



Classification of monoids by Condition (PWP_{ssc}) of right acts

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Abstract. Condition (PWP) which was introduced by Laan is related to the concept of flatness of acts over monoids. Golchin and Mohammadzadeh introduced Condition (PWP_E) as a generalization of Condition (PWP) .

In this paper, we introduce Condition (PWP_{ssc}) which is much easier to check than Conditions (PWP) and (PWP_E) and does not imply them. Also principally weakly flat is a generalization of this condition. At first, general properties of Condition (PWP_{ssc}) will be given. Finally a classification of monoids will be given for which all (cyclic, monocyclic) acts satisfy Condition (PWP_{ssc}) and also a classification of monoids S will be given for which all right S -acts satisfying some other flatness properties have Condition (PWP_{ssc}) .

1 Introduction

For a monoid S , with 1 as its identity, a set A (we consider nonempty) is called a right S -act, usually denoted by A_S (or simply A), if S acts on A

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unitarian from the right, that is, there exists a mapping $A \times S \rightarrow A$, $(a, s) \mapsto as$, satisfying the conditions $(as)t = a(st)$ and $a1 = a$, for all $a \in A$ and $s, t \in S$. Left acts are defined dually. The study of flatness properties of S -acts in general began in the early 1970s and a comprehensive survey of this research (up until the year 2000) is found in [14]. In [16], the principally weak form of Condition (P) is defined, called Condition (PWP), and in [8] the weak form of Condition (PWP) is defined, called Condition (PWP_E). In this paper we introduce Condition (PWP_{ssc}) and compare it with principally weakly flat, Condition (PWP) and Condition (PWP_E). At first general properties of this condition will be given and finally a classification of monoids will be given for which all (cyclic, monocyclic) acts satisfy Condition (PWP_{ssc}) and also a classification of monoids S will be given for which all right S -acts satisfying some other flatness properties have Condition (PWP_{ssc}).

From now on by S -act we mean right S -act. Throughout this paper, S will always denote a monoid and A an S -act. For basic definitions and terminologies relating to semigroups and acts over monoids we refer the reader to [12] and [14].

2 General properties

In this section we introduce Condition (PWP_{ssc}) and give some results of it. We show that this condition can be transferred from the product of S -acts to their components and we give equivalences for which S_G^n satisfies Condition (PWP_{ssc}). We show that Condition (PWP_{ssc}) implies principal weak flatness but the converse is not true. For left PSF monoid, we show that the converse is also true. Also we see that Condition (PWP_{ssc}) does not imply Condition (PWP) (Condition (PWP_E)) and vice versa.

An element s of S is called *right e-cancellable*, for an idempotent $e \in S$, if $s = es$ and $\ker \rho_s \leq \ker \rho_e$ (ρ_x is the right translation on S , for every $x \in S$, that is, $\rho_x : S \rightarrow S$, $t \mapsto tx$, for every $t \in S$). S is called *left PP* if every principal left ideal of S is projective as a left S -act. This is equivalent to saying that every element $s \in S$ is right e -cancellable for some idempotent $e \in S$ (see [5]). S is called *left PSF* if every principal left ideal of S is strongly flat as a left S -act. This is equivalent to saying that S is right semi-cancellative, that is, whenever $su = s'u$, for $s, s', u \in S$, there exists $r \in S$ such that $u = ru$ and $sr = s'r$ (see [17]).

We recall, from [14], that A satisfies *Condition (E)* if for all $a \in A$, $s, s' \in S$,

$$as = as' \Rightarrow (\exists a' \in A) (\exists u \in S) (a = a'u \text{ and } us = us'),$$

and A satisfies *Condition (P)* if for all $a, a' \in A$, $s, s' \in S$,

$$as = a's' \Rightarrow (\exists a'' \in A) (\exists u, u' \in S) (a = a''u, a' = a''u', \text{ and } us = u's'),$$

and A is *strongly flat* if and only if it satisfies both Conditions (P) and (E) .

Recall, from [16], that A satisfies *Condition (PWP)* if for all $a, a' \in A$, $s \in S$,

$$as = a's \Rightarrow (\exists a'' \in A) (\exists u, v \in S) (a = a''u, a' = a''v, \text{ and } us = vs).$$

Definition 2.1. We say that A satisfies *Condition strongly semi cancellative-(PWP)* or *Condition (PWP_{ssc})* if for all $a, a' \in A$, $s \in S$,

$$as = a's \Rightarrow (\exists u \in S) (au = a'u, \text{ and } us = s).$$

Theorem 2.2. *The following statements hold:*

- (1) *The one element act Θ_S satisfies Condition (PWP_{ssc}) .*
- (2) *If A satisfies Condition (PWP_{ssc}) , then every subact of A satisfies it.*
- (3) *If A satisfies Condition (PWP_{ssc}) , then every retract of A satisfies it.*
- (4) *If $\prod_{i \in I} A_i$, where $A_i, i \in I$, are S -acts, satisfies Condition (PWP_{ssc}) , then A_i satisfies it, for every $i \in I$.*
- (5) *$\prod_{i \in I} A_i$, where each A_i is an S -act, satisfies Condition (PWP_{ssc}) if and only if A_i satisfies Condition (PWP_{ssc}) , for every $i \in I$.*
- (6) *If $\{B_i \mid i \in I\}$ is a chain of subacts of A and every $B_i, i \in I$, satisfies Condition (PWP_{ssc}) , then $\bigcup_{i \in I} B_i$ satisfies it.*
- (7) *S_S satisfies Condition (PWP_{ssc}) if and only if S is right semi-cancellative.*

Proof. Proofs are obvious. □

Recall that for any nonempty set I , S_S^I is the product of a family of S in **Act-S**.

The converse of part (4) of Theorem 2.2 is not true. Consider the monoid S of Example 1.6 of [23]. S is left PSF and so S_S satisfies Condition (PWP_{ssc}) , by part (4) of Theorem 2.2, while S_S^I is not principally weakly flat, by [23, Proposition 1.5], and so it does not satisfy Condition (PWP_{ssc}) , by part (1) of Theorem 2.8.

An element $s \in S$ acts injectively on A if $as = a's$, $a, a' \in A$, implies $a = a'$. If every $s \in S$ acts injectively on A , then we say that S acts injectively on A (see [14]).

The proof of Propositions 2.3 is clear.

Proposition 2.3. *If S acts injectively on A_i , for every $i \in I$, then $\prod_{i \in I} A_i$ satisfies Condition (PWP_{ssc}) .*

Obviously, S acts injectively on S_S if and only if S is right cancellative.

Corollary 2.4. *If S is right cancellative, then S_S^I satisfies Condition (PWP_{ssc}) , for any nonempty set I .*

Since S acts injectively on $\Theta_i = \{\theta_i\}$, $\prod_{i \in I} \Theta_i$ satisfies Condition (PWP_{ssc}) .

Recall, from [2], that for S , the cartesian product $S \times S$, equipped with the right S -action $(s, t)u = (su, tu)$, for $s, t, u \in S$, is called the diagonal act of S and it is denoted by $D(S)$.

In the following theorem we obtain equivalent conditions for S_S^n to satisfy Condition (PWP_{ssc}) .

Theorem 2.5. *For a natural number $n \geq 2$, the following statements are equivalent:*

- (1) S_S^n satisfies Condition (PWP_{ssc}) ;
- (2) $D(S)$ satisfies Condition (PWP_{ssc}) ;
- (3) S is right semi-cancellative.

Proof. (1) \Rightarrow (2) It is clear.

(2) \Rightarrow (3) This is clear, by parts (4) and (7) of Theorem 2.2.

(3) \Rightarrow (1) Let $(x_1, x_2, \dots, x_n)s = (y_1, y_2, \dots, y_n)s$, for $x_1, y_1, \dots, x_n, y_n, s \in S$. Then $x_i s = y_i s$, for $1 \leq i \leq n$. Since S is right semi-cancellative, $x_1 s = y_1 s$ implies the existence of $v_1 \in S$ such that $v_1 s = s$ and $x_1 v_1 = y_1 v_1$. Then $x_2 v_1 s = y_2 v_1 s$ implies the existence of $v_2 \in S$ such that $v_2 s = s$ and $x_2 v_1 v_2 = y_2 v_1 v_2$. If $v = v_1 v_2$, then

$$vs = v_1 v_2 s = s, \quad x_1 v = x_1 v_1 v_2 = y_1 v_1 v_2 = y_1 v, \quad x_2 v = y_2 v.$$

Continuing this procedure, there exists $u \in S$ such that $us = s$ and $x_i u = y_i u$, for $1 \leq i \leq n$. Thus we have $(x_1, x_2, \dots, x_n)u = (y_1, y_2, \dots, y_n)u$ and so S_S^n satisfies Condition (PWP_{ssc}) , as required. \square

Now we give an equivalence for cyclic S -act to satisfy Condition (PWP_{ssc}) .

Theorem 2.6. *Let ρ be a right congruence on S . Then the S -act S/ρ satisfies Condition (PWP_{ssc}) if and only if*

$$(\forall x, y, s \in S)((xs)\rho(ys) \implies (\exists r \in S)((xr)\rho(yr) \wedge rs = s)).$$

Proof. Necessity. Suppose $(xs)\rho(ys)$, for $x, y, s \in S$. Thus $[x]_\rho s = [y]_\rho s$. By assumption, there exists $r \in S$ such that $[x]_\rho r = [y]_\rho r$ and $rs = s$. Hence $(xr)\rho(yr)$, as required.

Sufficiency. Let $[x]_\rho s = [y]_\rho s$, for $x, y, s \in S$. Thus $(xs)\rho(ys)$ and so, by assumption, there exists $r \in S$ such that $(xr)\rho(yr)$ and $rs = s$. Hence $[x]_\rho r = [y]_\rho r$. So S/ρ satisfies Condition (PWP_{ssc}) , as required. \square

Theorem 2.7. *Let $w \in S$ and $\rho = \rho(w, 1)$. The cyclic S -act S/ρ satisfies Condition (PWP_{ssc}) if and only if for every $x, y, s \in S$ and $m, n \in \mathbb{N}_0$*

$$w^m xs = w^n ys \implies (\exists p, q \in \mathbb{N}_0)(\exists r \in S)(w^p xr = w^q yr \wedge rs = s).$$

Proof. Necessity. Let $w^m xs = w^n ys$, for $x, y, s \in S$ and $m, n \in \mathbb{N}_0$. Then we have $(xs)\rho(ys)$, by [14, III, Corollary 8.7]. By Theorem 2.6, there exists $r \in S$ such that $rs = s$ and $(xr)\rho(yr)$. Thus, by [14, III, Corollary 8.7], there exist $p, q \in \mathbb{N}_0$ such that $w^p xr = w^q yr$.

Sufficiency. Let $(xs)\rho(ys)$, for $x, y, s \in S$. Then, by [14, III, Corollary 8.7], there exist $m, n \in \mathbb{N}_0$ such that $w^m xs = w^n ys$. By assumption, there exist $r \in S$ and $p, q \in \mathbb{N}_0$ such that $rs = s$ and $w^p xr = w^q yr$. Hence $(xr)\rho(yr)$, by [14, III, Corollary 8.7], and so S/ρ satisfies Condition (PWP_{ssc}) , by Theorem 2.6. \square

Notice that in the above theorem, if $w = 1$, then $S/\rho = S/\rho(w, 1) = S/\rho(1, 1) = S/\Delta_S \cong S_S$, which by placing w in the above theorem, the part (7) of Theorem 2.2 is obtained.

Theorem 2.8. *The following statements hold:*

- (1) *If A satisfies Condition (PWP_{ssc}) , then A is principally weakly flat.*
- (2) *For left PSF monoid S , A is principally weakly flat if and only if A satisfies Condition (PWP_{ssc}) .*

Proof. (1) Let $as = a's$, for $a \in A$ and $s, s' \in S$. By assumption, there exists $r \in S$ such that $ar = a'r$ and $rs = s$. Hence

$$a \otimes s = a \otimes rs = ar \otimes s = a'r \otimes s = a' \otimes rs = a' \otimes s$$

in $A \otimes Ss$. Thus A is principally weakly flat, as required.

(2) Necessity. Let $as = a's$, for $a \in A$ and $s, s' \in S$. By assumption, there exist $n \in \mathbb{N}$ and elements $a_1, \dots, a_n \in A$, $s_1, t_1, \dots, s_n, t_n \in S$ such that

$$\begin{array}{ll} a & = a_1s_1 \\ a_1t_1 & = a_2s_2 & s_1s & = t_1s \\ a_2t_2 & = a_3s_3 & s_2s & = t_2s \\ & \dots & \dots & \\ a_nt_n & = a' & s_ns & = t_ns. \end{array}$$

Since S is right semi-cancellative, $s_1s = t_1s$ implies the existence of $v_1 \in S$ such that $v_1s = s$ and $s_1v_1 = t_1v_1$. Then $s_2v_1s = t_2v_1s$ implies the existence of $v_2 \in S$ such that $v_2s = s$ and $s_2v_1v_2 = t_2v_1v_2$. If $v = v_1v_2$, then

$$vs = v_1v_2s = s, \quad s_1v = s_1v_1v_2 = t_1v_1v_2 = t_1v, \quad s_2v = t_2v.$$

Continuing this procedure, there exists $u \in S$ such that $us = s$ and $s_iu = t_iu$, for $1 \leq i \leq n$. Thus we have

$$au = (a_1s_1)u = a_1(s_1u) = a_1(t_1u) = (a_1t_1)u = \dots = (a_nt_n)u = a'u.$$

So A satisfies Condition (PWP_{ssc}) , as required.

Sufficiency. This is true, by (1). □

The following example shows that the converse of part (1) of Theorem 2.8 is not true in general.

Example 2.9. Let J be a proper right ideal of S . Let x, y, z be different symbols not belonging to S . Define $A(J) := (\{x, y\} \times (S \setminus J)) \cup (\{z\} \times J)$ and a right S -action on $A(J)$ by

$$(x, u)s = \begin{cases} (x, us) & \text{if } us \notin J, \\ (z, us) & \text{if } us \in J, \end{cases}$$

$$(y, u)s = \begin{cases} (y, us) & \text{if } us \notin J, \\ (z, us) & \text{if } us \in J, \end{cases}$$

$$(z, u)s = (z, us).$$

Clearly $A(J)$ is an S -act. Let $S = \{0, 1, e, s\}$ with the table

	0	1	e	s
0	0	0	0	0
1	0	1	e	s
e	0	e	e	0
s	0	s	s	0

If $J = \{0, e\}$, then J is a right ideal of S . Because $0 \in J0$ and $e \in Je$, $A(J) = \{(x, 1), (x, s), (y, 1), (y, s), (z, 0), (z, e)\}$, by [1, Proposition 2.2], is a flat S -act and so is principally weakly flat. But it does not satisfy Condition (PWP_{ssc}) . Otherwise $(x, s)s = (y, s)s$ implies that there exists $u \in S$ such that $(x, s)u = (y, s)u$ and $us = s$. Because 1 is the only element such that $1s = s$, $(x, s) = (y, s)$, which is a contradiction.

Recall, from [8], that A satisfies Condition (PWP_E) if for all $a, a' \in A$, $s \in S$

$$as = a's \Rightarrow (\exists a'' \in A) (\exists u, v \in S) (e, f \in E(S)) (ae = a''ue, a'f = a''vf, es = s = fs, \text{ and } us = vs).$$

Clearly Condition (PWP) implies Condition (PWP_E) .

Suppose S is not semi-cancellative. Then S_S does not satisfy Condition (PWP_{ssc}) while S_S satisfies Condition (PWP) . Therefore Condition (PWP) does not imply Condition (PWP_{ssc}) .

Now, in the following example, we show that Condition (PWP_{ssc}) does not imply Condition (PWP_E) and so does not imply Condition (PWP) . In this example we use the monoid S of Example 1.6 of [23].

Example 2.10. Let (I, \leq) be a totally ordered set with no successor for each element (as \mathbb{R}). Consider the commutative monoid $S = \{x_i^m \mid i \in I, m \in \mathbb{N}\} \cup \{1\}$ in which $x_i^m x_j^n$ equals to x_j^n if $i < j$ and equals to x_i^{m+n} if $i = j$. Then S is left PSF and, since S does not have any idempotent except 1, it is not left PP. Now let $A = \{a_1, a_2, a_3\}$ be a set such that $a_i 1 = a_i$ and $a_i s = a_3$, for $i = 1, 2, 3$ and any $s \in S$. Obviously A is an S -act which does not satisfy Condition (PWP_E) . Otherwise $a_1 x_i^m = a_2 x_i^m$ implies that there exist $u, v, f^2 = f, e^2 = e \in S$, and $b \in A$ such that $a_1 e = bue, a_2 f = bvf,$

$ex_i^m = x_i^m = fx_i^m$, and $ux_i^m = vx_i^m$. Because 1 is the only idempotent such that $1x_i^m = x_i^m$, $a_1 = bu$ implies $b = a_1$ and $u = 1$. Then $a_2 \neq a_1v$ for every $v \in S$, which is a contradiction. Therefore A does not satisfy Condition (PWP_E) and so does not satisfy Condition (PWP) . On the other hand, $a_jx_i^m = a_3 = a_kx_i^m$, for $j, k = 1, 2, 3$ and any $x_i^m \in S$. Let $l < i$. Then $a_jx_l^m = a_3 = a_kx_l^m$ and $x_l^mx_i^m = x_i^m$, that is, A satisfies Condition (PWP_{ssc}) .

3 Classification by Condition (PWP_{ssc}) of acts

In this section we give a classification of monoids for which all (cyclic, mono-cyclic) acts satisfy Condition (PWP_{ssc}) and give a classification of monoids for which all acts satisfying some other flatness properties have Condition (PWP_{ssc}) .

Theorem 3.1. *The following statements are equivalent:*

- (1) All S -acts satisfy Condition (PWP_{ssc}) ;
- (2) all S -acts generated by exactly two elements satisfy Condition (PWP_{ssc}) ;
- (3) all cyclic S -acts satisfy Condition (PWP_{ssc}) ;
- (4) all monocyclic S -acts of the form $S/\rho(s, s^2)$ ($s \in S$) satisfy Condition (PWP_{ssc}) ;
- (5) all Rees factor S -acts satisfy Condition (PWP_{ssc}) ;
- (6) all divisible S -acts satisfy Condition (PWP_{ssc}) ;
- (7) all cofree S -acts satisfy Condition (PWP_{ssc}) ;
- (8) S is regular.

Proof. Implications $(1) \Rightarrow (2)$, $(1) \Rightarrow (3) \Rightarrow (4)$, $(3) \Rightarrow (5)$ are obvious.

Implications $(1) \Rightarrow (6) \Rightarrow (7)$ are obvious because cofree \Rightarrow divisible.

$(2) \Rightarrow (8)$ Let $s \in S$. If $sS = S$, then there exists $z \in S$ such that $sz = 1$. Therefore $szs = s$ and s is regular. Now let $sS \neq S$. Put

$$A = S \coprod^{sS} S = \{(l, x) \mid l \in S \setminus sS\} \cup sS \cup \{(t, y) \mid t \in S \setminus sS\}.$$

Indeed $A = \langle (1, x), (1, y) \rangle = (1, x)S \cup (1, y)S$. Since A is generated by two different elements, by assumption, A satisfies Condition (PWP_{ssc}) . Now $s = (1, x)s = (1, y)s$ implies that there exists $r \in S$ such that $(1, x)r = (1, y)r$ and $rs = s$. Thus $r \in sS$ and so there exists $x \in S$ such that $r = sx$. Then $s = rs = xsx$, that is, s is regular and so S is regular.

(4) \Rightarrow (8) By part (1) of Theorem 2.8, all monocyclic S -acts of the form $S/\rho(s, s^2)$ ($s \in S$) are principally weakly flat. Thus, by [14, IV, Theorem 6.6], S is regular.

(5) \Rightarrow (8) By part (1) of Theorem 2.8, all Rees factor S -acts are principally weakly flat. Thus, by [14, IV, Theorem 6.6], S is regular.

(7) \Rightarrow (8) Every S -act can be embedded in a cofree S -act. By assumption, every S -act is a subact of an S -act satisfying Condition (PWP_{ssc}) . Thus every S -act satisfies Condition (PWP_{ssc}) , by part (2) of Theorem 2.2, and so every S -act is principally weakly flat, by part (1) of Theorem 2.8. Therefore, by [14, IV, Theorem 6.6], S is regular.

(8) \Rightarrow (1) Every S -act is principally weakly flat, by [14, IV, Theorem 6.6]. Every regular monoid is left PP and so is left PSF. Thus every S -act satisfies Condition (PWP_{ssc}) , by part (2) of Theorem 2.8. \square

Recall, from [10], [9], and [15], that A satisfies Condition $(E'P)$ if for all $a \in A, s, s', z \in S$,

$$as = as', sz = s'z \Rightarrow (\exists a' \in A) (\exists u, u' \in S) (a = a'u = a'u' \text{ and } us = u's').$$

A satisfies Condition (EP) if for all $a \in A, s, s' \in S$

$$as = as' \Rightarrow (\exists a' \in A) (\exists u, u' \in S) (a = a'u = a'u' \text{ and } us = u's').$$

A satisfies Condition (E') if for all $a \in A, s, s', z \in S$

$$as = as', sz = s'z \Rightarrow (\exists a' \in A) (\exists u \in S) (a = a'u \text{ and } us = us').$$

Theorem 3.2. *The following statements are equivalent:*

- (1) All S -acts satisfy Condition (PWP_{ssc}) ;
- (2) all S -acts satisfying Condition (E) satisfy Condition (PWP_{ssc}) ;
- (3) all S -acts satisfying Condition $(E'P)$ satisfy Condition (PWP_{ssc}) ;
- (4) S is regular.

Proof. Implications (1) \Rightarrow (3) \Rightarrow (2) are obvious, because $(E) \Rightarrow (E'P)$.

(2) \Rightarrow (4) Let $s \in S$. If $sS = S$, then there exists $x \in S$ such that $xs = 1$. Thus $sxs = s$, that is, s is regular and so S is regular. Let $sS \neq S$. Put

$$A = S \coprod^{sS} S = \{(l, x) \mid l \in S \setminus sS\} \cup sS \cup \{(t, y) \mid t \in S \setminus sS\}.$$

Indeed

$$B = \{(l, x) \mid l \in S \setminus sS\} \cup sS \cong S_S \cong \{(t, y) \mid t \in S \setminus sS\} \cup sS = C.$$

B and C are subacts of A which are generated by $(1, x)$ and $(1, y)$, respectively. A is generated by $(1, x)$ and $(1, y)$ because $A = B \cup C$. By the above isomorphisms, B and C satisfy Condition (E) and so A satisfies Condition (E). Thus, by assumption, A satisfies Condition (PWP_{ssc}) . Then $(1, x)s = (1, y)s$ implies that there exists $r \in S$ such that $rs = s$ and $(1, x)r = (1, y)r$. Therefore $r \in sS$ and so there exists $x \in S$ such that $r = sx$. Hence $s = rs = sxs$, that is, s is regular. Thus S is regular, as required.

(4) \Rightarrow (1) This is true, by Theorem 3.1. □

By the proof of Theorem 3.2, we conclude that the above theorem is true for finitely generated S -acts.

Theorem 3.3. *Let $(*)$ be a property on S -acts such that every S -act satisfying Property $(*)$ is principally weakly flat and S_S satisfies in Property $(*)$. Then the following statements are equivalent:*

- (1) *All S -acts satisfying Property $(*)$ satisfy Condition (PWP_{ssc}) ;*
- (2) *all finitely generated S -acts satisfying Property $(*)$ satisfy Condition (PWP_{ssc}) ;*
- (3) *all cyclic S -acts satisfying Property $(*)$ satisfy Condition (PWP_{ssc}) ;*
- (4) *S is left PSF.*

Proof. Implications (1) \Rightarrow (2) \Rightarrow (3) are obvious.

(3) \Rightarrow (4) S is a cyclic S -act satisfying Property $(*)$ and so, by assumption, satisfies Condition (PWP_{ssc}) . Therefore S is left PSF, by part (7) of Theorem 2.2.

(4) \Rightarrow (1) By assumption, every S -act satisfying Property $(*)$ is principally weakly flat. Since S is left PSF, principally weakly flat is equivalent to Condition (PWP_{ssc}) , by Theorem 2.8. Therefore all S -acts satisfying Property $(*)$ satisfy Condition (PWP_{ssc}) , as required. □

Theorem 3.4. *The following statements are equivalent:*

- (1) *All torsion free S -acts satisfy Condition (PWP_{ssc}) ;*
- (2) *all torsion free cyclic S -acts satisfy Condition (PWP_{ssc}) ;*

- (3) all torsion free Rees factor S -acts satisfy Condition (PWP_{ssc}) ;
- (4) S is left almost regular.

Proof. Implications $(1) \Rightarrow (2) \Rightarrow (3)$ are obvious.

$(3) \Rightarrow (4)$ All torsion free Rees factor S -acts are principally weakly flat, by part (1) of Theorem 2.8. Thus S is left almost regular, by [14, IV, Theorem 6.5].

$(4) \Rightarrow (1)$ All torsion free S -acts are principally weakly flat, by [14, IV, Theorem 6.5]. Also every left almost regular monoid is left PP, by the duality of [14, IV, Proposition 1.3], and so is left PSF. Thus all torsion free S -acts satisfy Condition (PWP_{ssc}) , by part (2) of Theorem 2.8. \square

Recall, from [21], that A is called \mathfrak{R} -torsion free if for any $a, b \in A$ and $c \in S$, c right cancellable, $ac = bc$ and $a \mathfrak{R} b$ (\mathfrak{R} is Green's equivalence) imply that $a = b$.

Theorem 3.5. *The following statements are equivalent:*

- (1) All \mathfrak{R} -torsion free S -acts satisfy Condition (PWP_{ssc}) ;
- (2) all \mathfrak{R} -torsion free S -acts generated by exactly two elements satisfy Condition (PWP_{ssc}) ;
- (3) S is regular.

Proof. Implication $(1) \Rightarrow (2)$ is obvious.

$(2) \Rightarrow (3)$ Let $s \in S$. If $sS = S$, then there exists $x \in S$ such that $sx = 1$. Thus $sxs = s$ and so s is regular. Let $sS \neq S$. Put

$$A = S \coprod_{sS} S = \{(l, x) \mid l \in S \setminus sS\} \cup sS \cup \{(t, y) \mid t \in S \setminus sS\}.$$

By the proof of part $(2) \Rightarrow (4)$ of Theorem 3.2, A is an S -act which is generated by two different elements $(1, x)$ and $(1, y)$ and also satisfy Condition (E) . Every S -act satisfying Condition (E) is \mathfrak{R} -torsion free, by [21, Proposition 1.2]. Thus A is \mathfrak{R} -torsion free and so, by assumption, satisfies Condition (PWP_{ssc}) . Hence, by the proof of part $(2) \Rightarrow (4)$ of Theorem 3.2, s is regular. Therefore S is regular, as required.

$(3) \Rightarrow (1)$ Every \mathfrak{R} -torsion free S -act is principally weakly flat, by [21, Theorem 4.5]. Since every regular monoid is left PP and so is left PSF, principally weakly flat is equivalent to Condition (PWP_{ssc}) , by part (2) of Theorem 2.8. Therefore every \mathfrak{R} -torsion free S -act satisfies Condition (PWP_{ssc}) . \square

We recall, from [14], that A is (strongly) faithful if for $s, t \in S$ the equality $as = at$, for (some) all $a \in A$, implies that $s = t$.

Theorem 3.6. *The following statements are equivalent:*

- (1) All faithful S -acts satisfy Condition (PWP_{ssc}) ;
- (2) all faithful S -acts generated by exactly two elements satisfy Condition (PWP_{ssc}) ;
- (3) S is regular.

Proof. Implication (1) \Rightarrow (2) is obvious.

(2) \Rightarrow (3) Let $s \in S$. If $sS = S$, then there exists $x \in S$ such that $sx = 1$. Thus $sxs = s$ and so s is regular. Let $sS \neq S$. Put $A = S \coprod^{sS} S$. By the proof of part (2) \Rightarrow (4) of Theorem 3.2, A is an S -act which is generated by two different elements $(1, x)$ and $(1, y)$. Indeed A is faithful and so, by assumption, satisfies Condition (PWP_{ssc}) . Hence, by the proof of part (2) \Rightarrow (4) of Theorem 3.2, s is regular. Therefore S is regular, as required.

(3) \Rightarrow (1) All S -acts satisfy Condition (PWP_{ssc}) , by Theorem 3.1. Thus all faithful S -acts satisfy Condition (PWP_{ssc}) . \square

Notation: C_l (C_r) is the set of all left (right) cancellable elements of S .

Lemma 3.7. *The following statements are equivalent:*

- (1) There exists at least one strongly faithful S -act;
- (2) sS as an S -act is strongly faithful, for every $s \in S$;
- (3) S as an S -act is strongly faithful;
- (4) $sS \subseteq C_l$, for every $s \in S$;
- (5) S is left cancellative.

Proof. Implications (2) \Rightarrow (1), (5) \Rightarrow (4) and (3) \Rightarrow (1) are obvious.

(1) \Rightarrow (5) Let A be a strongly faithful S -act and $sl = st$, for $s, l, t \in S$. Then $asl = ast$, for $a \in A$. Since A is strongly faithful and $as \in A$, $l = t$. Therefore S is left cancellative.

(5) \Rightarrow (3) Let S be left cancellative and $sl = st$, for $l, t, s \in S$. By assumption, $l = t$ and so S is strongly faithful, as an S -act.

(4) \Rightarrow (5) Let $sS \subseteq C_l$, for $s \in S$ and $rt = rl$, for $r, t, l \in S$. Then $(sr)t = (sr)l$. By assumption, $t = l$ and so S is left cancellative.

(5) \Rightarrow (2) Let $skt = skl$, for $sk \in sS$ and $t, l \in S$. By assumption, $t = l$ and so sS is strongly faithful as an S -act. \square

By the above lemma, for S there exists no strongly faithful S -act if and only if S is not left cancellative.

Theorem 3.8. *The following statements are equivalent:*

- (1) *All strongly faithful S -acts satisfy Condition (PWP_{ssc}) ;*
- (2) *all strongly faithful S -acts generated by exactly two elements satisfy Condition (PWP_{ssc}) ;*
- (3) *S is not left cancellative or it is a group.*

Proof. Implication (1) \Rightarrow (2) is obvious.

(2) \Rightarrow (3) If S is not left cancellative, then (3) is satisfied. Let S be left cancellative and $s \in S$. If $sS = S$, then there exists $x \in S$ such that $sx = 1$. Thus $sxs = s$ and so s is regular. Now let $sS \neq S$. Put

$$A = S \coprod^{sS} S = \{(l, x) \mid l \in S \setminus sS\} \cup sS \cup \{(t, y) \mid t \in S \setminus sS\}.$$

We have

$$B = \{(l, x) \mid l \in S \setminus sS\} \cup sS \cong S_S \cong \{(t, y) \mid t \in S \setminus sS\} \cup sS = C$$

and

$$A = \langle (1, x), (1, y) \rangle = B \cup C.$$

Since S is left cancellative, S_S is strongly faithful, by Lemma 3.7. By the above isomorphisms, B and C are strongly faithful as subacts of A . Thus A is strongly faithful. Since A is generated by two different elements $(1, x)$ and $(1, y)$, by assumption, A satisfies Condition (PWP_{ssc}) . By the proof of part (2) \Rightarrow (4) of Theorem 3.2, s is regular and so S is regular. Thus for every $s \in S$ there exists $x \in S$ such that $sxs = s$. Since S is left cancellative, $xs = 1$. Thus every element in S has left inverse and so S is a group.

(3) \Rightarrow (1) If S is not left cancellative, then there exists no strongly faithful S -act, by Lemma 3.7. Thus (1) is satisfied. If S is left cancellative, then there exists at least a strongly faithful S -act, by Lemma 3.7. Since S is a group, it is regular and so (1) is satisfied, by Theorem 3.1. \square

Lemma 3.9. *Let ρ be a right congruence on S . Then the following statements are equivalent:*

- (1) *S/ρ is a strongly faithful cyclic S -act;*
- (2) *$\rho = \Delta_S$ and S is left cancellative.*

Proof. (1) \Rightarrow (2) Since S/ρ is strongly faithful as an S -act, there exists at least one strongly faithful S -act. Hence S is left cancellative, by Lemma 3.7. Now let $(s, t) \in \rho$, for $s, t \in S$. Then $[1]_\rho \cdot s = [s]_\rho = [t]_\rho = [1]_\rho \cdot t$. Thus $s = t$, since S/ρ is strongly faithful, and so $\rho = \Delta_S$.

(2) \Rightarrow (1) $S/\rho = S/\Delta_S \cong S_S$. Since S is left cancellative, $S_S \cong S/\rho$ is strongly faithful, by Lemma 3.7. \square

Theorem 3.10. *The following statements are equivalent:*

- (1) Every strongly faithful cyclic S -act satisfies Condition (PWP_{ssc}) ;
- (2) S is not left cancellative or it is left PSF.

Proof. (1) \Rightarrow (2) If S is not left cancellative, then (2) is satisfied. Let S be left cancellative. Then S , as a cyclic S -act, is strongly faithful, by Lemma 3.7. Thus, by assumption, S_S satisfies Condition (PWP_{ssc}) and so S is left PSF, by part (7) of Theorem 2.2.

(2) \Rightarrow (1) If S is not left cancellative, then there exists no strongly faithful S -act, by Lemma 3.7. Thus (1) is satisfied. If S is left cancellative, then there exists at least one strongly faithful cyclic S -act, by Lemma 3.7. If S/ρ is a strongly faithful cyclic S -act, then $\rho = \Delta_S$, by Lemma 3.9, and so $S/\rho \cong S_S$. By assumption, S is left PSF and so $S/\rho \cong S_S$ satisfies Condition (PWP_{ssc}) , by part (7) of Theorem 2.2. \square

Theorem 3.11. *The following statements are equivalent:*

- (1) There exists at least one strongly faithful cyclic S -act satisfying Condition (PWP_{ssc}) ;
- (2) S is left cancellative and every strongly faithful cyclic S -act satisfies Condition (PWP_{ssc}) ;
- (3) S is left cancellative and left PSF.

Proof. (1) \Rightarrow (2) Since there exists at least one strongly faithful cyclic S -act, S is left cancellative, by Lemma 3.7. If S/ρ is a strongly faithful cyclic S -act satisfying Condition (PWP_{ssc}) , then $\rho = \Delta_S$, by Lemma 3.9. Thus $S/\rho \cong S_S$ satisfies Condition (PWP_{ssc}) and so S is left PSF, by part (7) of Theorem 2.2. Therefore every strongly faithful cyclic S -act satisfies Condition (PWP_{ssc}) , by Theorem 3.10.

(2) \Rightarrow (3) This is true, by Theorem 3.10.

(3) \Rightarrow (1) Since S is left cancellative, there exists at least one strongly faithful cyclic S -act, by Lemma 3.7. If S/ρ is a strongly faithful cyclic S -act,

then $\rho = \Delta_S$, by Lemma 3.9. Thus $S/\rho = S/\Delta_S \cong S_S$. Since S is left PSF, $S/\rho \cong S_S$ satisfies Condition (PWP_{ssc}) , by part (7) of Theorem 2.2. \square

We recall, from [22], that an act A is called *strongly torsion free* if for $a, b \in A$ and any $s \in S$, the equality $as = bs$ implies $a = b$. It is obvious that every strongly torsion free S -act satisfies Condition (PWP_{ssc}) .

Let K_S be a right ideal of S . Then the Rees factor S -act S/K_S is simple if and only if $K_S = S$, by [14, I, Proposition 5.28]. Thus S/K_S is simple if and only if $S/K_S = S/S_S \cong \Theta_S$. Since Θ_S satisfies Condition (PWP_{ssc}) , by part (1) of Theorem 2.2, every simple Rees factor S -act satisfies Condition (PWP_{ssc}) .

Recall, from [14], that a right ideal K_S of S satisfies *Condition (LU)* if for every $k \in K_S$ there exists $l \in K_S$ such that $lk = k$.

Lemma 3.12. *Let $S \neq C_r$. Then the following statements hold:*

- (1) $I = S \setminus C_r$ is a proper right ideal of S .
- (2) S/I ($I = S \setminus C_r$) is a torsion free S -act.
- (3) If S is left PSF, then the right ideal $I = S \setminus C_r$ satisfies Condition (LU) .

Proof. (1) Since $S \neq C_r$ and $1 \in C_r$, $\emptyset \neq I \subset S$. Let $i \in I$ and $s \in S$. Then there exist $l_1, l_2 \in S$ such that $l_1 \neq l_2$ and $l_1i = l_2i$. Thus $l_1is = l_2is$. If $is \in C_r$, then $l_1 = l_2$, which is a contradiction. Therefore $is \in I$ and so I is a proper right ideal of S , as required.

(2) Let $sc \in I$, for $s \in S$ and $c \in C_r$. We claim that $s \in I$. Since $sc \in I$, there exist $l_1, l_2 \in S$ such that $l_1 \neq l_2$ and $l_1sc = l_2sc$. Thus $l_1s = l_2s$. If $s \notin I$, then $s \in C_r$ and so $l_1 = l_2$, which is a contradiction. Thus $s \in I$ and so the S -act S/I is torsion free, by [14, III, Proposition 8.10].

(3) Let $i \in I$. Then i is not right cancellable. Thus there exist $l_1, l_2 \in S$ such that $l_1 \neq l_2$ and $l_1i = l_2i$. Since S is left PSF, there exists $r \in S$ such that $l_1r = l_2r$ and $ri = i$, by part (7) of Theorem 2.2. If $r \notin I$, then $l_1 = l_2$, which is a contradiction. Thus $r \in I$ and $ri = i$. Therefore I satisfies Condition (LU) . \square

We recall, from [20], that A is called *GP-flat* if for every $s \in S$ and $a, a' \in A$, $a \otimes s = a' \otimes s$ in $A \otimes S$ implies the existence of a natural number n such that $a \otimes s^n = a' \otimes s^n$ in $A \otimes Ss^n$.

Lemma 3.13. *Let S be right cancellative. Then for every S -act we have*

strongly torsion free \Leftrightarrow *torsion free* \Leftrightarrow *GP-flat* \Leftrightarrow *principally weakly flat* \Leftrightarrow
Condition (PWP) \Leftrightarrow *Condition (P')* \Leftrightarrow *Condition (PWP_E)* \Leftrightarrow
Condition (PWP_{ssc}) \Leftrightarrow *translation kernel flat* \Leftrightarrow *principally weakly kernel flat*.

Proof. Since S is right cancellative, strongly torsion free is equivalent to torsion free, by definition. We always have strongly torsion free \Rightarrow Condition (PWP) (Condition (PWP_E), Condition (PWP_{ssc}), Condition (P')) \Rightarrow principally weakly flat \Rightarrow GP-flat \Rightarrow torsion free. Thus strongly torsion free \Leftrightarrow torsion free \Leftrightarrow GP-flat \Leftrightarrow principally weakly flat \Leftrightarrow Condition (PWP) \Leftrightarrow Condition (PWP_{ssc}) \Leftrightarrow Condition (PWP_E) \Leftrightarrow Condition (P'). Also we always have principally weakly kernel flat \Rightarrow translation kernel flat \Rightarrow Condition (PWP). Since S is right cancellative, it is left PP and so translation kernel flat \Leftrightarrow principally weakly kernel flat, by [3, Proposition 27]. Let $a, a' \in A$ and $s, s' \in S$. $as = a's'$ in A if and only if $a \otimes s = a' \otimes s'$ in $A \otimes S$ (see [14, II, Proposition 5.13]). It is obvious that $a \otimes s = a' \otimes s'$ in $A \otimes S$ if and only if $a \otimes (s, s) = a' \otimes (s', s')$ in $A \otimes \Delta$. Since S is right cancellative, $\ker \rho_z = s\Delta$ for every $z \in S$. Now Condition (PWP) is equivalent to translation kernel flat, by [3, Proposition 5]. \square

Theorem 3.14. *Let (*) be a property on S-acts such that flat \Rightarrow Property (*) \Rightarrow torsion free. Then the following statements are equivalent:*

- (1) *S is left PSF and Property (*) in S-acts implies principally weakly kernel flat;*
- (2) *S is left PSF and Property (*) in S-acts implies translation kernel flat;*
- (3) *S is left PSF and Property (*) in S-acts implies Condition (PWP);*
- (4) *S is left PSF and Property (*) in S-acts implies Condition (P');*
- (5) *S is right cancellative.*

Proof. The implications (1) \Rightarrow (2) \Rightarrow (3) are obvious because principally weakly kernel flat \Rightarrow translation kernel flat \Rightarrow Condition (PWP), also implication (4) \Rightarrow (3) is obvious because Condition (P') \Rightarrow Condition (PWP).

(3) \Rightarrow (5) Let S be not right cancellative and $I = S \setminus C_r$. Then I is a proper right ideal of S which satisfies Condition (LU), by Lemma 3.12. Put

$$A = S \coprod_I S = \{(l, x) \mid l \in S \setminus I\} \cup I \cup \{(t, y) \mid t \in S \setminus I\}.$$

So A is flat, by [14, III, Proposition 12.19]. By assumption, A satisfies Condition (PWP) . Let $i \in I$, then the equality $(1, x)i = (1, y)i$ implies there exist $a \in A$ and $u, v \in S$ such that $(1, x) = au$, $(1, y) = av$, and $ui = vi$. Hence there exist $l, t \in S \setminus I$ such that $(l, x) = a = (t, y)$, which is a contradiction. Thus S is right cancellative, as required.

(5) \Rightarrow (1) Since S is right cancellative, it is left PP and so it is left PSF. Also torsion free is equivalent to principally weakly kernel flat, by Lemma 3.13. Thus, by assumption, every S -act satisfying $(*)$ is principally weakly kernel flat.

(5) \Rightarrow (4) Since S is right cancellative, it is left PP and so it is left PSF. Also torsion free is equivalent to Condition (P') , by Lemma 3.13. Thus, by assumption, every S -act satisfying the Property $(*)$ satisfies Condition (P') . \square

Using a similar argument as in the proof of the above theorem, we conclude that Theorem 3.14 is true for finitely generated S -acts. Furthermore the Property $(*)$ in Theorem 3.14 can be any property as flat, weakly flat, principally weakly flat, and GP-flat.

Notice that in general [6, Theorem 2.8], [18, Theorems 2.6 and 2.8], [19, Lemma 2.12], and [20, Theorem 3.11] follow for every S -act, by putting the Property $(*)$ in Theorem 3.14 with any property as flat, weakly flat, principally weakly flat, and GP-flat.

Corollary 3.15. *Let $(*)$ be a property on S -acts such that flat \Rightarrow Property $(*) \Rightarrow$ torsion free. Then the following statements are equivalent:*

- (1) *All S -acts satisfying Property $(*)$ are principally weakly kernel flat and satisfy Condition (PWP_{ssc}) ;*
- (2) *all S -acts satisfying Property $(*)$ are translation kernel flat and satisfy Condition (PWP_{ssc}) ;*
- (3) *all S -acts satisfying Property $(*)$ satisfy Conditions (PWP) and (PWP_{ssc}) ;*
- (4) *all S -acts satisfying Property $(*)$ satisfy Conditions (P') and (PWP_{ssc}) ;*
- (5) *S is right cancellative.*

Proof. Implications (1) \Rightarrow (2) \Rightarrow (3) are obvious, because principally weakly kernel flat \Rightarrow translation kernel flat \Rightarrow Condition (PWP) , also implication (4) \Rightarrow (3) is obvious because Condition $(P') \Rightarrow$ Condition (PWP) .

(3) \Rightarrow (5) Since S_S is flat, by assumption, it satisfies Property (*) and so it satisfies Condition (PWP_{ssc}) . Therefore S is left PSF, by part (7) of Theorem 2.2. Also, by assumption, every S -act satisfying Property (*) satisfies Condition (PWP) and thus S is right cancellative, by Theorem 3.14.

(5) \Rightarrow (1) Since S is right cancellative, every S -act satisfying Property (*) is principally weakly kernel flat, by Theorem 3.14. Also, since S is right cancellative, S is left almost regular and so every torsion free S -act satisfies Condition (PWP_{ssc}) , by Theorem 3.4. Therefore every S -act satisfying Property (*) satisfies Condition (PWP_{ssc}) .

(5) \Rightarrow (4) Since S is right cancellative, torsion free is equivalent to Condition (P') , by Lemma 3.13. Therefore, by assumption, every S -act satisfying Property (*) satisfies Condition (P') . Also, since S is right cancellative, S is left almost regular and so every torsion free S -act satisfies Condition (PWP_{ssc}) , by Theorem 3.4. Therefore every S -act satisfying Property (*) satisfies Condition (PWP_{ssc}) , as required. \square

Note that, since Theorem 3.14 holds for finitely generated S -acts, Corollary 3.15 is also true for finitely generated S -acts. Furthermore the Property (*) in the above corollary can be any property as flat, weakly flat, principally weakly flat, and GP-flat.

Lemma 3.16. *Let S be right cancellative. Then for every S -act we have*

$$\text{weakly pullback flat} \Leftrightarrow \text{weakly kernel flat} \Leftrightarrow (WP) \Leftrightarrow \text{Condition } (P) \Leftrightarrow \text{Condition } (P_E) \Leftrightarrow \text{flat} \Leftrightarrow \text{weakly flat.}$$

Proof. Since S is right cancellative, by [4, Proposition 1], Condition $(P) \Leftrightarrow \text{flat} \Leftrightarrow \text{weakly flat}$. Also, since Condition $(P) \Rightarrow (WP) \Rightarrow \text{weakly flat}$, Condition $(WP) \Leftrightarrow \text{weakly flat}$. We always have weakly pullback flat \Rightarrow weakly kernel flat \Rightarrow weakly flat. Thus weakly pullback flat \Leftrightarrow weakly kernel flat \Leftrightarrow weakly flat, by [19, Theorem 2.14]. Since S is right cancellative, it is left PP and so Condition $(P_E) \Leftrightarrow \text{weakly flat}$, by [11, Theorem 2.5]. \square

Theorem 3.17. *Let (*) be a property on S -acts such that flat \Rightarrow Property (*) \Rightarrow weakly flat. Then the following statements are equivalent:*

- (1) S is left PSF and Property (*) in S -acts implies weakly pullback flat;
- (2) S is left PSF and Property (*) in S -acts implies Condition (P) ;
- (3) S is right cancellative.

Proof. The implication $(1) \Rightarrow (2)$ is obvious because weakly pullback flat \Rightarrow Condition (P) .

$(2) \Rightarrow (3)$ All flat S -acts satisfy Condition (WP) , since flat implies Property $(*)$, by assumption, Property $(*)$ implies Condition (P) and Condition (P) implies Condition (WP) . Thus S is right cancellative, by [19, Theorem 2.14].

$(3) \Rightarrow (1)$ Since S is right cancellative, S is left PSF and every weakly flat S -act is weakly pullback flat, by [19, Theorem 2.14]. Since Property $(*)$ implies weakly flat, Property $(*)$ for S -acts implies weakly pullback flat, as required. \square

Using a similar argument as in the proof of the above theorem, we conclude that Theorem 3.17 is true for finitely generated S -acts. Furthermore Property $(*)$ in Theorem 3.17 can be properties as flat and weakly flat.

Notice that we have Theorem 2 of [4] and Theorem 5.4 of [13] as a corollary of Theorem 3.17. Furthermore these theorems are true for finitely generated S -acts.

Corollary 3.18. *Let $(*)$ be a property on S -acts such that flat \Rightarrow Property $(*) \Rightarrow$ weakly flat. Then the following statements are equivalent:*

- (1) *All S -acts satisfying Property $(*)$ are weakly pullback flat and satisfy Condition (PWP_{ssc}) ;*
- (2) *all S -acts satisfying Property $(*)$ are weakly kernel flat and satisfy Condition (PWP_{ssc}) ;*
- (3) *all S -acts satisfying Property $(*)$ satisfy Conditions (WP) and (PWP_{ssc}) ;*
- (4) *all S -acts satisfying Property $(*)$ satisfy Conditions (P) and (PWP_{ssc}) ;*
- (5) *S is right cancellative.*

Proof. The implications $(1) \Rightarrow (2) \Rightarrow (3)$ are obvious, because weakly pullback flat \Rightarrow weakly kernel flat \Rightarrow Condition (WP) , also implications $(1) \Rightarrow (4) \Rightarrow (3)$ are obvious, because weakly pullback flat \Rightarrow Condition $(P) \Rightarrow$ Condition (WP) .

$(3) \Rightarrow (5)$ Since S_S is flat, by assumption, satisfies Property $(*)$ and so satisfies Condition (PWP_{ssc}) . Therefore S is left PSF, by part (7) of Theorem 2.2. Also, by assumption, every S -act satisfying Property $(*)$ satisfies Conditions (WP) and thus S is right cancellative, by Theorem 3.17.

$(5) \Rightarrow (1)$ Since S is right cancellative, every S -act satisfying Property $(*)$ is weakly pullback flat, by Theorem 3.17. Also S is left PSF, by Theorem

3.17. Since Property (*) implies principally weakly flat, every S -act satisfying Property (*) satisfies Condition (PWP_{ssc}) , by Theorem 2.8(2). \square

Using a similar argument as in the proof of the above corollary, we conclude that Corollary 3.18 is true for finitely generated S -acts. Furthermore Property (*) in Corollary 3.18 can be properties as flat and weakly flat.

Theorem 3.19. *The following statements are equivalent:*

- (1) *Every Rees factor S -act satisfying Condition (P) satisfies Condition (PWP_{ssc}) ;*
- (2) *every free Rees factor S -act satisfies Condition (PWP_{ssc}) ;*
- (3) *S does not contain a left zero or it is left PSF.*

Proof. Implications (1) \Rightarrow (2) is obvious.

(2) \Rightarrow (3) If S does not contain a left zero, then (3) is true. Let z be a left zero of S . Then $K_S = zS = \{z\}$ is a right ideal of S . Since $|K_S| = 1$, $S/K_S \cong S_S$ is free, by [14, I, Proposition 5.22], and so, by assumption, $S/K_S \cong S_S$ satisfies Condition (PWP_{ssc}) . Thus S is left PSF, by part (7) of Theorem 2.2.

(3) \Rightarrow (1) Let K_S be a right ideal of S such that S/K_S satisfies Condition (P). If $K_S = S$, then $S/K_S = S/S_S \cong \Theta_S$ and so $S/K_S \cong \Theta_S$ satisfies Condition (PWP_{ssc}) , by part (1) of Theorem 2.2. If $K_S \neq S$, then $|K_S| = 1$, by [14, III, Proposition 13.9]. If $z \in K_S$, then $K_S = zS = \{z\}$, that is, z is a left zero of S . Since S contains a left zero, S is left PSF, by assumption, and so $S/K_S \cong S_S$ satisfies Condition (PWP_{ssc}) , by part (7) of Theorem 2.2. \square

Remark

The referee had told the new property lies between strong torsion freeness and principal flatness, and it looks very similar to the right semi-cancellativity of a monoid. So why not, for example, to say “ A is semi-cancellative” instead of “ A satisfies Condition (PWP_{ssc}) ”? We did not change the naming, since in our definition the first part ($as = a's$) is similar to Condition (PWP) and the last part ($au = a'u$ and $us = s$) is similar to semi-cancellative. Also semi-cancellativity of monoids is equivalent to PSF monoids, while the property PSF is not defined for S -acts. So we thought that the assertion “ A is semi-cancellative” is ambiguous.

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References

- [1] Bulman-Fleming, S., *Flat and strongly flat S-systems*, Comm. Algebra 20(9) (1992), 2553-2567.
- [2] Bulman-Fleming, S. and Gilmour, A., *Flatness properties of diagonal acts over monoids*, Semigroup Forum, 79 (2009), 298-314.
- [3] Bulman-Fleming, S., Kilp, M., and Laan, V., *Pullbacks and flatness properties of acts II*, Comm. Algebra 29(2) (2001), 851-878.
- [4] Bulman-Fleming, S. and McDowell, K., *A characterization of left cancellative monoids by flatness properties*, Semigroup Forum 40 (1990), 109-112.
- [5] Fountain, J., *Right PP monoids with central idempotents*, Semigroup Forum 13 (1977), 229-237.
- [6] Golchin, A. and Mohammadzadeh, H., *On Condition (P')* , Semigroup Forum 86 (2013), 413-430.
- [7] Golchin, A. and Mohammadzadeh, H., *On homological classification of monoids by Condition (P_E) of right acts*, Ital. J. Pure Appl. Math. 25 (2009), 175-186.
- [8] Golchin, A. and Mohammadzadeh, H., *On Condition (PWP_E)* , Southeast Asian Bull. Math. 33 (2009), 245-256.
- [9] Golchin, A. and Mohammadzadeh, H., *On Condition (EP)* , Int. Math. Forum 2(19) (2007), 911-918.
- [10] Golchin, A. and Mohammadzadeh, H., *On Condition $(E'P)$* , J. Sci. Islam. Repub. Iran 17(4) (2006), 343-349.
- [11] Golchin, A. and Renshaw, J., *A flatness property of acts over monoids*, Conference on Semigroups, University of St. Andrews (1997-1998), 72-77.
- [12] Howie, J.M., "Fundamentals of Semigroup Theory", London Math. Soc. Monographs, Oxford University Press, 1995.
- [13] Renshaw, J. and Golchin, A., *Flat acts that satisfy Condition (P)* , Semigroup Forum 59 (1999), 295-309.

- [14] Kilp, M., Knauer, U., and Mikhalev, A., "Monoids, Acts and Categories", Walter de Gruyter, Berlin, 2000.
- [15] Laan, V., "Pullbacks and Flatness Properties of Acts", Ph.D. Thesis, Tartu, 1999.
- [16] Laan, V., *Pullbacks and flatness properties of acts I*, Comm. Algebra 29(2) (2001), 829-850.
- [17] Liu, Z.K. and Yang, Y.B., *Monoids over which every flat right act satisfies Condition (P)*, Comm. Algebra 22(8) (1994), 2861-2875.
- [18] Qiao, H.S., *Some new characterizations of right cancellative monoids by Condition (PWP)*, Semigroup Forum 71 (2005), 134-139.
- [19] Qiao, H.S., Wang, L.M., and Liu, Z.K., *On some new characterizations of right cancellative monoids by flatness properties*, Arab. J. Sci. Eng. 32 (2007), 75-82.
- [20] Qiao, H.S. and Wei, C., *On a generalization of principal weak flatness property*, Semigroup Forum 85 (2012), 147-159.
- [21] Zare, A., Golchin, A., and Mohammadzadeh, H., *\mathfrak{R} -torsion free acts over monoids*, J. Sci. Islam. Repub. Iran 24(3) (2013), 275-285.
- [22] Zare, A., Golchin, A., and Mohammadzadeh, H., *Strongly torsion free acts over monoids*, Asian-Eur. J. Math. 6(3) (2013), 1-22.
- [23] Sedaghatjoo, M., Khosravi, R., and Ershad, M., *Principally weakly and weakly coherent monoids*, Comm. Algebra 37 (2009), 4281-4295.

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مشخص‌سازی تکواره‌ها با شرط (PWP_{SSC}) کنش‌ها

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شرط (PWP) که لان معرفی کرد، با مفهوم همواری کنش‌های روی تکواره‌ها مربوط است. گلچین و محمدزاده شرط (PWP_E) را معرفی کردند به طوری که شرط (PWP) آن را نتیجه می‌دهد، یعنی، شرط (PWP_E) یک تعمیم از شرط (PWP) است. در این مقاله شرط (PWP_{SSC}) را معرفی می‌کنیم که بررسی آن ساده‌تر از بررسی شرط‌های (PWP) و (PWP_E) است و هیچکدام را نتیجه نمی‌دهد. همچنین همواری به طور اصلی ضعیف‌تری از این شرط است. ابتدا خواص عمومی شرط (PWP_{SSC}) را ارائه می‌دهیم. در آخر یک مشخص‌سازی از تکواره‌ها ارائه می‌شود به طوری که همه سیستم‌های (دوری، تک‌مولدی) در شرط (PWP_{SSC}) صدق کنند و همچنین یک مشخص‌سازی از تکواره‌ی S ارائه می‌شود به طوری که همه S -سیستم‌هایی که در برخی خواص دیگر همواری صدق می‌کنند، شرط (PWP_{SSC}) را داشته باشند.