S$, $as^n = a's^n$ implies that

$$\begin{array}{ll}
 a = a_1 s_1 & \\
 a_1 t_1 = a_2 s_2 & s_1 s^n = t_1 s^n \\
 a_2 t_2 = a_3 s_3 & s_2 s^n = t_2 s^n \\
 \vdots & \vdots \\
 a_k t_k = a' & s_k s^n = t_k s^n,
 \end{array}$$

for some $k \in \mathbb{N}$ and elements $a_1, \dots, a_k \in A_S$, $s_1, t_1, \dots, s_k, t_k \in S$.

Proof. (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1) are obvious by Definition 2.1.

(2) \Leftrightarrow (4) This is obvious by [9, II, Lemma 5.5]. □

Corollary 2.4. *Suppose that S is an idempotent or right cancellative monoid. Then for a right S -act A_S the following statements are equivalent:*

- (i) A_S is principally weakly flat.
- (ii) A_S is GPW-flat.

Proof. This is obvious by Proposition 2.3. □

Proposition 2.5. *Every GPW-flat right S -act is torsion free.*

Proof. This is obvious by Proposition 2.3. □

Example 2.6. Let $S = \mathbb{N} \cup G$, where \mathbb{N} is the set of natural numbers and G is a nontrivial group with unit element e and define the multiplication on S as $gn = ng = n$ for every $g \in G$ and $n \in \mathbb{N}$. Clearly all right S -acts are torsion free by [9, IV, Theorem 6.1], but not all right S -acts are GPW-flat, see Theorem 4.5.

The following strict implications exist for different flatness properties of acts:

$$\text{Weakly flat} \Rightarrow \text{Principally weakly flat} \Rightarrow \text{GPW-flat} \Rightarrow \text{Torsion free.}$$

In 2001, Laan [10] gave equivalents of different flatness properties according to surjectivity of φ corresponding to some pullback diagram $P(M, N, f, g, Q)$ where $f : {}_S M \rightarrow {}_S Q$ and $g : {}_S N \rightarrow {}_S Q$ are homomorphisms of left S -acts.

Similar to [10, Proposition 2], we have the following proposition.

Proposition 2.7. *A right S -act A_S is GPW-flat if and only if for every $s \in S$ there exists $n = n_{(s, A_S)} \in \mathbb{N}$ such that the corresponding φ is surjective for the pullback diagram $P(Ss^n, Ss^n, \iota, \iota, S)$, where $\iota : {}_S(Ss^n) \rightarrow {}_S S$ is a monomorphism of left S -acts.*

Proposition 2.8. *The following statements hold:*

- (1) Any retract of a GPW-flat right S -act is GPW-flat.
- (2) If $A = \coprod_{i \in I} A_i$ is GPW-flat, then A_i is GPW-flat for every $i \in I$.
- (3) S_S is GPW-flat.
- (4) Θ_S is GPW-flat.

Proof. (3) and (4) are obvious.

(1) Suppose that B_S is a GPW-flat right S -act and A_S is a retract of B_S . Then there exist homomorphisms $f : B_S \rightarrow A_S$ and $f' : A_S \rightarrow B_S$, such that $ff' = id_{A_S}$. Let $s \in S$. Since B_S is GPW-flat, there exists $n \in \mathbb{N}$ such that the equality $b \otimes s^n = b' \otimes s^n$ in $B_S \otimes_S S$ implies that $b \otimes s^n = b' \otimes s^n$ in $B_S \otimes_S (Ss^n)$, for any $b, b' \in B_S$ by (2) of Proposition 2.3. Let $as^n = a's^n$ for $a, a' \in A_S$. Then $f'(as^n) = f'(a's^n)$ and so $f'(a)s^n = f'(a')s^n$. Since $f'(a), f'(a') \in B_S$, B_S is GPW-flat and $f'(a) \otimes s^n = f'(a') \otimes s^n$ in $B_S \otimes_S S$, we have

$$\begin{array}{ll} f'(a) = b_1s_1 & \\ b_1t_1 = b_2s_2 & s_1s^n = t_1s^n \\ b_2t_2 = b_3s_3 & s_2s^n = t_2s^n \\ \vdots & \vdots \\ b_kt_k = f'(a') & s_k s^n = t_k s^n, \end{array}$$

where $b_1, \dots, b_k \in B_S$, $s_1, t_1, \dots, s_k, t_k \in S$, by (4) of Proposition 2.3. Thus $f(f'(a)) = f(b_1s_1)$ and so $a = f(b_1)s_1$. Similarly, $f(b_{i-1})t_{i-1} = f(b_i)s_i$, $2 \leq i \leq k$, and $a' = f(b_k)t_k$. Hence

$$\begin{aligned} a \otimes s^n &= f(b_1)s_1 \otimes s^n = f(b_1) \otimes s_1s^n = f(b_1) \otimes t_1s^n = f(b_1)t_1 \otimes s^n \\ &= f(b_2)s_2 \otimes s^n = f(b_2) \otimes s_2s^n = \dots = f(b_k)t_k \otimes s^n = a' \otimes s^n \end{aligned}$$

in $A_S \otimes_S (Ss^n)$.

(2) Suppose that $A = \coprod_{i \in I} A_i$ is GPW-flat right S -act and let $s \in S$. Let $j \in I$. By assumption, there exists $n \in \mathbb{N}$ such that $as^n = a's^n$ for $a, a' \in A_S$ implies $a \otimes s^n = a' \otimes s^n$ in $A_S \otimes_S (Ss^n)$. Let $as^n = a's^n$ for $a, a' \in A_j$. Thus $a \otimes s^n = a' \otimes s^n$ in $A_S \otimes_S (Ss^n)$, by assumption. Hence $a \otimes s^n = a' \otimes s^n$ in $A_j \otimes_S (Ss^n)$, by [2, Corollary 2.3]. \square

A monoid S is called *left almost regular* if for every $s \in S$

$$\begin{aligned} s_1c_1 &= sr_1 \\ s_2c_2 &= s_1r_2 \\ &\vdots \\ s_m c_m &= s_{m-1}r_m \\ s &= s_m r_s, \end{aligned}$$

for some $r, r_1, \dots, r_m, s_1, s_2, \dots, s_m \in S$ and right cancellable elements $c_1, c_2, \dots, c_m \in S$. Clearly every regular and right cancellative monoid is left almost regular.

As, by [9, IV, Theorem 6.5], over a left almost regular monoid every torsion free right S -act is principally weakly flat, the following proposition is easily checked.

Proposition 2.9. *Let S be a left almost regular monoid, and A_S be a right S -act. Then the following statements are equivalent:*

- (1) A_S is principally weakly flat.
- (2) A_S is GPW-flat.
- (3) A_S is torsion free.

Theorem 2.10. *For a proper right ideal K of monoid S the following statements are equivalent:*

- (1) $(\forall s \in S, \exists n \in \mathbb{N})(\forall l \in S \setminus K)(ls^n \in K \Rightarrow (\exists k \in K, ls^n = ks^n))$.
- (2) $A(K)$ is GPW-flat.

Proof. (1) \Rightarrow (2) Let $s \in S$. Then, by assumption, there exists $n \in \mathbb{N}$ such that (1) is established. Let $as^n = a's^n$ for $a, a' \in A(K)$. Since $(x, 1)S \cong S_S \cong (y, 1)S$ (every free right S -act is GPW-flat), without loss of generality, we can take $a = (x, r_1), a' = (y, r_2)$, where $r_1, r_2 \in S \setminus K$. Since $(x, r_1)s^n = (y, r_2)s^n$, we have $r_1s^n = r_2s^n \in K$, and so there exists $k \in K$ such that $r_1s^n = ks^n = r_2s^n$. Hence

$$(x, r_1) \otimes s^n = (x, 1) \otimes r_1s^n = (x, 1) \otimes ks^n = (y, 1) \otimes ks^n = (y, r_2) \otimes s^n$$

in $A(K) \otimes_S(Ss^n)$.

(2) \Rightarrow (1) Let $A(K)$ be GPW-flat and suppose $s \in S$. Then there exists $n \in \mathbb{N}$ such that $A(K) \otimes_S-$ preserves the embedding $\iota : {}_S(Ss^n) \rightarrow {}_S S$. Now let $l \in S \setminus K$ such that $ls^n \in K$. Then clearly $(x, l)s^n = (y, l)s^n$. By Proposition 2.3, we have

$$\begin{array}{ll} (x, l) = (w_1, u_1)s_1 & \\ (w_1, u_1)t_1 = (w_2, u_2)s_2 & s_1s^n = t_1s^n \\ \vdots & \vdots \\ (w_{m-1}, u_{m-1})t_{m-1} = (w_m, u_m)s_m & s_{m-1}s^n = t_{m-1}s^n \\ (w_m, u_m)t_m = (y, l) & s_ms^n = t_ms^n, \end{array}$$

for some $m \in \mathbb{N}$, $u_1, \dots, u_m \in S$, $s_1, t_1, \dots, s_m, t_m \in S$, and $w_1, \dots, w_m \in \{x, y, z\}$. By definition of $A(K)$, there exists $i \in \{1, \dots, m-1\}$ such that $w_i \neq w_{i+1}$, and so there exists $k \in K$ such that $u_i t_i = u_{i+1} s_{i+1} = k$. Hence we have

$$l s^n = u_1 s_1 s^n = u_1 t_1 s^n = u_2 s_2 s^n = \dots = u_i t_i s^n = k s^n,$$

as required. □

Recall from [9, III, Definition 10.14] that an element s of a monoid S is called *right e-cancellable* for an idempotent $e \in S$ if $s = es$ and $\ker \rho_s \leq \ker \rho_e$. A monoid S is called *left PP* if every element $s \in S$ is right e -cancellable for some idempotent $e \in S$. It is obvious that every regular and every right cancellative monoid is left *PP*. An element $s \in S$ is *right semi-cancellable* if the equality $xs = ys$ for any $x, y \in S$ implies that there exists $r \in S$ such that $rs = s$ and $xr = yr$. A monoid S is called *left PSF* if every element $s \in S$ is right semi-cancellable. Clearly every left *PP* monoid is left *PSF*.

Proposition 2.11. *Suppose that S is a left PP monoid. An act A_S is GPW-flat if and only if for every $s \in S$ there exists $n \in \mathbb{N}$ such that for any $a, a' \in A_S$, $as^n = a's^n$ implies $es^n = s^n$ and $ae = a'e$ for some $e^2 = e \in S$.*

Proof. This is obvious by [9, III, Theorem 10.16]. □

For a left *PSF* monoid, similar to argument used in [11, Proposition 2.5], we can show the following proposition.

Proposition 2.12. *Suppose that S is a left PSF monoid. An act A_S is GPW-flat if and only if for every $s \in S$ there exists $n \in \mathbb{N}$ such that for any $a, a' \in A_S$, $as^n = a's^n$ implies $rs^n = s^n$ and $ar = a'r$ for some $r \in S$.*

Corollary 2.13. *For a left PSF monoid S , the following statements are equivalent:*

- (1) $\prod_{i=1}^k A_i$ is GPW-flat.
- (2) For every $s \in S$ there exists $n \in \mathbb{N}$ such that for any $\alpha_i, \alpha'_i \in A_i, 1 \leq i \leq k$, if $(\alpha_1, \alpha_2, \dots, \alpha_k)s^n = (\alpha'_1, \alpha'_2, \dots, \alpha'_k)s^n$, then $us^n = s^n$ and $(\alpha_1, \alpha_2, \dots, \alpha_k)u = (\alpha'_1, \alpha'_2, \dots, \alpha'_k)u$ for some $u \in S$.

Proof. This is obvious by Proposition 2.12. □

Proposition 2.14. For any family $\{A_i\}_{i \in I}$ of right S -acts, if $\prod_{i \in I} A_i$ is GPW -flat, then A_i is GPW -flat, for every $i \in I$.

Proof. Let $\prod_{i \in I} A_i$ be GPW -flat and let $s \in S$ and $i \in I$. By assumption there exists $n \in \mathbb{N}$ such that the functor $\prod_{i \in I} A_i \otimes_S -$ preserves the embedding $\iota : {}_S(Ss^n) \rightarrow {}_S S$. Let $a_i s^n = a'_i s^n$ for any $a_i, a'_i \in A_i$ and suppose $a_j \in (A_j)_S$ for $j \neq i$. If

$$a_k = \begin{cases} a_i & \text{if } k = i \\ a_j & \text{if } k \neq i \end{cases}$$

$$a'_k = \begin{cases} a'_i & \text{if } k = i \\ a_j & \text{if } k \neq i \end{cases}$$

Then $(a_k)_I s^n = (a'_k)_I s^n$ and so $(a_k)_I \otimes s^n = (a'_k)_I \otimes s^n$ in $\prod_{i \in I} A_i \otimes_S (Ss^n)$, by Proposition 2.3. Now we have $a_i \otimes s^n = a'_i \otimes s^n$ in $A_i \otimes_S (Ss^n)$, by [14, Remrak 3.1], and so A_i is GPW -flat. \square

Golchin in [3] showed that if $S = G \dot{\cup} I$ where G is a group and I is an ideal of S and A is a right S -act that is ((principally) weakly) flat, torsion free, satisfies Condition (P) or (P_E) as a right I^1 -act, then it has these properties as a right S -act. Similarly, we can show the following theorem for GPW -flatness.

Theorem 2.15. Let $S = G \dot{\cup} I$ and let A be a right S -act. If A is GPW -flat as a right I^1 -act, then it is GPW -flat as a right S -act.

Proof. This is obvious by Proposition 2.3. \square

3 GPW -flatness of (Rees factor) cyclic acts

In this section, we give conditions for a (Rees factor) cyclic act to be GPW -flat.

Proposition 3.1. Suppose that ρ is a right congruence on a monoid S . Then the following statements are equivalent:

- (i) S/ρ is GPW -flat.
- (ii) $(\forall s \in S)(\exists n \in \mathbb{N})(\forall u, v \in S) \left((us^n)\rho(vs^n) \Rightarrow u(\rho \vee \ker \rho_{s^n})v \right)$.

Proof. (i) \Rightarrow (ii) Let $s \in S$. Since the right S -act S/ρ is GPW-flat, there exists $n \in \mathbb{N}$ such that the functor $(S/\rho)_S \otimes_{S-}$ preserves the embedding $\iota : {}_S(Ss^n) \rightarrow {}_S S$. Now suppose that $(us^n)\rho(vs^n)$ for $u, v \in S$. Thus $[v]_\rho \otimes s^n = [v]_\rho \otimes s^n$ in $(S/\rho)_S \otimes {}_S S$ and so $[v]_\rho \otimes s^n = [v]_\rho \otimes s^n$ in $(S/\rho)_S \otimes {}_S (Ss^n)$. Hence $u(\rho \vee \ker \rho_{s^n})v$, by [9, III, Lemma 10.6].

(ii) \Rightarrow (i) Let $s \in S$. By assumption, there exists $n \in \mathbb{N}$ such that $(us^n)\rho(vs^n)$, for every $u, v \in S$, implies that $u(\rho \vee \ker \rho_{s^n})v$. Suppose $[v]_\rho \otimes s^n = [v]_\rho \otimes s^n$ in $(S/\rho)_S \otimes {}_S S$, thus $(us^n)\rho(vs^n)$. Now, by assumption, $u(\rho \vee \ker \rho_{s^n})v$ and so, by [9, III, Lemma 10.6], $[v]_\rho \otimes s^n = [v]_\rho \otimes s^n$ in $(S/\rho)_S \otimes {}_S (Ss^n)$. Hence S/ρ is GPW-flat, by Proposition 2.3. \square

Corollary 3.2. *The principal right ideal zS is GPW-flat if and only if for every $s \in S$, there exists $n \in \mathbb{N}$ such that for any $x, y \in S$, $zxs^n = zys^n$ implies that $x(\ker \lambda_z \vee \ker \rho_{s^n})y$.*

Proof. Since $zS \cong S/\ker \lambda_z$, by Proposition 3.1, it suffices to take $\rho = \ker \lambda_z$. \square

Theorem 3.3. *Suppose that K is a right ideal of S . Then S/K is GPW-flat if and only if for every $s \in S$ there exists a natural number $n \in \mathbb{N}$ such that $ls^n \in K$, for $l \in S \setminus K$ implies that $ls^n = ks^n$, for some $k \in K$.*

Proof. If $K = S$, then $S/K \cong \Theta_S$ is GPW-flat by (3) of Proposition 2.8. Thus suppose that K is a proper right ideal of S .

Necessity. Suppose that S/K is GPW-flat for the proper right ideal K of S and let $s \in S$. Then there exists $n \in \mathbb{N}$ such that the functor $A_S \otimes_{S-}$ preserves the embedding $\iota : {}_S(Ss^n) \rightarrow {}_S S$. Now suppose $ls^n \in K$ for $l \in S \setminus K$. Then $[l] \otimes s^n = [j] \otimes s^n$ in $S/K \otimes {}_S S$, for any $j \in K$ and so, by Proposition 2.3, there exist $m \in \mathbb{N}$, $p_1, \dots, p_m, s_1, t_1, \dots, s_m, t_m \in S$ such that

$$\begin{array}{ll} [l] = [p_1]s_1 & \\ [p_1]t_1 = [p_2]s_2 & s_1s^n = t_1s^n \\ [p_2]t_2 = [p_3]s_3 & s_2s^n = t_2s^n \\ \vdots & \vdots \\ [p_m]t_m = [j] & s_ms^n = t_ms^n. \end{array}$$

Since $j \in K$, we have $p_m t_m \in K$. Let q be the least number such that $q \in \{1, \dots, m\}$ and $p_q t_q \in K$. Let $k = p_q t_q$, then $p_{q-1} t_{q-1} = p_q s_q$, and so

$$\begin{aligned}
 ls^n &= p_1 s_1 s^n = p_1 t_1 s^n = p_2 s_2 s^n = \dots = \\
 &= p_{q-1} t_{q-1} s^n = p_q s_q s^n = p_q t_q s^n = ks^n.
 \end{aligned}$$

Sufficiency. Let K be a right ideal of S and let $s \in S$. Thus there exists $n \in \mathbb{N}$ such that $ls^n \in K$, for $l \in S \setminus K$ implies that $ls^n = ks^n$ for some $k \in K$, by assumption. Let for any $u, v \in S$, $[u] \otimes s^n = [v] \otimes s^n$ in $S/K \otimes_S S$. Thus there are four cases as follows:

Case 1. $u, v \in K$. Then it is clear that $[u] = [v]$ in S/K and so $[u] \otimes s^n = [v] \otimes s^n$ in $S/K \otimes_S (Ss^n)$.

Case 2. $u \in K, v \in S \setminus K$. Then there exists $k \in K$ such that $vs^n = ks^n$, by assumption. Then

$$[u] \otimes s^n = [k] \otimes s^n = [1] \otimes ks^n = [1] \otimes vs^n = [v] \otimes s^n$$

in $S/K \otimes_S (Ss^n)$.

Case 3. $u \in S \setminus K, v \in K$. It is similar to the Case 2.

Case 4. $u, v \in S \setminus K$. Then from $[u] \otimes s^n = [v] \otimes s^n$ in $S/K \otimes_S S$, we have either $us^n = vs^n$ or $us^n, vs^n \in K$. If $us^n = vs^n$, the result follows. Otherwise, $us^n = ks^n$ and $vs^n = ls^n$ for some $k, l \in K$, by assumption. So

$$\begin{aligned}
 [u] \otimes s^n &= [1] \otimes us^n = [1] \otimes ks^n = [k] \otimes s^n = \\
 [l] \otimes s^n &= [1] \otimes ls^n = [1] \otimes vs^n = [v] \otimes s^n
 \end{aligned}$$

in $S/K \otimes_S (Ss^n)$. □

4 Characterization of monoids by GPW-flatness of acts

Now we classify monoids over which all right S -acts are GPW-flat and also monoids over which some other properties imply GPW-flatness and vice versa.

A monoid S is called *regular* if for every $s \in S$ there exists $x \in S$ such that $s = sxs$.

Definition 4.1. An element $s \in S$ is called *eventually regular* if s^n is regular for some $n \in \mathbb{N}$. That is, $s^n = s^n x s^n$ for some $n \in \mathbb{N}$ and $x \in S$. A monoid S is called *eventually regular* if every $s \in S$ is eventually regular.

Obviously every regular monoid is eventually regular.

Definition 4.2. An element $s \in S$ is called *eventually left almost regular* if

$$\begin{aligned} s_1c_1 &= s^nr_1 \\ s_2c_2 &= s_1r_2 \\ &\vdots \\ s_m c_m &= s_{m-1}r_m \\ s^n &= s_m r s^n, \end{aligned}$$

for some $n \in \mathbb{N}$, elements $s_1, s_2, \dots, s_m, r, r_1, \dots, r_m \in S$ and right cancellable elements $c_1, c_2, \dots, c_m \in S$. In other words $s \in S$ is called *eventually left almost regular* if s^n is left almost regular for some $n \in \mathbb{N}$.

If every element of a monoid S is eventually left almost regular, then S is called *eventually left almost regular*.

It is clear that every left almost regular monoid is eventually left almost regular, and also every eventually regular monoid is eventually left almost regular.

Example 4.3. Let $S = \{1, 0, e, f, a\}$ be the monoid with the following table

	1	0	e	f	a
1	1	0	e	f	a
0	0	0	0	0	0
e	e	0	e	a	a
f	f	0	0	f	0
a	a	0	0	a	0

Clearly S is eventually regular and so it is eventually left almost regular. But S is not regular, because $a \in S$ is not regular. Also S is not left almost regular, since $a \in S$ is not left almost regular.

Theorem 4.4. *The following statements are equivalent:*

- (1) S is an eventually left almost regular monoid.
- (2) All torsion free right Rees factor acts over S are GPW-flat.
- (3) All torsion free cyclic right S -acts are GPW-flat.
- (4) All torsion free finitely generated right S -acts are GPW-flat.
- (5) All torsion free right S -acts are GPW-flat.

Proof. (5) \Rightarrow (4) \Rightarrow (3) \Rightarrow (2) are clear.

(2) \Rightarrow (1) Suppose that all torsion free right Rees factor S -acts are GPW -flat and let $s \in S$. Let $K(s)$ be the subset of S consisting of all elements $t \in S$ such that

$$\begin{aligned} s_1c_1 &= s^n r_1 \\ s_2c_2 &= s_1 r_2 \\ &\vdots \\ s_{m-1}c_{m-1} &= s_{m-2}r_{m-1} \\ tc_m &= s_{m-1}r_m, \end{aligned}$$

for some $n \in \mathbb{N}$, the elements $s_1, s_2, \dots, s_m, r, r_1, \dots, r_m \in S$ and the right cancellable elements $c_1, c_2, \dots, c_m \in S$. We see that $s^n \in K(s)$ for some $n \in \mathbb{N}$, and so $K(s)$ is non-empty, because, if $m = 1$ and $c_1 = r_1 = 1$, then $t = s^n$ has the required property mentioned. Now let $J = \bigcup_{t \in K(s)} tS$. Let $s'c \in J$, for $s' \in S$ and c right cancellable. Then $s'c \in tS$ for some $t \in K(s)$, and so we have

$$\begin{aligned} s_1c_1 &= s^n r_1 \\ s_2c_2 &= s_1 r_2 \\ &\vdots \\ s_{m-1}c_{m-1} &= s_{m-2}r_{m-1} \\ tc_m &= s_{m-1}r_m \\ s'c &= tr_{m+1}, \end{aligned}$$

for some $n \in \mathbb{N}$, the elements $s_1, s_2, \dots, s_m, r, r_1, \dots, r_m, r_{m+1} \in S$ and the right cancellable elements $c_1, c_2, \dots, c_m \in S$. Thus $s' \in J$, and so S/J is torsion free by [9, III, Proposition 8.10]. Hence S/J is GPW -flat by assumption and so by Theorem 3.3, for $s^n \in J$, there exists $tr \in J$ such that $s^n = trs^n$, where $t \in K(s)$, and $r \in S$. Now $s^n = trs^n$ and $t \in K(s)$ implies that s is eventually left almost regular.

(1) \Rightarrow (5) Let S be an eventually left almost regular monoid and suppose A_S is a torsion free right S -act and let $s \in S$. Since s is eventually left almost

regular, we have

$$\begin{aligned} s_1c_1 &= s^n r_1 \\ s_2c_2 &= s_1 r_2 \\ &\vdots \\ s_m c_m &= s_{m-1} r_m \\ s^n &= s_m r s^n, \end{aligned}$$

for some $n \in \mathbb{N}$, the elements $s_1, s_2, \dots, s_m, r, r_1, \dots, r_m \in S$ and the right cancellable elements $c_1, c_2, \dots, c_m \in S$. Let $as^n = a's^n$ for $a, a' \in A_S$. Using torsion freeness, it can easily be seen that $as_m r = a' s_m r$. Hence we have

$$a \otimes s^n = a \otimes s_m r s^n = as_m r \otimes s^n = a' s_m r \otimes s^n = a' \otimes s_m r s^n = a' \otimes s^n$$

in $A_S \otimes_S (Ss^n)$ and so A_S is GPW-flat, as required. \square

A right S -act A_S is a generator if for any distinct homomorphisms $\alpha, \beta : X_S \rightarrow Y_S$, there exists a homomorphism $f : A_S \rightarrow X_S$ such that $\alpha f \neq \beta f$. Equivalently, a right S -act A_S is a generator if and only if there exists an epimorphism $\pi : A_S \rightarrow S_S$ ([9, II, Theorem 3.16]).

As we know, $S \times A_S$ is a generator for each right S -act A_S .

Theorem 4.5. *The following statements are equivalent:*

- (1) S is an eventually regular monoid.
- (2) A right S -act A_S is GPW-flat if $\text{Hom}(A_S, S_S) \neq \emptyset$.
- (3) $S \times A_S$ is GPW-flat for every generator right S -act A_S .
- (4) $S \times A_S$ is GPW-flat for every right S -act A_S .
- (5) All generator right S -acts are GPW-flat.
- (6) All right Rees factor S -acts are GPW-flat.
- (7) All cyclic right S -acts are GPW-flat.
- (8) All right S -acts are GPW-flat.

Proof. (8) \Rightarrow (7) \Rightarrow (6), (8) \Rightarrow (5), (8) \Rightarrow (4) \Rightarrow (3), (2) \Rightarrow (4), (5) \Rightarrow (4) and (8) \Rightarrow (2) are obvious.

(4) \Rightarrow (8) This is valid by Proposition 2.14.

(6) \Rightarrow (1) If all right Rees factor acts over S are GPW-flat, then all right Rees factor acts over S are torsion free. So every right cancellable element

of S is right invertible, by [9, IV, Theorem 6.1], but by Theorem 4.4, S is eventually left almost regular. Now let $s \in S$. Then

$$\begin{aligned} s_1c_1 &= s^n r_1 \\ s_2c_2 &= s_1 r_2 \\ &\vdots \\ s_m c_m &= s_{m-1} r_m \\ s^n &= s_m r s^n, \end{aligned}$$

Multiplying both sides of the equalities in the above scheme by c_i^{-1} for $i \in \{1, \dots, m\}$, respectively, we get

$$\begin{aligned} s_1 &= s^n r_1 c_1^{-1} \\ s_2 &= s_1 r_2 c_2^{-1} \\ &\vdots \\ s_m &= s_{m-1} r_m c_m^{-1}. \end{aligned}$$

Thus

$$\begin{aligned} s^n &= s_m r s^n = s_{m-1} r_m c_m^{-1} r s^n = s_{m-2} r_{m-1} c_{m-1}^{-1} r_m c_m^{-1} r s^n = \dots \\ &= s^n r_1 c_1^{-1} \dots r_{m-1} c_{m-1}^{-1} r_m c_m^{-1} r s^n, \end{aligned}$$

and so s is eventually regular, as required.

(1) \Rightarrow (8) Suppose that A_S is a right S -act and let $s \in S$. By Proposition 2.3, we have to show that there exists $m \in \mathbb{N}$ such that for any $a, a' \in A_S$, if $a \otimes s^m = a' \otimes s^m$ in $A_S \otimes_S S$, then $a \otimes s^m = a' \otimes s^m$ in $A_S \otimes_S (Ss^m)$. Since s is eventually regular, there exist $n \in \mathbb{N}$ and $t \in S$, such that $s^n = s^n t s^n$. If $m = n$. Let $a \otimes s^n = a' \otimes s^n$ in $A_S \otimes_S S$ for any $a, a' \in A_S$, then

$$a \otimes s^n = a \otimes s^n t s^n = a s^n \otimes t s^n = a' s^n \otimes t s^n = a' \otimes s^n t s^n = a' \otimes s^n$$

in $A_S \otimes_S (Ss^n)$ and so A_S is GPW -flat.

(3) \Rightarrow (4) Suppose that A_S is a right act over S . As we show in the proof of (5) \Rightarrow (4), $S \times A_S$ is a generator and so, by assumption, $S \times (S \times A_S)$ is GPW -flat, which means that $S \times A_S$ is GPW -flat, by Proposition 2.14. \square

It is obvious that Condition (P) implies GPW-flatness, but the following example shows that this is not the case for Condition (E).

Example 4.6. Let $S = (\mathbb{N}, \cdot)$ be the monoid of natural numbers with multiplication and let $A_{\mathbb{N}} = \mathbb{N} \coprod^{2\mathbb{N}} \mathbb{N}$. Then $A_{\mathbb{N}}$ satisfies Condition (E), but it is not GPW-flat.

Now, the question is that: What is the structure of monoids over which Condition (E) of their acts implies GPW-flatness?

Theorem 4.7. For a right cancellative monoid S the following statements are equivalent:

- (1) $\prod_{i \in I} A_i$ is principally weakly flat, for any family $\{A_i\}_{i \in I}$ of right S -acts.
- (2) $\prod_{i \in I} A_i$ is GPW-flat, for any family $\{A_i\}_{i \in I}$ of right S -acts.
- (3) $\prod_{i \in I} A_i$ is torsion free, for any family $\{A_i\}_{i \in I}$ of right S -acts.
- (4) S is a group.

Proof. (1) \Rightarrow (2) \Rightarrow (3) and (4) \Rightarrow (1) are obvious.

(3) \Rightarrow (4) This is obvious, by [14, Remark 3.1] and [9, IV. Theorem 6.1]. □

An element $a \in A_S$ is called divisible by $s \in S$ if there exists $b \in A_S$, such that $bs = a$. An act A_S is said to be divisible if $Ac = A$, for any left cancellable element $c \in S$. It is clear that A_S is divisible if and only if every element of A_S is divisible by any left cancellable element of S .

Theorem 4.8. The following statements are equivalent:

- (1) All right S -acts are divisible.
- (2) All GPW-flat right S -acts are divisible.
- (3) All GPW-flat finitely generated right S -acts are divisible.
- (4) All GPW-flat cyclic right S -acts are divisible.
- (5) All GPW-flat monocyclic right S -acts are divisible.
- (6) All left cancellable elements of S are left invertible.

Proof. (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5) are obvious.

(5) \Rightarrow (6) For every $s \in S$ we have $S/\rho(s, s) = S_S/\Delta_S \cong S_S$, by (3) of Proposition 2.8, S_S is GPW-flat, and so it is divisible by assumption. Thus $Sc = S$, for any left cancellable element $c \in S$. Thus, there exists $s \in S$ such that $sc = 1$, and so c is left invertible, as required.

(6) \Rightarrow (1) It is clear from [9, III, Proposition 2.2]. □

Recall, from [15], that a right S -act A_S is called *strongly torsion free* if the equality $as = a's$, for $a, a' \in A_S$ and $s \in S$ implies $a = a'$. It is clear that every strongly torsion free right S -act is *GPW-flat*, but not the converse.

Theorem 4.9. *The following statements are equivalent:*

- (1) *All GPW-flat right S -acts are strongly torsion free.*
- (2) *All GPW-flat finitely generated right S -acts are strongly torsion free.*
- (3) *All GPW-flat cyclic right S -acts are strongly torsion free.*
- (4) *S is right cancellative monoid.*

Proof. This follows from [15, Theorem 3.1]. □

Recall from [6] that a right S -act A_S is *E -torsion free* if for any $a, a' \in A_S$ and $e \in E(S)$, $ae = a'e$ implies $a = a'$.

Theorem 4.10. *The following statements are equivalent:*

- (1) *All GPW-flat right S -acts are E -torsion free.*
- (2) *All GPW-flat finitely generated right S -acts are E -torsion free.*
- (3) *All GPW-flat cyclic right S -acts are E -torsion free.*
- (4) *$E(S) = \{1\}$.*

Proof. This is obvious by [6, Theorem 3.1]. □

Recall from [1, Definition 1] that a right S -act A_S is called *principally weakly kernel flat* (PWKF) if the corresponding φ is bijective for the pullback diagram $P(Ss, Ss, f, f, S)$ ($s \in S$), and A_S is *translation kernel flat* (TKF) if the corresponding φ is bijective for the pullback diagram $P(S, S, f, f, S)$.

Theorem 4.11. *The following statements on a monoid S are equivalent:*

- (1) *All GPW-flat right S -acts are PWKF and S is left PSF.*
- (2) *All GPW-flat right S -acts are TKF and S is left PSF.*
- (3) *All GPW-flat right S -acts satisfy Condition (PWP) and S is left PSF.*
- (4) *All GPW-flat right S -acts satisfy Condition (P') and S is left PSF.*
- (5) *S is right cancellative.*

Proof. (1) \Rightarrow (2) \Rightarrow (3) and (4) \Rightarrow (3) are clear.

(5) \Rightarrow (1) By [13, Theorem 2.12] and Corollary 2.4, it is clear.

(3) \Rightarrow (5) This is obvious, by [12, Theorem 2.8].

(5) \Rightarrow (4) Since S is right cancellative, S is left PSF. By [4, Theorem 2.8] and Corollary 2.4, it is clear. □

Theorem 4.12. *The following statements on a monoid S are equivalent:*

- (1) *All GPW-flat right S -acts are PWKF and there exists a regular left S -act.*
- (2) *All GPW-flat right S -acts are TKF and there exists a regular left S -act.*
- (3) *All GPW-flat right S -acts satisfy Condition (PWP) and there exists a regular left S -act.*
- (4) *All GPW-flat right S -acts satisfy Condition (P') and there exists a regular left S -act.*
- (5) *$|E(S)| = 1$ and there exists a regular left S -act.*
- (6) *S is right cancellative.*

Proof. (1) \Rightarrow (2) \Rightarrow (3) and (4) \Rightarrow (3) are clear.

(6) \Rightarrow (4) This is clear by [4, Theorem 2.9] and Corollary 2.4.

(5) \Leftrightarrow (6) This is clear by [12, Theorem 2.9].

(3) \Rightarrow (6) This is clear by [12, Theorem 2.9].

(6) \Rightarrow (1) This is clear by [13, Theorem 2.18] and Corollary 2.4. □

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