

## $(m, n)$ -Hyperideals in ordered semihypergroups

A. Mahboob\*, N.M. Khan, and B. Davvaz

**Abstract.** In this paper, first we introduce the notions of an  $(m, n)$ -hyperideal and a generalized  $(m, n)$ -hyperideal in an ordered semihypergroup, and then, some properties of these hyperideals are studied. Thereafter, we characterize  $(m, n)$ -regularity,  $(m, 0)$ -regularity, and  $(0, n)$ -regularity of an ordered semihypergroup in terms of its  $(m, n)$ -hyperideals,  $(m, 0)$ -hyperideals and  $(0, n)$ -hyperideals, respectively. The relations  ${}_m\mathcal{I}, \mathcal{I}_n, \mathcal{H}_m^n$ , and  $\mathcal{B}_m^n$  on an ordered semihypergroup are, then, introduced. We prove that  $\mathcal{B}_m^n \subseteq \mathcal{H}_m^n$  on an ordered semihypergroup and provide a condition under which equality holds in the above inclusion. We also show that the  $(m, 0)$ -regularity [(0, n)-regularity] of an element induce the  $(m, 0)$ -regularity [(0, n)-regularity] of the whole  $\mathcal{H}_m^n$ -class containing that element as well as the fact that  $(m, n)$ -regularity and  $(m, n)$ -right weakly regularity of an element induce the  $(m, n)$ -regularity and  $(m, n)$ -right weakly regularity of the whole  $\mathcal{B}_m^n$ -class and  $\mathcal{H}_m^n$ -class containing that element, respectively.

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## 1 Introduction and preliminaries

By an ordered semigroup, we mean an algebraic structure  $(S, \cdot, \leq)$ , which satisfies the following conditions: (1)  $S$  is a semigroup with respect to the multiplication “ $\cdot$ ”; (2)  $S$  is a partially ordered set by  $\leq$ ; (3) if  $a$  and  $b$  are elements of  $S$  such that  $a \leq b$ , then  $ac \leq bc$  and  $ca \leq cb$  for all  $c \in S$ . Many authors, especially Alimov [1], Clifford [2–4], Hion [13], Conrad [5], and Kehayopulu [15] studied such semigroups with some restrictions.

In 1934, Marty [21] introduced the concept of a hyperstructure and defined hypergroup. Later on several authors studied hyperstructure in various algebraic structures such as rings, semirings, semigroups, ordered semigroups,  $\Gamma$ -semigroups and Ternary semigroups, etc. The concept of a semihypergroup is a generalization of the concept of a semigroup and many classical notions such as of ideals, quasi-ideals and bi-ideals defined in semigroups and regular semigroups have been generalized to semihypergroups (see [8, 9] for other related notions and results on semihypergroups). In [14], Heidari and Davvaz introduced the notion of an ordered semihypergroup as a generalization of the notion of an ordered semigroup. Davvaz et al. in [6, 7, 14, 22, 23, 25, 26] studied some properties of hyperideals and bi-hyperideals in ordered semihypergroups. Lajos [16] introduced the concept of  $(m, n)$ -ideals in semigroups (see also [17–19]). In [12], the authors defined the notion of an  $(m, n)$ -quasi-hyperideal in a semihypergroup and investigated several properties of these  $(m, n)$ -quasi-hyperideals.

A *hyperoperation* on a non-empty set  $H$  is a map  $\circ : H \times H \rightarrow \mathcal{P}^*(H)$  where  $\mathcal{P}^*(H) = \mathcal{P}(H) \setminus \{\emptyset\}$  (the set of all non-empty subsets of  $H$ ). In such a case,  $H$  is called a *hypergroupoid*. Let  $H$  be a hypergroupoid and  $A, B$  be any non-empty subsets of  $H$ . Then

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b.$$

We shall write, in whatever follows,  $A \circ x$  instead of  $A \circ \{x\}$  and  $x \circ A$  instead of  $\{x\} \circ A$ , for any  $x \in H$ . Also, for simplicity, throughout the paper, we shall write  $A^n$  for  $A \circ A \circ \dots \circ A$  ( $n$  – copies of  $A$ ) for any  $n \in \mathbb{Z}^+$ . Also the integers  $m, n$  will stand for positive integers throughout the paper until and unless otherwise specified. Moreover, the hypergroupoid  $H$  is called a *semihypergroup* if, for all  $x, y, z \in H$ ,

$$(x \circ y) \circ z = x \circ (y \circ z)$$

that is,

$$\bigcup_{u \in x \circ y} u \circ z = \bigcup_{v \in y \circ z} x \circ v.$$

A non-empty subset  $T$  of a semihypergroup  $H$  is called a *subsemihypergroup* of  $H$  if  $T \circ T \subseteq T$ .

Let  $H$  be a non-empty set, the triplet  $(H, \circ, \leq)$  is called an *ordered semihypergroup* if  $(H, \circ)$  is a semihypergroup and  $(H, \leq)$  is a partially ordered set such that

$$x \leq y \Rightarrow x \circ z \leq y \circ z \text{ and } z \circ x \leq z \circ y$$

for all  $x, y, z \in H$ . Here, if  $A$  and  $B$  are non-empty subsets of  $H$ , then we say that  $A \leq B$  if for every  $a \in A$  there exists  $b \in B$  such that  $a \leq b$ .

Let  $H$  be an ordered semihypergroup. For a non-empty subset  $A$  of  $H$ , we denote  $(A] = \{x \in H \mid x \leq a \text{ for some } a \in A\}$ . A non-empty subset  $A$  of  $H$  is called *idempotent* if  $A = (A \circ A]$ . A non-empty subset  $A$  of  $H$  is called *left (right)-hyperideal* [7] of  $H$  if  $H \circ A \subseteq A$  ( $A \circ H \subseteq A$ ) and  $(A] \subseteq A$ . A non-empty subset  $J$  of  $H$  is called a *hyperideal* of  $H$  if  $J$  is both a left hyperideal and a right hyperideal of  $H$ . A subsemihypergroup (non-empty subset)  $B$  of an ordered semihypergroup  $H$  is called a *bi-hyperideal (generalized bi-hyperideal)* of  $H$  if  $B \circ H \circ B \subseteq B$  and  $(B] \subseteq B$ . An ordered semihypergroup  $H$  is called *regular (left-regular, right-regular)* [7] if for each  $x \in H$ ,  $x \in (x \circ H \circ x]$  ( $x \in (H \circ x \circ x]$ ,  $x \in (x \circ x \circ H]$ ).

**Lemma 1.1.** [7] *Let  $H$  be an ordered semihypergroup and  $A, B$  be any non-empty subsets of  $H$ . Then the following conditions hold:*

- (i)  $A \subseteq (A]$ ;
- (ii)  $A \subseteq B \Rightarrow (A] \subseteq (B]$ ;
- (iii)  $(A] \circ (B] \subseteq (A \circ B]$ ;
- (iv)  $((A] \circ (B]) = (A \circ B]$ ;
- (v)  $(A] \cup (B] = (A \cup B]$ .

## 2 $(m, 0)$ -hyperideals, $(0, n)$ -hyperideals and $(m, n)$ -hyperideals in ordered semihypergroups

In this section, the notions of  $(m, n)$ -hyperideals and generalized  $(m, n)$ -hyperideals in ordered semihypergroups are introduced. Moreover, important some properties of these hyperideals are studied.

**Definition 2.1.** Let  $H$  be an ordered semihypergroup and  $m, n$  be the positive integers. Then a subsemihypergroup (respectively, non-empty subset)  $A$  of  $H$  is called an (respectively, *generalized*)  $(m, n)$ -hyperideal of  $H$  if

- (i)  $A^m \circ H \circ A^n \subseteq A$ ; and
- (ii)  $(A] \subseteq A$ .

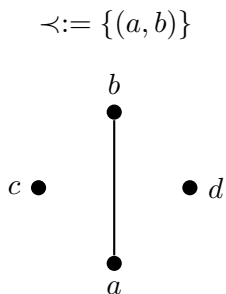
Note that in Definition 2.1, if  $m = 1 = n$ , then  $A$  is called a (*generalized*) *bi-hyperideal* of  $H$ . Moreover, a (generalized) bi-hyperideal of an ordered semihypergroup  $H$  is an (generalized)  $(m, n)$ -hyperideal of  $H$  for all positive integers  $m$  and  $n$ . It is clear that, for positive integers  $m$  and  $n$ , the notion of (generalized)  $(m, n)$ -hyperideal of  $H$  is a generalization of the notion of (generalized) bi-hyperideal of  $H$ . The following example shows that a generalized  $(m, n)$ -hyperideal of  $H$  need not be an  $(m, n)$ -hyperideal and generalized bi-hyperideal of  $H$ .

**Example 2.2.** Let  $H = \{a, b, c, d\}$ . Define the hyperoperation  $\circ$  and order  $\leq$  on  $H$  as follows:

$\circ$	$a$	$b$	$c$	$d$
$a$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
$b$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
$c$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a, b\}$
$d$	$\{a\}$	$\{a\}$	$\{a, b\}$	$\{a, b, c\}$

$\leq := \{(a, a), (b, b), (c, c), (d, d), (a, b)\}$ .

The covering relation  $\prec$  and the figure of  $H$  are as follows:



Then  $H$  is an ordered semihypergroup. The subset  $\{a, d\}$  of  $H$  is a generalized  $(m, n)$ -hyperideal of  $H$  for all integers  $m, n \geq 2$  which is neither an  $(m, n)$ -hyperideal nor a generalized bi-hyperideal of  $H$ .

**Definition 2.3.** [20] Let  $H$  be an ordered semihypergroup and  $m, n$  be positive integers. Then a subsemihypergroup  $A$  of  $H$  is called an  $(m, 0)$ -hyperideal (respectively,  $(0, n)$ -hyperideal) of  $H$  if

- (i)  $A^m \circ H \subseteq A$  (respectively,  $H \circ A^n \subseteq A$ ); and
- (ii)  $(A) \subseteq A$ .

In Definition 2.3, if  $m = 1 = n$ , then  $A$  is called a *right hyperideal* (*left hyperideal*) of  $H$ . Clearly, each right hyperideal (respectively, left hyperideal) of  $H$  is an  $(m, 0)$ -hyperideal for each positive integer  $m$  (respectively,  $(0, n)$ -hyperideal for each positive integer  $n$ ), that is, the notion of an  $(m, 0)$ -hyperideal ( $(0, n)$ -hyperideal) of  $H$  is a generalization of the notion of a right hyperideal (respectively, left hyperideal) of  $H$ . Conversely, an  $(m, 0)$ -hyperideal (respectively,  $(0, n)$ -hyperideal) of  $H$  need not be a right hyperideal (respectively, left hyperideal) of  $H$ . We illustrate it by the following example.

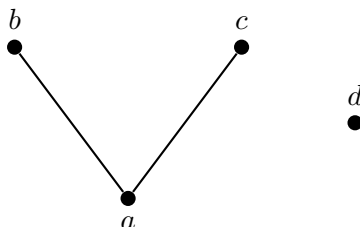
**Example 2.4.** Let  $H = \{a, b, c, d\}$ . Define the hyperoperation  $\circ$  and order  $\leq$  on  $H$  as follows:

$\circ$	$a$	$b$	$c$	$d$
$a$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
$b$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
$c$	$\{a\}$	$\{a\}$	$\{a, b\}$	$\{a, b\}$
$d$	$\{a\}$	$\{a\}$	$\{a, b\}$	$\{a\}$

$$\leq := \{(a, a), (b, b), (c, c), (d, d), (a, b), (a, c)\}.$$

The covering relation  $\prec$  and the figure of  $H$  are as follows:

$$\prec := \{(a, b), (a, c)\}$$



Then  $H$  is an ordered semihypergroup. It is easy to verify that the subset  $A = \{a, d\}$  of  $H$  is an  $(m, 0)$ -hyperideal and a  $(0, n)$ -hyperideal of  $H$  for all integers  $m, n \geq 2$ , but it is neither a right hyperideal nor a left hyperideal of  $H$ .

**Remark 2.5.** Let  $H$  be an ordered semihypergroup,  $m \geq 2$  be any positive integer and  $B$  be any non-empty subset of  $H$ . Then  $(B^m \cup B \circ H \circ B^m)$  is a (generalized) bi-hyperideal of  $H$ . Indeed,  $(B^m \cup B \circ H \circ B^m) \circ (B^m \cup B \circ H \circ B^m) \subseteq ((B^m \cup B \circ H \circ B^m) \circ (B^m \cup B \circ H \circ B^m)) = (B^m \circ B^m \cup B^m \circ B \circ H \circ B^m \cup B \circ H \circ B^m \circ B^m \cup B \circ H \circ B^m \circ B \circ H \circ B^m) \subseteq (B \circ H \circ B^m) \subseteq (B^m \cup B \circ H \circ B^m)$  and  $(B^m \cup B \circ H \circ B^m) \circ H \circ (B^m \cup B \circ H \circ B^m) \subseteq (B^m \circ H \cup B \circ H \circ B^m \circ H) \circ (B^m \cup B \circ H \circ B^m) \subseteq (B^m \circ H \circ B^m \cup B^m \circ H \circ B \circ H \circ B^m \cup B \circ H \circ B^m \circ H \circ B^m \cup B \circ H \circ B^m \circ H \circ B \circ H \circ B^m) \subseteq (B \circ H \circ B^m) \subseteq (B^m \cup B \circ H \circ B^m)$ .

Note that in Remark 2.5, if  $m = 1$ , then  $(B \cup B \circ H \circ B)$  is a generalized bi-hyperideal of  $H$  which is not a bi-hyperideal of  $H$ . Thus  $(B^m \cup B \circ H \circ B^m)$  is a generalized bi-hyperideal of  $H$  for each positive integer  $m$ .

**Theorem 2.6.** Let  $B$  be a non-empty subset of an ordered semihypergroup  $H$  and let  $m \geq 2$  be any positive integer. Then the following are equivalent:

- (i)  $B$  is a  $(1, m)$ -hyperideal of  $H$ ;
- (ii)  $B$  is a left hyperideal of some bi-hyperideals of  $H$ ;
- (iii)  $B$  is a bi-hyperideal of some left hyperideals of  $H$ ;
- (iv)  $B$  is a  $(0, m)$ -hyperideal of some right hyperideals of  $H$ ;
- (v)  $B$  is a right hyperideal of some  $(0, m)$ -hyperideals of  $H$ .

*Proof.* (i)  $\Rightarrow$  (ii) Let  $B$  be a  $(1, m)$ -hyperideal of  $H$ . So  $B \circ B \subseteq B, (B] \subseteq B$  and  $B \circ H \circ B^m \subseteq B$ . Therefore,  $(B^m \cup B \circ H \circ B^m) \circ B = (B^m \cup B \circ H \circ B^m) \circ (B] \subseteq (B^{m+1} \cup B \circ H \circ B^{m+1}) \subseteq (B^{m+1} \cup B \circ H \circ B^m) \subseteq (B] = B$ . If  $b \in B$ , then  $h \in (B^m \cup B \circ H \circ B^m)$  such that  $h \leq b$ . As  $h \in H$  and  $B$  is a  $(1, m)$ -hyperideal of  $H$ ,  $h \in B$ . Hence,  $B$  is a left hyperideal of the bi-hyperideal  $(B^m \cup B \circ S \circ B^m)$  of  $H$ .

(ii)  $\Rightarrow$  (iii) Let  $B$  be a left hyperideal of a bi-hyperideal  $A$  of  $H$ . So  $B \subseteq A, A \circ B \subseteq B$  and  $A \circ H \circ A \subseteq A$ . Therefore,  $B \circ B \subseteq A \circ B \subseteq B$  and  $B \circ (B \cup H \circ B) \circ B = (B] \circ (B \cup H \circ B) \circ (B] \subseteq (B \circ (B \cup H \circ B)) \circ (B] \subseteq (B \circ (B \cup H \circ B) \circ B) \subseteq (B^3 \cup B^2 \circ H \circ B^2) \subseteq (A^2 \circ B \cup A \circ A \circ H \circ A \circ B) \subseteq (A \circ B \cup A \circ A \circ B) \subseteq (B \cup A \circ B) \subseteq (B] = B$ . Let  $b \in B, h \in (B \cup H \circ B)$  such that  $h \leq b$ . As  $b \in B \subseteq A, b \in A$ . So  $h \in A$ . Thus  $h \in B$ . Hence,  $B$  is a bi-hyperideal of the left hyperideal  $(B \cup H \circ B)$  of  $H$ .

(iii)  $\Rightarrow$  (iv) Let  $B$  be a bi-hyperideal of a left hyperideal  $L$  of  $H$ . Then  $B \subseteq L$ ,  $B \circ L \circ B \subseteq B$  and  $H \circ L \subseteq L$ . Therefore,  $(B \cup B \circ H] \circ B^m \subseteq (B \cup B \circ H] \circ (B^m) \subseteq (B^{m+1} \cup B \circ H \circ B^m) \subseteq (B \cup B \circ (H \circ B^{m-2}) \circ L \circ B] \subseteq (B \cup B \circ H \circ L \circ B] = (B \cup B \circ (H \circ L) \circ B] \subseteq (B \cup B \circ L \circ B] = (B] = B$ . Let  $b \in B$ ,  $h \in (B \cup B \circ H]$  such that  $h \leq b$ . As  $B \subseteq L$ ,  $b \in L$ . So  $h \in L$  and, thus,  $h \in B$ . Hence,  $B$  is a  $(0, m)$ -hyperideal of the right hyperideal  $(B \cup B \circ H]$  of  $H$ .

(iv)  $\Rightarrow$  (v) Let  $B$  be a  $(0, m)$ -hyperideal of a right hyperideal  $R$  of  $H$ . So  $B \subseteq R$ ,  $R \circ B^m \subseteq B$  and  $R \circ H \subseteq R$ . Therefore,  $B \circ (B \cup H \circ B^m) \subseteq (B] \circ (B \cup H \circ B^m) \subseteq (B^2 \cup B \circ H \circ B^m) \subseteq (B \cup R \circ H \circ B^m) \subseteq (B \cup R \circ B^m) = (B] = B$ . Let  $b \in B$ ,  $h \in (B \cup H \circ B^m]$  be such that  $h \leq b$ . As  $B \subseteq R$ ,  $b \in R$ . So  $h \in R$  which implies that  $h \in B$ . Hence,  $B$  is a right hyperideal of the  $(0, m)$ -hyperideal  $(B \cup H \circ B^m]$  of  $H$ .

(v)  $\Rightarrow$  (i) Let  $B$  be a right hyperideal of a  $(0, m)$ -hyperideal  $A$  of  $H$ . Thus  $B \subseteq A$ ,  $B \circ A \subseteq B$  and  $H \circ A^m \subseteq A$ . Therefore,  $B \circ H \circ B^m \subseteq B \circ H \circ A^m \subseteq B \circ A \subseteq B$ . Let  $b \in B$ ,  $h \in H$  be such that  $h \leq b$ . As  $B \subseteq R$ , we have  $b \in R$ . Therefore,  $h \in R$  and, thus,  $h \in B$ . Hence,  $B$  is a  $(1, m)$ -hyperideal of  $H$ .  $\square$

**Definition 2.7.** Let  $H$  be an ordered semihypergroup,  $m, n$  be positive integers and  $A$  be any (generalized)  $(m, n)$ -hyperideal of  $H$ . Then  $A$  is said to be a *minimal (generalized)  $(m, n)$ -hyperideal of  $H$*  if for every (generalized)  $(m, n)$ -hyperideal  $B$  of  $H$ ,  $B \subseteq A$  implies  $B = A$ .

Similarly, a *minimal  $(m, 0)$ -hyperideal* and a *minimal  $(0, n)$ -hyperideal* of  $H$  may be defined.

**Lemma 2.8.** Let  $H$  be an ordered semihypergroup,  $m \geq 2$  be any positive integer and  $B$  be a non-empty subset of  $H$ . Then  $B$  is a minimal (generalized)  $(m, m - 1)$ -hyperideal of  $H$  if and only if  $B$  is a minimal (generalized) bi-hyperideal of  $H$ .

*Proof.* Let  $H$  be an ordered semihypergroup and  $B$  be a minimal  $(m, m - 1)$ -hyperideal of  $H$ . Since  $(B^m \circ H \circ B^{m-1}] \circ (B^m \circ H \circ B^{m-1}] \subseteq (B^m \circ H \circ B^{m-1}]$ ,  $((B^m \circ H \circ B^{m-1})^m \circ H \circ ((B^m \circ H \circ B^{m-1})^{m-1}) \subseteq (B^m \circ H \circ B^{m-1})$  and  $((B^m \circ H \circ B^{m-1})] \subseteq (B^m \circ H \circ B^{m-1})$ . Therefore,  $(B^m \circ H \circ B^{m-1}]$  is a  $(m, m - 1)$ -hyperideal of  $H$  such that  $(B^m \circ H \circ B^{m-1}] \subseteq B$ . So by minimality of  $(m, m - 1)$ -hyperideal  $B$  of  $H$ ,  $(B^m \circ H \circ B^{m-1}) = B$ . Now  $B \circ B =$

$(B^m \circ H \circ B^{m-1}] \circ (B^m \circ H \circ B^{m-1}] \subseteq ((B^m \circ H \circ B^{m-1}) \circ (B^m \circ H \circ B^{m-1})) \subseteq (B^m \circ H \circ B^{m-1}] = B$  and  $B \circ H \circ B = (B^m \circ H \circ B^{m-1}] \circ H \circ (B^m \circ H \circ B^{m-1}] \subseteq (B^m \circ H \circ B^{m-1}] = B$ . Therefore,  $B$  is bi-hyperideal of  $H$ . It remains to show that  $B$  is a minimal bi-hyperideal of  $H$ , so assume that  $A$  is any bi-hyperideal of  $H$  contained in  $B$ . Therefore,  $A$  is  $(m, m - 1)$ -hyperideal of  $H$ . Since  $B$  is a minimal  $(m, m - 1)$ -hyperideal of  $H$ ,  $B = A$ . Hence,  $B$  is a minimal bi-hyperideal of  $H$ . For the converse, assume that  $B$  is a minimal bi-hyperideal of  $H$ . As  $B^m \circ H \circ B^{m-1} = B \circ (B^{m-1} \circ H \circ B^{m-2}) \circ B \subseteq B \circ H \circ B \subseteq B$ ,  $B$  is an  $(m, m - 1)$ -hyperideal of  $H$ . To show that  $B$  is a minimal  $(m, m - 1)$ -hyperideal of  $H$ , let  $A$  be any  $(m, m - 1)$ -hyperideal of  $H$  such that  $A \subseteq B$ . As  $(A^m \circ H \circ A^{m-1}] \circ (A^m \circ H \circ A^{m-1}] \subseteq ((A^m \circ H \circ A^{m-1}) \circ (A^m \circ H \circ A^{m-1})) = (A^m \circ (H \circ A^{m-1} \circ A^m \circ H) \circ A^{m-1}] \subseteq (A^m \circ H \circ A^{m-1}]$  and  $(A^m \circ H \circ A^{m-1}] \circ H \circ (A^m \circ H \circ A^{m-1}] \subseteq ((A^m \circ H \circ A^{m-1}) \circ H \circ (A^m \circ H \circ A^{m-1})) = (A^m \circ (H \circ A^{m-1} \circ H \circ A^m \circ H) \circ A^{m-1}] \subseteq (A^m \circ H \circ A^{m-1}]$ ,  $(A^m \circ H \circ A^{m-1}]$  is a bi-hyperideal of  $H$ . Since  $B$  is a minimal bi-hyperideal of  $H$  and  $(A^m \circ H \circ A^{m-1}] \subseteq B$ ,  $(A^m \circ H \circ A^{m-1}] = B$ . As  $(A^m \circ H \circ A^{m-1}] \subseteq A$ ,  $B \subseteq A$ . Now, as  $A \subseteq B$ , we have  $A = B$ . Hence,  $B$  is a minimal  $(m, m - 1)$ -hyperideal of  $H$ .  $\square$

**Theorem 2.9.** *Let  $H$  be an ordered semihypergroup and  $\{A_i \mid i \in I\}$  be a set of  $(m, n)$ -hyperideals of  $H$ . If  $\bigcap_{i \in I} A_i \neq \emptyset$ , then  $\bigcap_{i \in I} A_i$  is an  $(m, n)$ -hyperideal of  $H$ .*

*Proof.* Assume that  $\bigcap_{i \in I} A_i \neq \emptyset$ . Let  $x, y \in \bigcap_{i \in I} A_i$ . Then,  $x, y \in A_i$  for each  $i \in I$ . As for each  $i \in I$ ,  $A_i$  is an  $(m, n)$ -hyperideal,  $x \circ y \subseteq A_i$ . Therefore,  $x \circ y \subseteq \bigcap_{i \in I} A_i$ . Thus,  $\bigcap_{i \in I} A_i$  is a subsemihypergroup of  $H$ . Next we show that  $(\bigcap_{i \in I} A_i)^m \circ H \circ (\bigcap_{i \in I} A_i)^n \subseteq \bigcap_{i \in I} A_i$ . We have

$$\begin{aligned} & (\bigcap_{i \in I} A_i)^m \circ H \circ (\bigcap_{i \in I} A_i)^n \\ & \subseteq (A_i)^m \circ H \circ (A_i)^n \quad (\text{as } \bigcap_{i \in I} A_i \subseteq A_i, \forall i \in I) \\ & \subseteq A_i \quad (\text{as } A_i\text{'s are } (m, n)\text{-hyperideals}). \end{aligned}$$

Thus  $(\bigcap_{i \in I} A_i)^m \circ H \circ (\bigcap_{i \in I} A_i)^n \subseteq \bigcap_{i \in I} A_i$ . Finally, we show that  $(\bigcap_{i \in I} A_i] \subseteq \bigcap_{i \in I} A_i$ . Let  $a \in \bigcap_{i \in I} A_i, h \in H$  such that  $h \leq a$ . As  $a \in A_i$  for each  $i \in I$  and  $A_i$ 's



are  $(m, n)$ -hyperideals,  $h \in A_i$  for each  $i \in I$ . Therefore,  $h \in \bigcap_{i \in I} A_i$ , as required.  $\square$

**Theorem 2.10.** [20] *Let  $H$  be an ordered semihypergroup. Then the following conditions hold:*

- (i) *Let  $\{L_i \mid i \in I\}$  be a set of  $(m, 0)$ -hyperideals of  $H$ . If  $\bigcap_{i \in I} L_i \neq \emptyset$ , then  $\bigcap_{i \in I} L_i$  is an  $(m, 0)$ -hyperideal of  $H$ .*
- (ii) *Let  $\{R_i \mid i \in I\}$  be a set of  $(0, n)$ -hyperideals of  $H$ . If  $\bigcap_{i \in I} R_i \neq \emptyset$ , then  $\bigcap_{i \in I} R_i$  is a  $(0, n)$ -hyperideal of  $H$ .*

Let  $H$  be an ordered semihypergroup and  $A$  be any non-empty subset of  $H$ . We denote  $\mathcal{P} = \{J \mid J \text{ is an } (m, n)\text{-hyperideal of } H \text{ containing } A\}$ . Clearly,  $\mathcal{P} \neq \emptyset$  since  $H \in \mathcal{P}$ . Let  $[A]_{m,n} = \bigcap_{J \in \mathcal{P}} J$ . As  $A \subseteq J$  for each  $J \in \mathcal{P}$ ,  $[A]_{m,n} \neq \emptyset$ . By Theorem 1.9,  $[A]_{m,n}$  is an  $(m, n)$ -hyperideal of  $H$  containing  $A$ . The  $(m, n)$ -hyperideal  $[A]_{m,n}$  is called the  $(m, n)$ -hyperideal of  $H$  generated by  $A$ . Similarly,  $[A]_{m,0}$  and  $[A]_{0,n}$  are called  $(m, 0)$ -hyperideal and  $(0, n)$ -hyperideal of  $H$  generated by  $A$ , respectively.

**Theorem 2.11.** *Let  $H$  be an ordered semihypergroup and  $A$  be a non-empty subset of  $H$ . Then*

$$[A]_{m,n} = (\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n)$$

for any positive integers  $m, n$ .

*Proof.* Clearly  $(\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n) \neq \emptyset$ . Now we have

$$\begin{aligned} & \left( \bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n \right) \circ \left( \bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n \right) \\ & \subseteq \left( \left( \bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n \right) \circ \left( \bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n \right) \right) \\ & = \left( \left( \bigcup_{i=1}^{m+n} A^i \right) \circ \left( \bigcup_{i=1}^{m+n} A^i \right) \cup \left( \bigcup_{i=1}^{m+n} A^i \right) \circ A^m \circ H \circ A^n \cup \left( A^m \circ H \circ A^n \right) \circ \left( \bigcup_{i=1}^{m+n} A^i \right) \right. \\ & \quad \left. \cup \left( A^m \circ H \circ A^n \right) \circ \left( A^m \circ H \circ A^n \right) \right) \\ & \subseteq \left( \left( \bigcup_{i=1}^{m+n} A^i \right) \circ \left( \bigcup_{i=1}^{m+n} A^i \right) \cup A^m \circ H \circ A^n \right) \end{aligned} \tag{1}$$

Let  $x \in (\bigcup_{i=1}^{m+n} A^i) \circ (\bigcup_{i=1}^{m+n} A^i)$ . Then,  $x \in z_1 \circ z_2$  for some  $z_1, z_2 \in$

$\bigcup_{i=1}^{m+n} A^i$ . Then,  $z_1 = A^p, z_2 = A^q$  for some  $1 < p, q \leq m + n$ . There are two cases arising. If  $p + q \leq m + n$ , then  $z_1 \circ z_2 \subseteq \bigcup_{i=1}^{m+n} A^i$ . If  $m + n \leq p + q$ , then  $z_1 \circ z_2 \subseteq A^m \circ H \circ A^n$ . Therefore, in both cases  $z_1 \circ z_2 \subseteq \bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n$ . As  $x \in z_1 \circ z_2, x \in \bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n$ . Thus,  $(\bigcup_{i=1}^{m+n} A^i) \circ (\bigcup_{i=1}^{m+n} A^i) \subseteq \bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n$ . Therefore, from (1),  $(\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n) \circ (\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n) \subseteq (\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n)$ . Hence,  $(\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n)$  is a subsemihypergroup of  $H$  containing

A. We have

$$\begin{aligned} & ((\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n)^m \circ H \\ &= (\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n)^{m-1} \circ (\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n) \circ H \\ &\subseteq (\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n)^{m-1} \circ (\bigcup_{i=1}^{m+n} A^i \circ H \cup A^m \circ H \circ A^n \circ H) \\ &\subseteq (\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n)^{m-1} \circ (A \circ H) \\ &= (\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n)^{m-2} \circ (\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n) \circ (A \circ H) \\ &\subseteq (\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n)^{m-2} \circ (\bigcup_{i=1}^{m+n} A^i \circ A \circ H \cup A^m \circ H \circ A^n \circ A \circ H) \\ &\subseteq (\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n)^{m-1} \circ (A^2 \circ H) \\ &\vdots \\ &= (A^m \circ H). \end{aligned}$$

Similarly,  $H \circ ((\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n)^n) \subseteq (H \circ A^n)$ . Therefore, we have

$$\begin{aligned} & ((\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n)^m \circ H \circ ((\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n)^n) \\ &\subseteq (A^m \circ H \circ A^n) \\ &\subseteq (\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n). \end{aligned}$$

Also  $((\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n)) \subseteq (\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n)$ . Therefore,  $(\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n)$  is an  $(m, n)$ -hyperideal of  $H$  containing  $A$ . It follows that  $[a]_{m,n} \subseteq (\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n)$ . For the reverse inclusion, let  $x \in (\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n)$ , that is, there exist  $z \in \bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n$

such that  $x \leq z$ . If  $z \in \bigcup_{i=1}^{m+n} A^i$ , then  $z = A^p$  for some  $1 \leq p \leq m + n$ . Therefore,  $x \in [A]_{m,n}$ . If  $z \in A^m \circ H \circ A^n$ , then

$$A^m \circ H \circ A^n \subseteq ([A]_{m,n})^m \circ H \circ ([A]_{m,n})^n \subseteq [A]_{m,n}.$$

Therefore,  $z \in [A]_{m,n}$  implies  $x \in [A]_{m,n}$ . Hence,  $[A]_{m,n} = (\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n)$ , as required.  $\square$

**Theorem 2.12.** [20] *Let  $H$  be an ordered semihypergroup and  $A$  be any non-empty subset of  $H$ . Then:*

- (i)  $[A]_{m,0} = (\bigcup_{i=1}^m A^i \cup A^m \circ H)$ ;
- (ii)  $[A]_{0,n} = (\bigcup_{i=1}^n A^i \cup H \circ A^n)$ .

**Theorem 2.13.** *Let  $H$  be an ordered semihypergroup and  $A$  be a non-empty subset of  $H$ . Then*

$$((([A]_{m,n})^m \circ H \circ ([A]_{m,n})^n) = (A^m \circ H \circ A^n)$$

for any positive integers  $m, n$ .

*Proof.* We have

$$\begin{aligned} & ([A]_{m,n})^m \circ H \\ &= ((\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n)^m \circ H \\ &= ((\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n)^{m-1} \circ (\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n) \circ H \\ &\subseteq ((\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n)^{m-1} \circ (\bigcup_{i=1}^{m+n} A^i \circ H \cup A^m \circ H \circ A^n \circ H) \\ &\subseteq ((\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n)^{m-1} \circ (A \circ H) \\ &= ((\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n)^{m-2} \circ (\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n) \circ (A \circ H) \\ &\subseteq ((\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n)^{m-2} \circ (\bigcup_{i=1}^{m+n} A^i \circ A \circ H \cup A^m \circ H \circ A^n \circ A \circ H) \\ &\subseteq ((\bigcup_{i=1}^{m+n} A^i \cup A^m \circ H \circ A^n)^{m-1} \circ (A^2 \circ H) \end{aligned}$$

$$\begin{aligned} & \vdots \\ & = (A^m \circ H). \end{aligned}$$

Similarly,  $H \circ ([A]_{m,n})^n \subseteq H \circ A^n$ . Therefore,  $(([A]_{m,n})^m \circ H \circ ([A]_{m,n})^n) \subseteq (A^m \circ H \circ A^n)$ . The reverse inclusion is obvious, that is,  $(A^m \circ H \circ A^n) \subseteq (([A]_{m,n})^m \circ H \circ ([A]_{m,n})^n)$ . Hence,  $(([A]_{m,n})^m \circ H \circ ([A]_{m,n})^n) = (A^m \circ H \circ A^n)$ .  $\square$

**Theorem 2.14.** [20] *Let  $H$  be an ordered semihypergroup and  $A$  be a non-empty subset of  $H$ . Then*

- (i)  $(([A]_{m,0})^m \circ H) = (A^m \circ H)$  for any positive integer  $m$ .
- (ii)  $(H \circ ([A]_{(0,n)})^n) = (H \circ A^n)$  for any positive integer  $n$ .

### 3 $(m, n)$ -regularity in ordered semihypergroups

In this section, we characterize  $(m, n)$ -regular,  $(m, 0)$ -regular and  $(0, n)$ -regular ordered semihypergroup in terms of its  $(m, n)$ -hyperideals,  $(m, 0)$ -hyperideals and  $(0, n)$ -hyperideals.

**Definition 3.1.** Let  $H$  be an ordered semihypergroup and  $m, n$  be non-negative integers. An element  $a$  of  $H$  is said to be an  $(m, n)$ -regular element if  $a \in (a^m \circ H \circ a^n)$ . The ordered semihypergroup  $H$  is said to be  $(m, n)$ -regular if each element of  $H$  is  $(m, n)$ -regular, equivalently, for each subset  $A$  of  $H$  we have  $A \subseteq (A^m \circ H \circ A^n)$ . Here,  $A^0 \circ H = H \circ A^0 = H$ .

It is clear from Definition 3.1 that, for each non-negative integers  $m$  and  $n$  every  $(m, n)$ -regular ordered semihypergroup is  $(r, s)$ -regular ( $r \leq m, s \leq n$  are non-negative integers). In particular, for any positive integers  $m$  and  $n$ , an  $(m, n)$ -regular ordered semihypergroup is regular. Indeed,  $a \in (a^m \circ H \circ a^n) \subseteq (a \circ H \circ a)$ . On the other hand, for each positive integer  $m$ , an  $(m, 0)$ -regular ordered semihypergroup need not be a regular ordered semihypergroup.

**Proposition 3.2.** *Let  $H$  be an  $(m, n)$ -regular ordered semihypergroup and  $A$  be a generalized  $(m, n)$ -hyperideal of  $H$  for any positive integers  $m, n$ . Then  $A$  is an  $(m, n)$ -hyperideal of  $H$ .*

*Proof.* Let  $a, b \in A$ . Since  $H$  is an  $(m, n)$ -regular ordered semihypergroup, there exist  $x, y \in H$  such that  $a \leq a^m \circ x \circ a^n, b \leq b^m \circ y \circ b^n$ . Therefore,  $a \circ b \leq a^m \circ x \circ a^n \circ b^m \circ y \circ b^n = a^m \circ (x \circ a^n \circ b^m \circ y) \circ b^n \subseteq A^m \circ H \circ A^m \subseteq A$  whence  $a \circ b \subseteq (A] = A$ . Thus  $A$  is a subsemihypergroup of  $H$ . Hence,  $A$  is an  $(m, n)$ -hyperideal of  $H$ .  $\square$

**Theorem 3.3.** *Let  $H$  be an ordered semihypergroup and  $m, n$  be non-negative integers. The set of all  $(m, 0)$ -hyperideals,  $(0, n)$ -hyperideals, and  $(m, n)$ -hyperideals will be denoted by  $I_{(m,0)}, I_{(0,n)}$  and  $I_{(m,n)}$ , respectively. Then, we have*

- (i)  $H$  is  $(m, 0)$ -regular if and only if  $I_{(m,0)}$  is  $(m, 0)$ -regular;
- (ii)  $H$  is  $(0, n)$ -regular if and only if  $I_{(0,n)}$  is  $(0, n)$ -regular;
- (iii)  $H$  is  $(m, n)$ -regular if and only if  $I_{(m,n)}$  is  $(m, n)$ -regular.

*Proof.* (i) When  $m = 0$ , the statement holds trivially because  $H$  is the only  $(0, 0)$ -hyperideal of  $H$ . So, let  $m \neq 0$  and  $A \in I_{(m,0)}$ . Therefore  $(A^m \circ H] \subseteq A$ . As  $S$  is  $(m, 0)$ -regular,  $A \subseteq (A^m \circ H]$ . Thus,  $A = (A^m \circ H]$ . Since  $H \in I_{(m,0)}$ ,  $A$  is a  $(m, 0)$ -regular element of  $I_{(m,0)}$ . Hence  $I_{(m,0)}$  is  $(m, 0)$ -regular. For the converse, assume that  $I_{(m,0)}$  is  $(m, 0)$ -regular. Take any  $a \in S$ . As  $[a]_{m,0}$  is in  $I_{(m,0)}$  and  $I_{(m,0)}$  is  $(m, 0)$ -regular, there exists  $B \in I_{(m,0)}$  such that  $[a]_{m,0} = ([a]_{m,0})^m \circ B \subseteq ([a]_{m,0})^m \circ H \subseteq (([a]_{m,0})^m \circ H]$ . By Theorem 2.14,  $([a]_{m,0})^m \circ H = (a^m \circ H]$ . As  $\{a\} \subseteq [a]_{m,0}$ , we have  $a \in (a^m \circ H]$ . Hence  $H$  is  $(m, 0)$ -regular.

(ii) On the similar lines to (i), we may prove (ii).

(iii) If  $m = n = 0$ , then the statement is true because  $I_{(0,0)} = \{H\}$ . If  $m \neq 0$  and  $n = 0$  or  $m = 0$  and  $n \neq 0$ , then the statement follows by (i) and (ii), respectively. So, let  $m \neq 0, n \neq 0$  and  $A \in I_{(m,n)}$ . Therefore  $(A^m \circ H \circ A^n] \subseteq A$ . As  $H$  is  $(m, n)$ -regular,  $A \subseteq (A^m \circ H \circ A^n]$ . Thus,  $A = (A^m \circ H \circ A^n]$ . Since  $H \in I_{(m,n)}$ ,  $A$  is an  $(m, n)$ -regular element of  $I_{(m,n)}$ . Hence,  $I_{(m,n)}$  is  $(m, n)$ -regular. For the converse, assume that  $I_{(m,n)}$  is  $(m, n)$ -regular and  $a \in H$ . As  $[a]_{m,n}$  is in  $I_{(m,n)}$  and  $I_{(m,n)}$  is  $(m, n)$ -regular, there exists  $B \in I_{(m,n)}$  such that  $[a]_{m,n} = ([a]_{m,n})^m \circ B \circ ([a]_{m,n})^n \subseteq ([a]_{m,n})^m \circ H \circ ([a]_{m,n})^n \subseteq (([a]_{m,n})^m \circ H \circ ([a]_{m,n})^n]$ . By Theorem 4.1, we have  $(([a]_{m,n})^m \circ H \circ ([a]_{m,n})^n) = (a^m \circ H \circ a^n]$ . As  $\{a\} \subseteq [a]_{m,n}$ ,  $a \in (a^m \circ H \circ a^n]$ . This implies that  $a$  is an  $(m, n)$ -regular element of  $H$ . Hence,  $H$  is  $(m, n)$ -regular.  $\square$

**Lemma 3.4.** [20] *Let  $H$  be an ordered semihypergroup. If the sets of all  $(m, 0)$ -hyperideals and  $(0, n)$ -hyperideals are denoted by  $I_{(m,0)}$  and  $I_{(0,n)}$  respectively, then*

(i)  *$H$  is  $(m, 0)$ -regular if and only if  $R = (R^m \circ H)$  ( $\forall R \in I_{(m,0)}$ ), where  $m$  is any positive integer;*

(ii)  *$H$  is  $(0, n)$ -regular if and only if  $L = (H \circ L^n)$  ( $\forall L \in I_{(0,n)}$ ), where  $n$  is any positive integer.*

**Theorem 3.5.** *Let  $H$  be an ordered semihypergroup and  $m, n$  be non-negative integers. The set of all  $(m, n)$ -hyperideals will be denoted by  $I_{(m,n)}$ . Then  $H$  is  $(m, n)$ -regular if and only if  $A = (A^m \circ H \circ A^n)$  for all  $A \in I_{(m,n)}$ .*

*Proof.* If  $m = n = 0$ , then the statement is true because  $I_{(0,0)} = \{H\}$ . If  $m \neq 0$  and  $n = 0$  or  $m = 0$  and  $n \neq 0$ , then the statement follows by Lemma 3.4. So, let  $m \neq 0, n \neq 0$  and  $A \in I_{(m,n)}$ . Then, by definition of  $(m, n)$ -regularity, we have  $A \subseteq (A^m \circ H \circ A^n)$  and, by definition of  $(m, n)$ -hyperideal, we have  $(A^m \circ H \circ A^n) \subseteq (A) = A$ . Hence,  $A = (A^m \circ H \circ A^n)$ .

For the converse, assume that  $A = (A^m \circ H \circ A^n)$  for each  $A \in I_{(m,n)}$ . Take any  $a \in H$ , so  $[a]_{m,n} \in I_{(m,n)}$ . From Theorem 4.1 and by the assumption,  $[a]_{m,n} = (([a]_{m,n})^m \circ H \circ [a]_{m,n}^n) = (a^m \circ H \circ a^n)$ . As  $\{a\} \subseteq [a]_{m,n}$ ,  $a \in (a^m \circ H \circ a^n)$ . Hence,  $H$  is  $(m, n)$ -regular.  $\square$

**Theorem 3.6.** *Let  $H$  be an ordered semihypergroup and  $m, n$  be non-negative integers. Then,  $H$  is  $(m, n)$ -regular if and only if  $L \cap R = (R^m \circ L^n)$  for each  $(m, 0)$ -hyperideal  $R$  and for each  $(0, n)$ -hyperideal  $L$  of  $H$ .*

*Proof.* The statement is trivially true for  $m = 0 = n$ . If  $m = 0$  and  $n \neq 0$  or  $m \neq 0$  and  $n = 0$ , then the result follows by Lemma 3.4. So, let  $m \neq 0, n \neq 0$ ,  $R$  be any  $(m, 0)$ -hyperideal and  $L$  be any  $(0, n)$ -hyperideal of  $H$ . Therefore  $(R^m \circ L^n) \subseteq (R^m \circ H) \subseteq (R) = R$  and  $(R^m \circ L^n) \subseteq (H \circ L^n) \subseteq (L) = L$ . Therefore,  $(R^m \circ L^n) \subseteq R \cap L$ . As  $H$  is  $(m, n)$ -regular, we have

$$\begin{aligned} & (R \cap L) \\ & \subseteq ((R \cap L)^m \circ H \circ (R \cap L)^n) \\ & \subseteq (R^m \circ H \circ L^n) \\ & \subseteq (R^m \circ H \circ L^{n-1} \circ (L^m \circ H \circ L^n)) \quad (\text{as } H \text{ is } (m, n)\text{-regular}) \\ & = (R^m \circ H \circ L^{n-1} \circ L^m \circ H \circ L^n) \quad (\text{by Lemma 1.1}) \\ & \subseteq (R^m \circ H \circ L^{n-1} \circ L^{m-1} \circ (L^m \circ H \circ L^n) \circ H \circ L^n) \quad (\text{as } H \text{ is } (m, n)\text{-regular}) \\ & \subseteq (R^m \circ H \circ L^{n-1} \circ L^{m-1} \circ (L^m \circ H \circ L^n) \circ (H \circ L^n)) \quad (\text{as } H \circ L^n \subseteq (H \circ L^n)) \end{aligned}$$

$$\begin{aligned}
 &\subseteq (R^m \circ H \circ L^{n-1} \circ L^{m-1} \circ (L^m \circ H \circ L^n \circ H \circ L^n]) \quad (\text{by Lemma 1.1}) \\
 &\subseteq (R^m \circ H \circ L^{n-1} \circ L^{m-1} \circ L^m \circ H \circ L^n \circ H \circ L^n] \quad (\text{by Lemma 1.1}) \\
 &\subseteq (R^m \circ H \circ L^{n-1} \circ L^{m-1} \circ L^{m-1} \circ (L^m \circ H \circ L^n] \circ H \circ L^n \circ H \circ L^n] \\
 &\quad \vdots \\
 &\subseteq (R^m \circ H \circ L^{n-1} \circ \underbrace{L^{m-1} \circ L^{m-1} \circ \dots \circ L^{m-1}}_{n-1\text{-times}} \circ (L^m \circ H \circ L^n) \\
 &\quad \circ \underbrace{H \circ L^n \circ H \circ L^n \circ \dots \circ H \circ L^n}_{n-1\text{-times}}] \\
 &= (R^m \circ H \circ L^{n-1} \circ (L^{m-1})^{n-1} \circ L^m \circ \underbrace{H \circ L^n \circ H \circ L^n \circ \dots \circ H \circ L^n}_{n\text{-times}}] \\
 &= (R^m \circ H \circ (L^{n-1} \circ L^{mn-m-n+1} \circ L^m) \circ \underbrace{H \circ L^n \circ H \circ L^n \circ \dots \circ H \circ L^n}_{n\text{-times}}] \\
 &= (R^m \circ (H \circ L^{mn}) \circ \underbrace{H \circ L^n \circ H \circ L^n \circ \dots \circ H \circ L^n}_{n\text{-times}}] \\
 &\subseteq (R^m \circ H \circ \underbrace{H \circ L^n \circ H \circ L^n \circ \dots \circ H \circ L^n}_{n\text{-times}}] \\
 &\subseteq (R^m \circ \underbrace{H \circ L^n \circ H \circ L^n \circ \dots \circ H \circ L^n}_{n\text{-times}}] \\
 &\subseteq (R^m \circ (H \circ L^n)^n] \\
 &\subseteq (R^m \circ L^n].
 \end{aligned}$$

Therefore,  $L \cap R = (R^m \circ L^n]$ .

Conversely, assume that  $L \cap R = (R^m \circ L^n]$  for each  $(m, 0)$ -hyperideal  $R$  and for each  $(0, n)$ -hyperideal  $L$  of  $H$ . Let  $a \in S$ . As  $[a]_{m,0}$  is an  $(m, 0)$ -hyperideal and  $H$  is a  $(0, n)$ -hyperideal of  $H$ , we have

$$\begin{aligned}
 [a]_{m,0} &= [a]_{m,0} \cap H = (([a]_{m,0})^m \circ H^n] \\
 &\subseteq (([a]_{m,0})^m \circ H) = (a^m \circ H) \quad (\text{by Theorem 2.14})
 \end{aligned}$$

Similarly,  $[a]_{0,n} \subseteq (H \circ a^n]$ . As  $(a^m \circ H]$  and  $(H \circ a^n]$  are an  $(m, 0)$ -hyperideal and  $(0, n)$ -hyperideal of  $H$ , by hypothesis we get

$$\begin{aligned}
 \{a\} &\subseteq [a]_{m,0} \cap [a]_{0,n} \subseteq (a^m \circ H) \cap (H \circ a^n] \\
 &= (((a^m \circ H)^m \circ ((H \circ a^n)^n] \quad (\text{by hypothesis}) \\
 &\subseteq (a^m \circ H \circ a^n].
 \end{aligned}$$

Hence,  $H$  is  $(m, n)$ -regular. □

**Theorem 3.7.** *Let  $H$  be an ordered semihypergroup and  $m, n$  be positive integers (either  $m \geq 2$  or  $n \geq 2$ ). Then, the following are equivalent:*

- (i) Each  $(m, n)$ -hyperideal of  $H$  is idempotent;
- (ii) For each  $(m, n)$ -hyperideals  $A, B$  of  $H$ ,  $A \cap B \subseteq (A^m \circ B^n)$ ;
- (iii)  $[a]_{m,n} \cap [b]_{m,n} \subseteq (([a]_{m,n})^m \circ ([b]_{m,n})^n) \forall a, b \in H$ ;
- (iv)  $[a]_{m,n} \subseteq (([a]_{m,n})^m \circ ([a]_{m,n})^n) \forall a \in H$ ;
- (v)  $H$  is  $(m, n)$ -regular.

*Proof.* (i)  $\Rightarrow$  (ii) Assume that each  $(m, n)$ -hyperideal of  $H$  is idempotent. Let  $A$  and  $B$  be any  $(m, n)$ -hyperideals of  $H$ . As  $A \cap B$  is an  $(m, n)$ -hyperideal of  $H$ , we have

$$\begin{aligned} A \cap B &= ((A \cap B)^2] = ((A \cap B) \circ ((A \cap B)]^2] \\ &= ((A \cap B)^3] = \dots = ((A \cap B)^{m+n}] \\ &= ((A \cap B)^m \circ (A \cap B)^n] \subseteq (A^m \circ B^n]. \end{aligned}$$

(ii)  $\Rightarrow$  (iii) and (iii)  $\Rightarrow$  (iv) are obvious.

(iv)  $\Rightarrow$  (v) Take any  $a \in A$ . Then, by (iv), we have

$$\begin{aligned} [a]_{m,n} &\subseteq (([a]_{m,n})^m \circ ([a]_{m,n})^n] \\ &\subseteq ((([a]_{m,n})^m \circ ([a]_{m,n})^{n-1} \circ (([a]_{m,n})^m \circ ([a]_{m,n})^n)] \\ &= ((([a]_{m,n})^m \circ ([a]_{m,n})^{n-1} \circ ([a]_{m,n})^m \circ ([a]_{m,n})^n] \text{ (by Lemma 1.1)} \\ &\subseteq ((([a]_{m,n})^m \circ H \circ ([a]_{m,n})^n] \\ &= ((([a]_{m,n})^m \circ H] \circ ([a]_{m,n})^n] \text{ (by Lemma 1.1)} \\ &= ((a^m \circ H] \circ ([a]_{m,n})^n] \text{ (by Theorem 4.1)} \\ &= (a^m \circ H \circ ([a]_{m,n})^n] \text{ (by Lemma 1.1)} \\ &= (a^m \circ (H \circ ([a]_{m,n})^n)] \text{ (by Lemma 1.1)} \\ &= (a^m \circ (H \circ a^n)] \text{ (by Theorem 4.1)} \\ &= (a^m \circ H \circ a^n] \text{ (by Lemma 1.1)} \end{aligned}$$

As  $\{a\} \subseteq [a]_{m,n}$ ,  $a \in (a^m \circ H \circ a^n]$ . Hence  $H$  is  $(m, n)$ -regular.

(v)  $\Rightarrow$  (i) Take any  $(m, n)$ -hyperideal  $A$  of  $H$ . As  $H$  is  $(m, n)$ -regular and  $A$  is an  $(m, n)$ -hyperideal,  $A = (A^m \circ H \circ A^n]$ . Now

$$(A \circ A] = ((A^m \circ H \circ A^n] \circ (A^m \circ H \circ A^n)] \subseteq (A^m \circ H \circ A^n] = A$$

and

$$\begin{aligned} A &= (A^m \circ H \circ A^n] = (((A^m \circ H \circ A^n])^m \circ H \circ A^n] \\ &= \underbrace{((A^m \circ H \circ A^n] \circ \dots \circ (A^m \circ H \circ A^n]) \circ H \circ A^n]}_{m\text{-times}} \\ &= ((A^m \circ H \circ A^n] \circ (A^m \circ H \circ A^n]) \circ \end{aligned}$$



$$\begin{aligned}
 & \underbrace{(A^m \circ H \circ A^n) \circ \dots \circ (A^m \circ H \circ A^n) \circ H \circ A^n}_{(m-2)\text{-times}} \\
 & \subseteq ((A^m \circ H \circ A^n) \circ (A^m \circ H \circ A^n) \circ H \circ H \circ A^n) \\
 & \subseteq ((A^m \circ H \circ A^n) \circ (A^m \circ H \circ A^n) \circ H \circ A^n) \\
 & \subseteq ((A^m \circ H \circ A^n) \circ (A^m \circ H \circ A^n) \circ (H \circ A^n)) \\
 & \subseteq ((A^m \circ H \circ A^n) \circ (A^m \circ H \circ A^n \circ H \circ A^n)) \\
 & \subseteq ((A^m \circ H \circ A^n) \circ (A^m \circ H \circ A^n)) \\
 & = (A \circ A).
 \end{aligned}$$

Therefore,  $A = (A \circ A)$ . Hence, each  $(m, n)$ -hyperideal of  $H$  is an idempotent. □

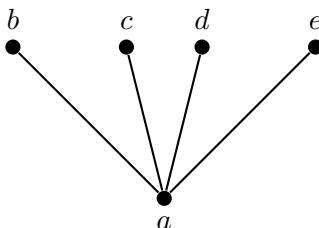
The following example shows that the condition  $m \geq 2$  or  $n \geq 2$  in Theorem 3.7 is necessary.

**Example 3.8.** [24] Let  $H = \{a, b, c, d, e\}$ . Define a hyperoperation  $\circ$  on  $H$  by the table

$\circ$	$a$	$b$	$c$	$d$	$e$
$a$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
$b$	$\{a\}$	$\{a, b\}$	$\{a\}$	$\{a, d\}$	$\{a\}$
$c$	$\{a\}$	$\{a, e\}$	$\{a, c\}$	$\{a, c\}$	$\{a, e\}$
$d$	$\{a\}$	$\{a, b\}$	$\{a, d\}$	$\{a, d\}$	$\{a, b\}$
$e$	$\{a\}$	$\{a, e\}$	$\{a\}$	$\{a, c\}$	$\{a\}$

and the order  $\leq$  on  $H$  as  $\leq := \{(a, a), (b, b), (c, c), (d, d), (e, e), (a, b), (a, c), (a, d), (a, e)\}$ . The covering relation  $\prec$  and the figure of  $H$  are as

$$\prec := \{(a, b), (a, c), (a, d), (a, e)\}$$



Now,  $(H, \circ, \leq)$  is a regular ordered semihypergroup. One may easily check that  $A = \{a, e\}$  is a bi-hyperideal of  $H$ , but  $A \neq (A^2)$ .

#### 4 Relations $\mathcal{I}_n, {}_m\mathcal{I}, \mathcal{H}_m^n$ and $\mathcal{B}_m^n$ on ordered semihypergroups

In this section, the relations  $\mathcal{I}_n, {}_m\mathcal{I}, \mathcal{H}_m^n$  and  $\mathcal{B}_m^n$  on an ordered semihypergroup are introduced. Then, some related properties of these relations are studied.

**Definition 4.1.** Let  $H$  be an ordered semihypergroup and  $m, n$  be positive integers. We define the relations  $\mathcal{I}_n, {}_m\mathcal{I}, \mathcal{H}_m^n$  and  $\mathcal{B}_m^n$  as

$$\begin{aligned} \mathcal{I}_n &= \{(a, b) \in S \times S \mid [a]_{0,n} = [b]_{0,n}\}; \\ {}_m\mathcal{I} &= \{(a, b) \in S \times S \mid [a]_{m,0} = [b]_{m,0}\}; \\ \mathcal{H}_m^n &= {}_m\mathcal{I} \cap \mathcal{I}_n; \\ \mathcal{B}_m^n &= \{(a, b) \in S \times S \mid [a]_{m,n} = [b]_{m,n}\}. \end{aligned}$$

Clearly, all the relations defined above are equivalence relations on  $H$ .

**Lemma 4.2.** Let  $H$  be an ordered semihypergroup and  $a, b \in H$  be  ${}_m\mathcal{I}$ -related (respectively,  $\mathcal{I}_n$ -related). Then,  $(a^m \circ H) = (b^m \circ H)$  (respectively,  $(H \circ a^n) = (H \circ b^n)$ ).

*Proof.* Suppose that  $(a, b) \in {}_m\mathcal{I}$ . Then, by definition,  $[a]_{m,0} = [b]_{m,0}$ , i.e.  $(\bigcup_{i=1}^m a^i \cup a^m \circ H) = (\bigcup_{i=1}^m b^i \cup b^m \circ H)$ . Therefore,  $\{a\} \subseteq (\bigcup_{i=1}^m b^i \cup b^m \circ H)$  and  $\{b\} \subseteq (\bigcup_{i=1}^m a^i \cup a^m \circ H)$ . Thus,  $(a^m \circ H) \subseteq ((\bigcup_{i=1}^m b^i \cup b^m \circ H)^m \circ H) = ((b^m \circ H)^m \circ H) = (b^m \circ H)$  (by Theorem 4.1). Similarly, from  $\{b\} \subseteq (\bigcup_{i=1}^m a^i \cup a^m \circ H)$ , we have  $(b^m \circ H) \subseteq (a^m \circ H)$ . Hence  $(a^m \circ H) = (b^m \circ H)$ . Similarly, we may show that  $(a, b) \in \mathcal{I}_n$  implies  $(H \circ a^n) = (H \circ b^n)$ .  $\square$

**Lemma 4.3.** Let  $H$  be an ordered semihypergroup and  $a, b \in H$  be  $\mathcal{H}_m^n$ -related. Then,  $(a^m \circ H) = (b^m \circ H)$ ,  $(H \circ a^n) = (H \circ b^n)$  and  $(a^m \circ H \circ a^n) = (b^m \circ H \circ b^n)$ .

*Proof.* Suppose that  $(a, b) \in \mathcal{H}_m^n$ . Then, by definition,  $(a, b) \in {}_m\mathcal{I}$  and  $(a, b) \in \mathcal{I}_n$ . By Lemma 4.1,  $(a^m \circ H) = (b^m \circ H)$  and  $(H \circ a^n) = (H \circ b^n)$ . Therefore, we have  $(a^m \circ H \circ a^n) = ((a^m \circ H) \circ a^n) = ((b^m \circ H) \circ a^n) = (b^m \circ H \circ a^n) = (b^m \circ (H \circ a^n)) = (b^m \circ (H \circ b^n)) = (b^m \circ H \circ b^n)$ .  $\square$

**Lemma 4.4.** Let  $H$  be an ordered semihypergroup. Then,  $\mathcal{B}_m^n \subseteq \mathcal{H}_m^n$ .

*Proof.* Let  $(a, b) \in \mathcal{B}_m^n$ . Then,  $[a]_{m,n} = [b]_{m,n}$ , i.e.  $(\bigcup_{i=1}^{m+n} a^i \cup a^m \circ H \circ a^n) = (\bigcup_{i=1}^{m+n} b^i \cup b^m \circ H \circ b^n)$ . So  $a^i \subseteq (\bigcup_{i=1}^{m+n} b^i \cup b^m \circ H \circ b^n)$  and  $b^i \subseteq (\bigcup_{i=1}^{m+n} a^i \cup a^m \circ H \circ a^n)$  for each  $i \in \{1, 2, \dots, m+n\}$ . It follows that  $\bigcup_{i=1}^m a^i \subseteq (\bigcup_{i=1}^{m+n} b^i \cup b^m \circ H \circ b^n)$  and  $\bigcup_{i=1}^m b^i \subseteq (\bigcup_{i=1}^{m+n} a^i \cup a^m \circ H \circ a^n)$ . Now  $(a^m \circ H) \subseteq (((\bigcup_{i=1}^{m+n} b^i \cup b^m \circ H \circ b^n))^m \circ H)$  and  $(b^m \circ H) \subseteq (((\bigcup_{i=1}^{m+n} a^i \cup a^m \circ H \circ a^n))^m \circ H)$ . Therefore, by Theorem 4.1,  $(a^m \circ H) \subseteq (b^m \circ H)$  and  $(b^m \circ H) \subseteq (a^m \circ H)$ . Now

$$\begin{aligned} [a]_{m,0} &= \left( \bigcup_{i=1}^m a^i \cup a^m \circ H \right) \\ &\subseteq \left( \left( \bigcup_{i=1}^{m+n} b^i \cup b^m \circ H \circ b^n \right) \cup a^m \circ H \right) \quad \left( \text{since } \bigcup_{i=1}^m a^i \subseteq \left( \bigcup_{i=1}^{m+n} b^i \cup b^m \circ H \circ b^n \right) \right) \\ &\subseteq \left( \left( \bigcup_{i=1}^{m+n} b^i \cup b^m \circ H \circ b^n \right) \cup (a^m \circ H) \right) \quad \left( \text{as } a^m \circ H \subseteq (a^m \circ H) \right) \\ &\subseteq \left( \left( \bigcup_{i=1}^{m+n} b^i \cup b^m \circ H \circ b^n \right) \cup (b^m \circ H) \right) \quad \left( \text{as } (a^m \circ H) \subseteq (b^m \circ H) \right) \\ &= \left( \left( \bigcup_{i=1}^{m+n} b^i \cup b^m \circ H \circ b^n \cup b^m \circ H \right) \right) \quad \left( \text{by Lemma 1.1} \right) \\ &= \left( \bigcup_{i=1}^{m+n} b^i \cup b^m \circ H \circ b^n \cup b^m \circ H \right) \quad \left( \text{by Lemma 1.1} \right) \\ &\subseteq \left( \bigcup_{i=1}^m b^i \cup b^m \circ H \cup b^m \circ H \circ b^n \cup b^m \circ H \right) \quad \left( \text{since } \bigcup_{i=1}^{m+n} b^i \subseteq \bigcup_{i=1}^m b^i \cup b^m \circ H \right) \\ &= \left( \bigcup_{i=1}^m b^i \cup b^m \circ H \right) \quad \left( \text{as } b^m \circ H \circ b^n \subseteq b^m \circ H \right) \\ &= [b]_{m,0}; \end{aligned}$$

and

$$\begin{aligned} [b]_{m,0} &= \left( \bigcup_{i=1}^m b^i \cup b^m \circ H \right) \\ &\subseteq \left( \left( \bigcup_{i=1}^{m+n} a^i \cup a^m \circ H \circ a^n \right) \cup b^m \circ H \right) \quad \left( \text{since } \bigcup_{i=1}^m b^i \subseteq \left( \bigcup_{i=1}^{m+n} a^i \cup a^m \circ H \circ a^n \right) \right) \\ &\subseteq \left( \left( \bigcup_{i=1}^{m+n} a^i \cup a^m \circ H \circ a^n \right) \cup (b^m \circ H) \right) \quad \left( \text{as } b^m \circ H \subseteq (b^m \circ H) \right) \\ &\subseteq \left( \left( \bigcup_{i=1}^{m+n} a^i \cup a^m \circ H \circ a^n \right) \cup (a^m \circ H) \right) \quad \left( \text{as } (b^m \circ H) \subseteq (a^m \circ H) \right) \end{aligned}$$

$$\begin{aligned}
 &= \left( \left( \bigcup_{i=1}^{m+n} a^i \cup a^m \circ H \circ a^m \cup a^m \circ H \right) \right] \text{ (by Lemma 1.1)} \\
 &= \left( \bigcup_{i=1}^{m+n} a^i \cup a^m \circ H \circ a^m \cup a^m \circ H \right) \text{ (by Lemma 1.1)} \\
 &\subseteq \left( \bigcup_{i=1}^m a^i \cup a^m \circ H \cup a^m \circ H \circ a^n \cup a^m \circ H \right) \text{ (since } \bigcup_{i=1}^{m+n} a^i \subseteq \bigcup_{i=1}^m a^i \cup a^m \circ H) \\
 &\subseteq \left( \bigcup_{i=1}^m a^i \cup a^m \circ H \right) \text{ (as } a^m \circ H \circ a^n \subseteq a^m \circ H) \\
 &= [a]_{m,0}.
 \end{aligned}$$

Therefore,  $[a]_{m,0} = [b]_{m,0}$ . Similarly, one can show that  $[a]_{0,n} = [b]_{0,n}$ . Thus,  $(a, b) \in \mathcal{H}_m^n$ . Hence,  $\mathcal{B}_m^n \subseteq \mathcal{H}_m^n$ .  $\square$

**Theorem 4.5.** *Let  $H$  be an  $(m, n)$ -regular ordered semihypergroup. Then,  $\mathcal{B}_m^n = \mathcal{H}_m^n$ .*

*Proof.* Let  $(a, b) \in \mathcal{H}_m^n$ . Therefore, by Lemma 4.2,  $(a^m \circ H \circ a^n) = (b^m \circ H \circ b^n)$ . As  $S$  is  $(m, n)$ -regular,  $a \in (a^m \circ H \circ a^m)$  and  $b \in (b^m \circ H \circ b^n)$ . So  $a^i \subseteq (a^m \circ H \circ a^m)$  for each  $i \in \{1, 2, \dots, m+n\}$ , it follows that  $\bigcup_{i=1}^{m+n} a^i \subseteq (a^m \circ H \circ a^m)$ . Thus,  $[a]_{m,n} = (\bigcup_{i=1}^{m+n} a^i \cup a^m \circ H \circ a^m) = (a^m \circ H \circ a^m)$  and similarly  $[b]_{m,n} = (b^m \circ H \circ b^n)$ . Thus,  $[a]_{m,n} = [b]_{m,n}$ , i.e.  $(a, b) \in \mathcal{B}_m^n$ . This implies that  $\mathcal{H}_m^n \subseteq \mathcal{B}_m^n$ . Hence, by Lemma 4.3,  $\mathcal{B}_m^n = \mathcal{H}_m^n$ .  $\square$

**Lemma 4.6.** *If  $B_x$  and  $B_y$  are two  $(m, n)$ -regular  $\mathcal{B}_m^n$ -classes contained in the same  $\mathcal{H}_m^n$ -class of ordered semihypergroup  $H$ , then  $B_x = B_y$ .*

*Proof.* As  $x$  and  $y$  are  $(m, n)$ -regular elements of  $H$ ,  $x \in (x^m \circ H \circ x^n)$  and  $y \in (y^m \circ H \circ y^n)$ ,  $\{x\}^i \subseteq (x^m \circ H \circ x^n)$  and  $\{y\}^i \subseteq (y^m \circ H \circ y^n)$  for each  $i \in \{1, 2, \dots, m+n\}$ . It follows that  $\bigcup_{i=1}^{m+n} x^i \subseteq (x^m \circ H \circ x^n)$  and  $\bigcup_{i=1}^{m+n} y^i \subseteq (y^m \circ H \circ y^n)$ . Therefore,  $[x]_{m,n} = (x^m \circ H \circ x^n)$  and  $[y]_{m,n} = (y^m \circ H \circ y^n)$ . Since  $x$  and  $y$  are contained in the same  $\mathcal{H}_m^n$ -class, by Lemma 4.2,  $(x^m \circ H \circ x^n) = (y^m \circ H \circ y^n)$ . So  $[x]_{m,n} = [y]_{m,n}$ . Therefore,  $x\mathcal{B}_m^n y$ . Hence,  $B_x = B_y$ .  $\square$

## 5 $(m, 0)$ -regularity [ $(0, n)$ -regularity] and $(m, n)$ -right weakly regularity of a $\mathcal{B}_m^n$ -class, $\mathcal{Q}_m^n$ -class and $\mathcal{H}_m^n$ -class

In this section, the  $(m, 0)$ -regular,  $(0, n)$ -regular,  $(m, n)$ -regular and  $(m, n)$ -right weakly regular class of the relations  $\mathcal{H}_m^n$  and  $\mathcal{B}_m^n$  are studied.

**Lemma 5.1.** *An  $\mathcal{H}_m^n$ -class  $H$  of an ordered semihypergroup is  $(m, 0)$ -regular  $[(0, n)$ -regular] if it contains an  $(m, 0)$ -regular  $[(0, n)$ -regular] element.*

*Proof.* Let  $a$  be an  $(m, 0)$ -regular element and  $c$  be an element of  $\mathcal{H}_m^n$ -class  $H$ . This implies  $[b]_{m,0} = [a]_{m,0}$  and  $a \in (a^m \circ H)$ . Therefore,  $\{a\}^i \subseteq (a^m \circ H)$  for each  $i \in \{1, 2, \dots, m\}$ . Then  $\bigcup_{i=1}^m a^i \subseteq (a^m \circ H)$  implies  $(\bigcup_{i=1}^m a^i) \subseteq ((a^m \circ H)) = (a^m \circ H)$ . Thus,  $[b]_{m,0} = [a]_{m,0} = (\bigcup_{i=1}^m a^i \cup a^m \circ H) = (\bigcup_{i=1}^m a^i) \cup (a^m \circ H) = (a^m \circ H)$ . By Lemma 4.2,  $(a^m \circ H) = (b^m \circ H)$ . This implies that  $[b]_{m,0} \subseteq (b^m \circ H)$ . Hence,  $b \in (b^m \circ H)$ . So  $b$  is an  $(m, 0)$ -regular element of  $\mathcal{H}_m^n$ -class  $H$ . Hence, the  $\mathcal{H}_m^n$ -class  $H$  is  $(m, 0)$ -regular. The dual statement follows on the similar lines.  $\square$

**Lemma 5.2.** *An  $\mathcal{H}_m^n$ -class  $H$  of an ordered semihypergroup is  $(m, n)$ -regular if it contains an  $(m, n)$ -regular element.*

*Proof.* The proof is similar to the proof of Lemma 5.1.  $\square$

**Lemma 5.3.** *A  $\mathcal{B}_m^n$ -class  $B$  of an ordered semihypergroup is  $(m, n)$ -regular if it contains an  $(m, n)$ -regular element.*

*Proof.* Let  $a \in B$  be an  $(m, n)$ -regular element and  $b \in B$ . Then,  $a \in (a^m \circ H \circ a^n)$  so that  $\{a\}^i \subseteq (a^m \circ H \circ a^n)$  for each  $i \in \{1, 2, \dots, m+n\}$ , so  $\bigcup_{i=1}^{m+n} a^i \subseteq (a^m \circ H \circ a^n)$  implies  $(\bigcup_{i=1}^m a^i) \subseteq ((a^m \circ H \circ a^n)) = (a^m \circ H \circ a^n)$ . Since  $a, b \in B$ ,  $[b]_{m,n} = [a]_{m,n} = (\bigcup_{i=1}^{m+n} a^i \cup a^m \circ H \circ a^n) = (\bigcup_{i=1}^{m+n} a^i) \cup (a^m \circ H \circ a^n) = (a^m \circ H \circ a^n)$ . By Lemmas 4.3 and 4.2, we have  $(a^m \circ H \circ a^n) = (b^m \circ H \circ b^n)$ . This implies that  $[b]_{m,n} \subseteq (b^m \circ H \circ b^n)$ . So  $b \in (b^m \circ H \circ b^n)$ . Thus,  $b$  is an  $(m, n)$ -regular element of  $B$ . Hence,  $B$  is  $(m, n)$ -regular.  $\square$

**Definition 5.4.** Let  $H$  be an ordered semihypergroup and  $m, n$  be positive integers. An element  $a$  of  $H$  is said to be an  $(m, n)$ -right weakly regular element if  $a \in (a^m \circ H \circ a^n \circ H)$ . The ordered semihypergroup  $H$  is said to be  $(m, n)$ -right weakly regular if each element of  $H$  is  $(m, n)$ -right weakly regular, equivalently, for each subset  $A$  of  $H$ ,  $A \subseteq (A^m \circ H \circ A^n \circ H)$ .

**Lemma 5.5.** *A  $\mathcal{B}_m^n$ -class  $B$  of an ordered semihypergroup  $H$  is  $(m, n)$ -right weakly regular if it contains an  $(m, n)$ -right weakly regular element.*

*Proof.* Let  $a \in B$  be an  $(m, n)$ -right weakly regular element and  $b \in B$ . Then,  $a \in (a^m \circ H \circ a^n \circ H)$ . This implies that  $\{a\}^i \subseteq (a^m \circ H \circ a^n \circ H)$

for each  $i \in \{1, 2, \dots, m+n\}$ , so  $\bigcup_{i=1}^{m+n} a^i \subseteq (a^m \circ H \circ a^n \circ H)$  implies  $(\bigcup_{i=1}^m a^i) \subseteq ((a^m \circ H \circ a^n \circ H]) = (a^m \circ H \circ a^n \circ H)$ . So,  $(a^m \circ H \circ a^n) \subseteq (((a^m \circ H \circ a^n \circ H]) \circ H \circ ((a^m \circ H \circ a^n \circ H])) \subseteq (a^m \circ H \circ a^n \circ H)$ . Since  $a, b \in B$ ,  $[b]_{m,n} = [a]_{m,n} = (\bigcup_{i=1}^{m+n} a^i \cup a^m \circ H \circ a^n) = (\bigcup_{i=1}^{m+n} a^i) \cup (a^m \circ H \circ a^n) \subseteq (\bigcup_{i=1}^{m+n} a^i) \cup (a^m \circ H \circ a^n \circ H) = (a^m \circ H \circ a^n \circ H)$  (since  $(\bigcup_{i=1}^{m+n} a^i) \subseteq (a^m \circ H \circ a^n \circ H)$ ). By Lemmas 4.3 and 4.2,  $(a^m \circ H \circ a^n) = (b^m \circ H \circ b^n)$ . This implies that  $[b]_{m,n} \subseteq (a^m \circ H \circ a^n \circ H) = ((a^m \circ H \circ a^n) \circ H) = ((b^m \circ H \circ b^n) \circ H) = (b^m \circ H \circ b^n \circ H)$ . So  $b \in (b^m \circ H \circ b^n \circ H)$ . Thus,  $b$  is an  $(m, n)$ -right weakly regular element of  $B$ . Hence,  $B$  is  $(m, n)$ -right weakly regular.  $\square$

**Corollary 5.6.** *An ordered semihypergroup  $H$  is  $(m, n)$ -regular ( $(m, n)$ -right weakly regular) if and only if each  $\mathcal{B}_m^n$ -class of  $H$  contains an  $(m, n)$ -regular ( $(m, n)$ -right weakly regular) element.*

**Lemma 5.7.** *An  $\mathcal{H}_m^n$ -class  $H$  of an ordered semihypergroup is  $(m, n)$ -right weakly regular if it contains an  $(m, n)$ -right weakly regular element.*

*Proof.* Let  $a$  be an  $(m, n)$ -right weakly regular element and  $b$  be an element of  $\mathcal{H}_m^n$ -class  $H$ . Then,  $a \in (a^m \circ H \circ a^n \circ H)$ . This gives that  $\{a\}^i \subseteq (a^m \circ H \circ a^n \circ H)$  for each  $i \in \{1, 2, \dots, m+n\}$ , and so  $\bigcup_{i=1}^{m+n} a^i \subseteq (a^m \circ H \circ a^n \circ H)$  implies  $(\bigcup_{i=1}^{m+n} a^i) \subseteq ((a^m \circ H \circ a^n \circ H]) = (a^m \circ H \circ a^n \circ H)$ . Therefore,  $(a^m \circ H) \subseteq ((a^m \circ H \circ a^n \circ H) \circ H) = (a^m \circ H \circ a^n \circ H \circ H) \subseteq (a^m \circ H \circ a^n \circ H)$ . Since  $a, b \in H$ ,  $[b]_{m,0} = [a]_{m,0} = (\bigcup_{i=1}^{m+n} a^i \cup a^m \circ H) = (\bigcup_{i=1}^{m+n} a^i) \cup (a^m \circ H) = (a^m \circ H) \subseteq (a^m \circ H \circ a^n \circ H)$ . So, by Lemma 4.2,  $(a^m \circ H \circ a^n) = (b^m \circ H \circ b^n)$ . This implies that  $[b]_{m,0} \subseteq (a^m \circ H \circ a^n \circ H) = ((a^m \circ H \circ a^n) \circ H) = ((b^m \circ H \circ b^n) \circ H) = (b^m \circ H \circ b^n \circ H)$ . Therefore,  $b \in (b^m \circ H \circ b^n \circ H)$  and thus,  $b$  is an  $(m, n)$ -right weakly regular element of  $\mathcal{H}_m^n$ -class  $H$ . Hence,  $H$  is  $(m, n)$ -right weakly regular.  $\square$

**Corollary 5.8.** *An ordered semihypergroup  $H$  is (respectively,  $(m, 0)$ -regular,  $(0, n)$ -regular,  $(m, n)$ -regular)  $(m, n)$ -right weakly regular if and only if each  $\mathcal{H}_m^n$ -class of  $H$  contains a (respectively,  $(m, 0)$ -regular,  $(0, n)$ -regular,  $(m, n)$ -regular)  $(m, n)$ -right weakly regular element.*

## 6 Conclusion

The main purpose of the present paper is to introduce the equivalence relations  ${}_m\mathcal{I}, \mathcal{I}_n, \mathcal{B}_m^n$  and  $\mathcal{H}_m^n$  on an ordered semihypergroup and enhance the un-

derstanding of different classes of ordered semihypergroups ( $(m, n)$ -regular,  $(m, 0)$ -regular,  $(0, n)$ -regular,  $(m, n)$ -right weakly regular) by considering the structural influence of the equivalence relations  ${}_m\mathcal{I}, \mathcal{I}_n, \mathcal{B}_m^n$ , and  $\mathcal{H}_m^n$ . In particular, if we take  $m = 1 = n$ , the equivalence relations  ${}_m\mathcal{I}, \mathcal{I}_n$  and  $\mathcal{H}_m^n$  are reduced to the equivalence relations  $\mathcal{R}, \mathcal{L}$  and  $\mathcal{H}$  in ordered semihypergroup, respectively, which mimic the definition of the usual Green's relations  $\mathcal{R}, \mathcal{L}$  and  $\mathcal{H}$  in plain semihypergroups [11]. Also when we take  $m = 1 = n$  in Theorems 1.9, 1.11, 4.1, 3.6, and 4.2, and Lemmas 4.1, 4.2, 4.3, 4.3, 5.1, and 5.2, then we obtain all the results for bi-hyperideals in an ordered semihypergroup and some characterizations of regular ordered semihypergroups, which is the main application of the results presented in this paper.

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## $(m, n)$ - ابر ایده‌آل‌ها در نیم‌ابری‌های مرتب

احسن محبوب، نور محمد خان، و بیژن دواز

در این مقاله، ابتدا مفاهیم  $(m, n)$ -ابر ایده‌آل و  $(m, n)$ -ایده‌آل تعمیم‌یافته را در یک نیم‌ابری گروه مرتب معرفی می‌کنیم، و سپس، برخی از ویژگی‌های این ابر ایده‌آل‌ها را مطالعه می‌کنیم. پس از آن،  $(m, n)$ -منظم بودن،  $(m, \circ)$ -منظم بودن، و  $(\circ, n)$ -منظم بودن نیم‌ابری گروه مرتب را، به ترتیب، بر حسب  $(m, n)$ -ابر ایده‌آل‌ها،  $(m, \circ)$ -ابر ایده‌آل‌ها، و  $(\circ, n)$ -ابر ایده‌آل‌ها، مشخص می‌کنیم. پس از آن، رابطه‌های  $\mathcal{I}_n, \mathcal{I}_m, \mathcal{H}_m^n$  و  $\mathcal{B}_m^n$  را روی نیم‌ابری گروه‌های مرتب معرفی می‌کنیم. ثابت می‌کنیم که  $\mathcal{B}_m^n \subseteq \mathcal{H}_m^n$ ، و شرایطی ارائه می‌کنیم که تحت آنها تساوی در این رابطه‌ی شمول برقرار است. همچنین، نشان می‌دهیم که  $(m, \circ)$ -منظم بودن  $(\circ, n)$ -منظم بودن یک عضو  $(m, \circ)$ -منظم بودن  $(\circ, n)$ -منظم بودن تمام  $\mathcal{H}_m^n$ -کلاس شامل آن عضو را نتیجه می‌دهد. همچنین، این واقعیت را که  $(m, n)$ -منظم بودن و  $(m, n)$ -منظم بودن راست ضعیف یک عضو، به ترتیب،  $(m, n)$ -منظم بودن و  $(m, n)$ -منظم بودن راست ضعیف تمام کلاس  $\mathcal{B}_m^n$  و تمام کلاس  $\mathcal{H}_m^n$  شامل آن عضو را ایجاب می‌کند، نشان می‌دهیم.