

Amenability and Weak Amenability of the Semigroup Algebra $\ell^1(S_T)$

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Abstract

Let S be a semigroup with a left multiplier T on S . A new product on S is defined by T related to S and T such that S and the new semigroup S_T have the same underlying set as S . It is shown that if T is injective then $\ell^1(S_T) \cong \ell^1(S)_{\tilde{T}}$ where, \tilde{T} is the extension of T on $\ell^1(S)$. Also, we show that if T is bijective, then $\ell^1(S)$ is amenable if and only if $\ell^1(S_T)$ is so. Moreover, if S completely regular, then $\ell^1(S_T)$ is weakly amenable.

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Introduction

Let S be a semigroup and T be a left multiplier on S . We present a general method of defining a new product on S which makes S a semigroup. Let S_T denote S with the new product. These two semigroups are sometimes different and we try to find conditions on S and T such that the semigroups S and S_T have the same properties. This idea has started by Birtel in [1] for Banach algebras and continued by Larsen in [11]. Recently, this notion developed by some authors, for more details see [1], [10], [11], [12] and [15]. One of the best result in this work expresses that $L^1(G)_T$ is Arens regular if and only if G is a compact group [10]. We continue this direction on the regularity of S and S_T and the amenability of their semigroup algebras.

The term of semigroup will be a non-empty set S endowed with an associative binary operation on S , defined by $(s, t) \rightarrow st$. If S is also a Hausdorff topological space and the binary operation is jointly continuous, then S is called a topological semigroup.

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Let $p \in S$. Then p is an idempotent if $p^2 = p$. The set of all idempotents of S is denoted by $E(S)$.

An element e is a left (right) identity if $es = s$ (resp. $se = s$) for all $s \in S$. An element $e \in S$ is an identity if it is a left and a right identity. An element z is a left (resp. right) zero if $zs = z$ (resp. $sz = z$) for all $s \in S$. An element $z \in S$ is a zero if it is a left and a right zero. We denote any zero of S by 0_S (or z_S). An element $p \in S$ is a regular element of S if there exists $t \in S$ such that $p = ptp$ and p is completely regular if it is regular and $pt = tp$. We say that $p \in S$ has an inverse if there exists $t \in S$ such that $p = ptp$ and $t = tpt$. Note that the inverse of element $p \in S$ need not be unique. If $p \in S$ has an inverse, then p is regular and vice versa. Since, if $p \in S$ is regular, there exists $s \in S$ such that $p = psp$. Let $t = sps$. Then

$$p = psp = (psp)s = p(sps)p = ptp, \quad t = sps = s(psp)s = (sps)p(sps) = tpt.$$

So p has an inverse. We say that S is a regular (resp. completely regular) semigroup if each $p \in S$ is regular (resp. completely regular). Also S is an inverse semigroup if each $p \in S$ has a unique inverse. The map $T : S \rightarrow S$ is called a left (resp. right) multiplier if

$$T(st) = T(s)t \quad (\text{resp. } T(st) = sT(t)) \quad (s, t \in S).$$

The map $T : S \rightarrow S$ is a multiplier if it is a left and right multiplier. Let S be a topological semigroup. The net $(e_\alpha) \subseteq S$ is a left (resp. right) approximate identity if $\lim_\alpha e_\alpha t = t$ (resp. $\lim_\alpha t e_\alpha = t$) ($t \in S$). The net $(e_\alpha) \subseteq S$ is an approximate identity if it is a left and a right approximate identity.

Let S be a discrete semigroup. We denote by $\ell^1(S)$ the Banach space of all complex function $f : S \rightarrow \mathbb{C}$ having the form

$$f = \sum_{s \in S} f(s) \delta_s,$$

such that $\sum_{s \in S} |f(s)| = \|f\|_1$ is finite, where δ_s is the point mass at $\{s\}$. For $f, g \in \ell^1(S)$ we define the convolution product on $\ell^1(S)$ as follow:

$$f * g(s) = \sum_{t_1 t_2 = s} f(t_1) g(t_2) \quad (s \in S),$$

with this product $\ell^1(S)$ becomes a Banach algebra and is called the semigroup algebra on S .

Remark 1.1. If $f \in \ell^1(S)$ then $f = 0$ on S except at most on a countable subset of S . In other words, the set $A = \{s \in S : f(s) \neq 0\}$ is at most countable. Since, if $A_n = \{s \in S : |f(s)| \geq \frac{1}{n}\}$, $A = \bigcup_{n \in \mathbb{N}} A_n$. Set $\|f\|_1 = M$ and $n \in \mathbb{N}$ is fixed. Then we have

$$M = \sum_{s \in S} |f(s)| \geq \sum_{s \in A_n} |f(s)| \geq \sum_{s \in A_n} \frac{1}{n} = \frac{1}{n} |A_n|,$$

where $|A_n|$ is the cardinality of A_n . So $|A_n| \leq nM$. Hence A_n is a finite subset of S and thus A is at most countable.

Semigroup S_T

Let $T \in \text{Mul}_l(S)$. Then we define a new binary operation " \circ " on S as follow :

$$s \circ t = s T(t) \quad (s, t \in S).$$

The set S equipt with the new operation " \circ " is denoted by S_T and sometimes called "induced semigroup of S ". Now we have the following results.

Theorem 2.1. Let S be a Semigroup. Then (i) if $T \in \text{Mul}_l(S)$ then S_T is a semigroup .

The converse is true if S is left cancellative and T is surjective .

(ii) If S_T is left cancellative and T is surjective, then $T^{-1} \in \text{Mul}_l(S)$.

(iii) If S is a topological semigroup and S_T has a left approximate identity then $T^{-1} \in \text{Mul}_l(S)$.

Proof. i) Let $T \in \text{Mul}_l(S)$ and take $r, s, t \in S$. Then

$$\begin{aligned} r \circ (s \circ t) &= r T(s \circ t) = r T(s T(t)) = r T(s) T(t) = (r T(s)) T(t) \\ &= (r \circ s) \circ t \end{aligned}$$

So, S_T is a semigroup.

Conversely, suppose that S is left cancellative and take $r, s, t \in S$. Since T is surjective, there exists $u \in S$ such that $T(u) = t$. Then

$$\begin{aligned} r T(st) &= r T(s T(u)) = r \circ (s \circ u) = (r \circ s) \circ u = (r T(s)) T(u) \\ &= r(T(s)t). \end{aligned}$$

By the left cancellativity of S , we have $T(st) = T(s)t \quad (r, s \in S)$. So, T is a left multiplier.

ii) We must prove that T is injective. To do this end, take $r, s, u \in S$ and let $T(r) = T(s)$.

Then $u \circ r = u T(r) = u T(s) = u \circ s$. So $r = s$, since S_T is left cancellative. Hence T^{-1} exists.

Now, we show that $T^{-1} \in \text{Mul}_l(S)$. Take $r, s \in S$. Then

$$\begin{aligned} T^{-1}(rs) &= T^{-1}[T T^{-1}(r)s] = T^{-1}[T(T^{-1}(r)s)] \\ &= (T^{-1}T)[T^{-1}(r)s] = T^{-1}(r)s . \end{aligned}$$

iii) It is enough to show that T is injective. Take $r, s \in S$ and suppose that $T(r) = T(s)$. Then

$$r = \lim_{\alpha} e_{\alpha} \circ r = \lim_{\alpha} e_{\alpha} T(r) = \lim_{\alpha} e_{\alpha} T(s) = \lim_{\alpha} e_{\alpha} \circ s = s.$$

There are many properties that induced from \mathbf{S} to semigroup \mathbf{S}_T . But sometimes they are different.

Theorem 2.2. Let \mathbf{S} be a Hausdorff topological semigroup and $T \in \text{Mul}_l(\mathbf{S})$. If \mathbf{S} is commutative then so is \mathbf{S}_T . The converse is true if $\overline{T(\mathbf{S})} = \mathbf{S}$.

Proof. Suppose \mathbf{S} is commutative and take $r, s \in \mathbf{S}$. Then

$$r \circ s = r T(s) = T(s)r = T(sr) = T(rs) = T(r) s = s T(r) = s \circ r.$$

So, \mathbf{S}_T is commutative.

Conversely, Let \mathbf{S}_T be commutative and take $r, s \in \mathbf{S}$. Then there exist nets (r_α) and (s_β) in \mathbf{S} such that $\lim_\alpha T(r_\alpha) = r$ and $\lim_\beta T(s_\beta) = s$.

So, we have

$$rs = \lim_\alpha \lim_\beta T(r_\alpha \circ s_\beta) = \lim_\alpha \lim_\beta T(s_\beta \circ r_\alpha) = \lim_\alpha \lim_\beta T(s_\beta) T(r_\alpha) = s r.$$

Thus \mathbf{S} is commutative.

In the sequel, we investigate some relations between two semigroup \mathbf{S} and \mathbf{S}_T according to the role of the left multiplier T .

Theorem 2.3. Let \mathbf{S} be a semigroup and $T \in \text{Mul}_l(\mathbf{S})$. Then

- (i) If T is surjective and \mathbf{S}_T is an inverse semigroup then \mathbf{S} is an inverse semigroup and $T(s^{-1}) = T(s)^{-1}$ for all $s \in \mathbf{S}$.
- (ii) If \mathbf{S}_T is an inverse semigroup and T is injective then $T(\mathbf{S})$ is an inverse subsemigroup of \mathbf{S} .
- (iii) If T is bijective then \mathbf{S}_T is an inverse semigroup if and only if \mathbf{S} is an inverse semigroup.

Proof. i) Suppose that \mathbf{S}_T is an inverse semigroup and T is surjective. Define the map $\varphi: \mathbf{S}_T \rightarrow \mathbf{S}$ by $\varphi(s) = T(s)$. Take $r, s \in \mathbf{S}$, then

$$\varphi(r \circ s) = T(r \circ s) = T(r)T(s) = \varphi(r)\varphi(s).$$

So, φ is an epimorphism from \mathbf{S}_T onto \mathbf{S} , since T is surjective. By theorem 5.1.4[7], \mathbf{S} is an inverse semigroup and $T(s^{-1}) = T(s)^{-1}$ for all $s \in \mathbf{S}$.

ii) Suppose that T is injective and \mathbf{S}_T is an inverse semigroup. Evidently, $T(\mathbf{S})$ is a subsemigroup of \mathbf{S} . We show that it is an inverse semigroup. Take $s \in T(\mathbf{S})$. There exists $t \in \mathbf{S}$ such that $s = T(t)$. Also, there exists a unique element $u \in \mathbf{S}$ such that $t = t \circ u \circ t$, since \mathbf{S}_T is an inverse semigroup. Therefore, $T(t) = T(t)T(u)T(t)$, or $s = s \circ T(u) \circ s$. Of course, $T(u)$ is unique because $u \in \mathbf{S}$ is unique and T is injective. Hence $T(\mathbf{S})$ is an inverse subsemigroup of \mathbf{S} .

iii) Suppose that T is bijective and let S_T be an inverse semigroup. Since T is injective and surjective, by (i) and (ii), $S = T(S)$ is an inverse semigroup.

Conversely, suppose that S is an inverse semigroup. Since T is bijective, by theorem 2.1(ii), $T^{-1} \in \text{Mul}_l(S)$. So $\varphi^{-1}: S \rightarrow S_T$ defined by $\varphi^{-1}(s) = T^{-1}(s)$ is an epimorphism. Hence by (i) S_T is an inverse semigroup.

We say that $T \in \text{Mul}_l(S)$ is an inner left multiplier if it has the form $T = L_s$ for some $s \in S$ where $L_s(t) = s t$ ($t \in S$).

If $T \in \text{Mul}_l(S)$ is inner, then each ideal of S is permanent under T ; that is $T(I) \subseteq I$ for all ideal I of S . It is easily to see that if S has an identity, then each $T \in \text{Mul}_l(S)$ is inner.

Let S be a semigroup. Then S is called semisimple if $I^2 = I$ for all ideal I of S (see [9], page 95 for more details).

Theorem 2.4. Let S be a semigroup with an identity and $T \in \text{Mul}_l(S)$. If S_T is semisimple, then S is so. The converse is true if S_T is left cancellative and T is surjective.

Proof. Since S is unital there exists $\mu \in S$ such that $T = L_\mu$. Suppose that S_T is semisimple and I is an ideal of S . Then

$$I \circ S = I T(S) \subseteq I S \subseteq I.$$

Similarly, $S \circ I \subseteq I$. It follows that I is an ideal of S_T . By the hypothesis $(I_T)^2 = I \circ I = I$. Now, take $r \in I$ then there are $s, t \in I$ such that

$$r = s \circ t = s T(t) = s(\mu t) \in I^2.$$

So we show that $I^2 = I$ and hence S is semisimple.

Conversely, assume that S_T is left cancellative and $T \in \text{Mul}_l(S)$ is surjective then by theorem 2.1(ii), $T^{-1} \in \text{Mul}_l(S)$. So, there exists $b \in S$ such that $T^{-1} = L_b$. Suppose that $\tilde{S} = S_{T^{-1}}$. Then we have.

$$S = S_{T T^{-1}} = (S_T)_{T^{-1}} = \tilde{S}_{T^{-1}}.$$

By hypothesis and above the proof, $\tilde{S} = S_{T^{-1}}$ is semisimple.

Semigroup Algebra $\ell^1(S_T)$

We say that a discrete semigroup S is amenable if there exists a positive linear functional on $\ell^\infty(S)$ called a mean such that $m(\mathbf{1}) = 1$ and $m(l_s f) = m(f)$, $m(r_s f) = m(f)$ for each $s \in S$, where $l_s f(t) = f(st)$ and $r_s f(t) = f(ts)$ for all $t \in S$. The definition of amenable group is similar to semigroup case. Refer to [12] for more details.

Let \mathfrak{A} be a Banach algebra and let X be a Banach \mathfrak{A} –bimodule. A derivation from \mathfrak{A} to X is a linear map $D: \mathfrak{A} \rightarrow X$ such that

$$D(ab) = D(a) \cdot b + a \cdot D(b) \quad (a, b \in \mathfrak{A}).$$

A derivation D is inner if there exists $x \in X$ such that

$$D(a) = a \cdot x - x \cdot a \quad (a \in \mathfrak{A}).$$

The Banach algebra \mathfrak{A} is amenable if every bounded derivation $D: \mathfrak{A} \rightarrow X^*$ is inner for all Banach \mathfrak{A} –bimodule X . Where X^* is the dual space of X . We say that the Banach algebra \mathfrak{A} is weakly amenable if any bounded derivation D from \mathfrak{A} to \mathfrak{A}^* is inner. For more details see [12], [16].

If \mathbf{S} is a commutative semigroup, by theorem 5.8 of [8] $\ell^1(\mathbf{S})$ is called semisimple if and only if for all $x, y \in \mathbf{S}$, $x^2 = y^2 = xy$ implies $x = y$.

Theorem 3.1. Let \mathbf{S} be a commutative semigroup and let $T \in \text{Mul}_l(\mathbf{S})$ be injective. Then $\ell^1(\mathbf{S})$ is semisimple if and only if $\ell^1(\mathbf{S}_T)$ is semisimple.

Proof. Take $r, s \in \mathbf{S}$. Then $r^2 = s^2 = rs$ if and only if $T(r^2) = T(s^2) = T(r)T(s)$ or equivalently $r_0 r = s_0 s = r_0 s$, because T is injective. So, by theorem 5.8 [8], $\ell^1(\mathbf{S})$ is semisimple if and only if $\ell^1(\mathbf{S}_T)$ is semisimple.

Theorem 3.2. Let \mathbf{S} be a discrete semigroup and $T \in \text{Mul}_l(\mathbf{S})$. Then

- (i) The left multiplier T has an extension $\tilde{T} \in \text{Mul}_l(\ell^1(\mathbf{S}))$ with the norm decreasing.
- (ii) The left multiplier T is injective if and only if so is \tilde{T} .
- (iii) If T is injective then \tilde{T} is an isometry and also $\ell^1(\mathbf{S}_T)$ and $(\ell^1(\mathbf{S}))_T$ are isomorphic.

Proof. (i) An arbitrary element $f \in \ell^1(\mathbf{S})$ is of the form $f: \mathbf{S} \rightarrow \mathbb{C}$ such that $f(x) = 0$ except at the most countable subset A of \mathbf{S} . If A is a finite subset of \mathbf{S} then $f = \sum_{k=1}^n f(x_k) \delta_{x_k}$ for some fixed $n \in \mathbb{N}$. So in general we have

$$f = \sum_{x \in \mathbf{S}} f(x) \delta_x = \sum_{x \in A} f(x) \delta_x = \sum_{k=1}^{\infty} f(x_k) \delta_{x_k}.$$

Now, for each $n \in \mathbb{N}$, let $f_n = \sum_{k=1}^n f(x_k) \delta_{x_k}$ and define $\tilde{T}: \ell^1(\mathbf{S}) \rightarrow \ell^1(\mathbf{S})$ by

$$\begin{aligned} \tilde{T}(\delta_x) &= \delta_{T(x)} \quad (x \in \mathbf{S}), \\ \tilde{T}(f_n) &= \sum_{k=1}^n f(x_k) \tilde{T}(\delta_{x_k}) = \tilde{f}_n. \end{aligned}$$

For each $m, n \in \mathbb{N}$ where $n \geq m$, we have

$$\begin{aligned} \|\tilde{T}(f_n) - \tilde{T}(f_m)\|_1 &= \|\tilde{f}_n - \tilde{f}_m\|_1 = \left\| \sum_{k=m+1}^n f(x_k) \tilde{T}(\delta_{x_k}) \right\|_1 = \left\| \sum_{k=m+1}^n f(x_k) \delta_{T(x_k)} \right\|_1 \\ &\leq \sum_{k=m+1}^n |f(x_k)| = \|f_n - f_m\|_1. \end{aligned}$$

So $\{\tilde{T}(f_n)\}_n$ is a Cauchy sequence and it is convergent. Now, we define $\tilde{T}(f) = \lim_n \tilde{f}_n$. Then the definition is well defined. Hence

$$\tilde{T}(f) = \sum_{k=1}^{\infty} f(x_k) \tilde{T}(\delta_{x_k}) = \tilde{f},$$

also

$$\|\tilde{f}\|_1 \leq \sum_{x_k \in A} |f(x_k)| = \|f\|_1 \quad \text{or} \quad \|\tilde{T}(f)\|_1 \leq \|f\|_1.$$

It shows that \tilde{T} is norm decreasing.

In the following, we extend \tilde{T} by linearity. Let $f, g \in \ell^1(S)$. Then there are two at most countable sub set A, B of S such that

$$f = \sum_{x \in A} f(x) \delta_x, \quad g = \sum_{x \in B} g(x) \delta_x.$$

Suppose that $D = A \cup B$. So we have $f + g = \sum_{x \in D} (f(x) + g(x)) \delta_x$.

Then, it follows that

$$\begin{aligned} \tilde{T}(f + g) &= \widetilde{f + g} = \sum_{x \in D} (f(x) + g(x)) \tilde{T}(\delta_x) = \sum_{x \in A} f(x) \tilde{T}(\delta_x) + \sum_{x \in B} g(x) \tilde{T}(\delta_x) \\ &= \tilde{f} + \tilde{g}. \end{aligned}$$

Also, if $\alpha \in \mathbb{C}$, we have

$$\tilde{T}(\alpha f) = \widetilde{\alpha f} = \sum_{x \in A} \alpha f(x) \tilde{T}(\delta_x) = \alpha \sum_{x \in A} f(x) \tilde{T}(\delta_x) = \alpha \tilde{T}(f).$$

Therefore, \tilde{T} is a bounded linear isometry.

Now, we prove that $\tilde{T} \in \text{Mul}_l(\ell^1(S))$. Take $x, y \in S$. Then

$$\tilde{T}(\delta_x * \delta_y) = \tilde{T}(\delta_{xy}) = \delta_{T(xy)} = \delta_{T(x)y} = \delta_{T(x)} * \delta_y = \tilde{T}(\delta_x) * \delta_y.$$

Let $y \in S$ be fixed and $f, g \in \ell^1(S)$. Then

$$\begin{aligned} \tilde{T}(f * \delta_y) &= \tilde{T}\left(\sum_{x \in A} f(x) \delta_{xy}\right) = \sum_{x \in A} f(x) \tilde{T}(\delta_{xy}) \\ &= \left(\sum_{x \in A} \tilde{T}(\delta_x)\right) * \delta_y = \tilde{f} * \delta_y = \tilde{T}(f) * \delta_y. \end{aligned}$$

In the general case, we have

$$\begin{aligned} \tilde{T}(f * g) &= \tilde{T}\left(\sum_{x \in A} f(x) \left(\sum_{y \in B} g(y) \delta_{xy}\right)\right) = \sum_{x \in A} f(x) \sum_{y \in B} g(y) \tilde{T}(\delta_x) * \delta_y \\ &= \sum_{x \in A} f(x) \tilde{T}(\delta_x) * \sum_{y \in B} g(y) \delta_y = \tilde{T}(f) * g. \end{aligned}$$

This shows that \tilde{T} is a multiplier on $\ell^1(S)$.

(ii) Let T be injective. Take $x, y \in S$ and suppose that $\tilde{T}(\delta_x) = \tilde{T}(\delta_y)$. Then $\delta_{T(x)} = \tilde{T}(\delta_x) = \tilde{T}(\delta_y) = \delta_{T(y)}$.

Therefore, $T(x) = T(y)$. Since T is injective, we have $x = y$. It follows that $\delta_x = \delta_y$, consequently \tilde{T} is injective.

Conversely, the same argument shows that the converse holds.

(iii) Let T be injective and $f \in \ell^1(S)$. Then there exists at most a countable subset $A \subseteq S$ such that

$$f = \sum_{x \in A} f(x) \delta_x$$

Since A and $T(A)$ have the same cardinal number, $\|\tilde{T}(f)\| = \|\sum_{x \in A} f(x) \delta_x\| = \sum_{x \in A} |f(x)| = \|f\|_1$, so \tilde{T} is an isometry.

Now, we can define a new multiplication " \boxtimes " on $\ell^1(\mathbf{S})$ as follow

$$f \boxtimes g = f * \tilde{T}g \quad (f, g \in \ell^1(\mathbf{S})).$$

By a similar argument in theorem 1.31 [10], $\ell^1(\mathbf{S})$ with the new product is a Banach algebra that is denoted it by $\ell^1(\mathbf{S})_{\tilde{T}}$. We define the map $\Psi: \ell^1(\mathbf{S}_T) \rightarrow \ell^1(\mathbf{S})_{\tilde{T}}$, by

$$\Psi(\delta_x) = \delta_x \quad (x \in \mathbf{S}).$$

Take $x, y \in \mathbf{S}$. Then

$$\begin{aligned} \Psi(\delta_x * \delta_y) &= \Psi(\delta_{x \circ y}) = \delta_{xT(y)} = \delta_x * \delta_{T(y)} \\ &= \delta_x * \tilde{T}(\delta_y) = \delta_x \boxtimes \delta_y \\ &= \Psi(\delta_x) \boxtimes \Psi(\delta_y). \end{aligned}$$

So, in general case, we have

$$\Psi(f * g) = \Psi(f) \boxtimes \Psi(g) \quad (f, g \in \ell^1(\mathbf{S})).$$

Thus, Ψ is an isomorphism. Therefore $\ell^1(\mathbf{S}_T)$ and $\ell^1(\mathbf{S})_{\tilde{T}}$ are isomorphic

Theorem 3.3. Let \mathbf{S} be a semigroup and $T \in \text{Mul}_l(\mathbf{S})$ be bijective. Then $\ell^1(\mathbf{S})$ is amenable if and only if $\ell^1(\mathbf{S}_T)$ is amenable.

Proof. By theorem 3.2, we have $\ell^1(\mathbf{S}_T) \cong \ell^1(\mathbf{S})_{\tilde{T}}$. Suppose that $\ell^1(\mathbf{S}_T)$ is amenable and define $\varphi: \ell^1(\mathbf{S})_{\tilde{T}} \rightarrow \ell^1(\mathbf{S})$ by $\varphi(f) = \tilde{T}(f)$. Take $x, y \in \mathbf{S}$. Then

$$\begin{aligned} \varphi(\delta_x \boxtimes \delta_y) &= \tilde{T}(\delta_x \boxtimes \delta_y) = \tilde{T}(\delta_{xT(y)}) = \tilde{T}(\delta_x * \delta_{T(y)}) = \tilde{T}(\delta_x) * \delta_{T(y)} \\ &= \tilde{T}(\delta_x) * \tilde{T}(\delta_y) = \varphi(\delta_x) * \varphi(\delta_y). \end{aligned}$$

Now, by induction and continuity of \tilde{T} , we have

$$\varphi(f \boxtimes g) = \varphi(f) * \varphi(g).$$

If T is bijective, \tilde{T} is bijective. Therefore φ is an epimorphism of $\ell^1(\mathbf{S}_T)$ onto $\ell^1(\mathbf{S})$.

Hence, by proposition 2.3.1 [16] $\ell^1(\mathbf{S})$ is amenable.

Conversely, suppose that $\ell^1(\mathbf{S})$ is amenable. Since T is bijective, \tilde{T} is bijective. Therefore \tilde{T}^{-1} exists. Now define $\theta: \ell^1(\mathbf{S}) \rightarrow \ell^1(\mathbf{S}_T) [\cong \ell^1(\mathbf{S})_{\tilde{T}}]$ by $\theta(f) = \tilde{T}^{-1}(f)$.

Take $x, y \in \mathbf{S}$. Then

$$\begin{aligned} \theta(\delta_x * \delta_y) &= \tilde{T}^{-1}(\delta_{xy}) = \tilde{T}^{-1}(\delta_x) \tilde{T} \tilde{T}^{-1}(\delta_y) = \tilde{T}^{-1}(\delta_x) \boxtimes \tilde{T}^{-1}(\delta_y) \\ &= \theta(\delta_x) \boxtimes \theta(\delta_y). \end{aligned}$$

Similarly θ is an epimorphism from $\ell^1(\mathbf{S})$ onto $\ell^1(\mathbf{S}_T)$. By proposition 2.3.1 [16] $\ell^1(\mathbf{S}_T)$ is amenable.

Note that, in general, it is not known when $\ell^1(\mathbf{S})$ is weakly amenable. For more details see [2].

Theorem 3.4. Let \mathbf{S} be a semigroup and $T \in \text{Mul}_l(\mathbf{S})$ be bijective. Then, if \mathbf{S} is completely regular then $\ell^1(\mathbf{S}_T)$ is weakly amenable.

Proof. It is enough to prove that \mathbf{S}_T is completely regular, then by theorem 3.6 [2], $\ell^1(\mathbf{S}_T)$ can be weakly amenable. Take $s \in \mathbf{S}$. Then there exists $r \in \mathbf{S}$ such that $T(s) = T(s)T(r)T(s)$, $T(r)T(s) = T(s)T(r)$, since T is bijective and $\mathbf{S} = T(\mathbf{S})$ is completely regular. So we have $T(s) = T(s \circ r \circ s)$ and $T(r \circ s) = T(s \circ r)$. Hence $s = s \circ r \circ s$ and $r \circ s = s \circ r$ for some $r \in \mathbf{S}$, since T is injective. Therefore \mathbf{S}_T is completely regular.

Corollary 3.5. Suppose that \mathbf{S} is a commutative completely regular semigroup and $T \in \text{Mul}_l(\mathbf{S})$ is injective. Then $\ell^1(T(\mathbf{S})_T)$ is weakly amenable.

Proof. [2, theorem 3.6] $\ell^1(\mathbf{S})$ is weakly amenable. Define $\varphi: \mathbf{S} \rightarrow \ell^1(\mathbf{S})_T$ by

$$\varphi(s) = T^{-1}(s) \quad (s \in \mathbf{S}).$$

We show that φ is a homomorphism. Take $s \in \mathbf{S}$, then we have

$$\varphi(rs) = T^{-1}(rs) = T^{-1}(r)s = T^{-1}(r) \circ (T^{-1}s).$$

So φ is a homomorphism. Then by proposition 2.1[7], $\ell^1(T(\mathbf{S})_T)$ is weakly amenable. In the case that \mathbf{S} is a group, it is easy to see that the amenability of \mathbf{S} implies the amenability of $\ell^1(\mathbf{S}_T)$. Indeed, when \mathbf{S} is a group, by theorem 2.1, \mathbf{S}_T is a semigroup and one can easily prove that \mathbf{S}_T is also a group. On the other hand, $\text{Mul}_l(\mathbf{S}) \cong \mathbf{S}$ because \mathbf{S} is a unital semigroup, so each $T \in \text{Mul}_l(\mathbf{S})$ is inner and of the form $T = L_s$ for some $s \in \mathbf{S}$. Also $T^{-1} = L_{a^{-1}}$ exists, since \mathbf{S} is a group. Then the map $\theta: \mathbf{S}_T \rightarrow \mathbf{S}$ defined by $\theta(s) = T(s)$ is an isomorphism; that is $\mathbf{S} \cong \mathbf{S}_T$. Thus we have the following result:

Corollary 3.6. Let \mathbf{S} be a cancellative regular discrete semigroup. Then $\ell^1(\mathbf{S})$ is amenable if and only if $\ell^1(\mathbf{S}_T)$ is amenable.

Proof. By [9, Exercise 2.6.11] \mathbf{S} is a group. So the assertion holds by [15, theorem 2.1.8]

Examples

In this section we present some examples which either comments on our results or indicate necessary condition in our theorems.

4.1. There are semigroups \mathcal{S} and $T \in \text{Mul}_l(\mathcal{S})$ such that the background semigroups \mathcal{S} are not commutative but their induced semigroups \mathcal{S}_T are commutative.

This example shows that the condition $\overline{T(\mathcal{S})} = \mathcal{S}$, in theorem 2.2, can not be omitted.

Let \mathcal{S} be the set $\{a, b, c, d, e\}$ with operation table given by

.	a	b	c	d	e
a	a	a	a	d	d
b	a	b	c	d	d
c	a	c	b	d	d
d	d	d	d	a	a
e	d	e	e	a	a

Clearly $(\mathcal{S}, .)$ is a non-commutative semigroup. Now, put $T = L_a$ where $L_a(x) = ax$ for all $x \in \mathcal{S}$. One can get easily the operation table of \mathcal{S}_T as follow:

o	a	b	c	d	e
a	a	a	a	d	d
b	a	a	a	d	d
c	a	a	a	d	d
d	d	d	d	a	a
e	d	d	d	a	a

The operation table shows that the induced semigroup \mathcal{S}_T is commutative and $T(\mathcal{S}) \neq \mathcal{S}$. Also the other induced semigroup \mathcal{S}_T is commutative for $T = L_d$ analogously.

Now we present some important theorems from [14] that we need in the following examples:

Theorem 4. 2. Let \mathcal{S} be a semigroup. Suppose that $\ell^1(\mathcal{S})$ is amenable. Then

- (i) \mathcal{S} is amenable
- (ii) \mathcal{S} is regular.
- (iii) $E(\mathcal{S})$ is finite.
- (iv) $\ell^1(\mathcal{S})$ has an identity.

Proof. (i) That is lemma 3 in [5].

(ii) and (iii) See theorem 2 in [6].

(iv) That is corollary 10.6 in [4].

Theorem 4.3. Let S be a finite semigroup. Then the following statements are equivalent:

- (i) $\ell^1(S)$ is amenable.
- (ii) S is regular and $\ell^1(S)$ is unital.
- (ii) S is regular and $\ell^1(S)$ is semisimple.

Proof. Refer to [3].

4.4. There are semigroups S and $T \in \text{Mul}_l(S)$ such that S and $\ell^1(S)$ are amenable but S_T is not regular and also, $\ell^1(S_T)$ is not amenable.

This example shows that two semigroup algebras $\ell^1(S)$ and $\ell^1(S_T)$ can be different in some properties. Also, it notifies that the bijectivity of T in the theorem 3.3 is essential. Put $S = \{x_0, x_1, x_2, \dots, x_n\}$ with the operation $x_i x_j = x_{\text{Max}\{i,j\}}$ ($0 \leq i, j \leq n$, $n \geq 2$).

Then S is a semigroup. Since

$$\text{Max}\{i, \text{Max}\{j, k\}\} = \text{Max}\{\text{Max}\{i, j\}, k\} = \text{Max}\{i, j, k\}.$$

We denote it by S_V . This semigroup is commutative. So by (0.18) in [12], it is amenable. S_V is a unital semigroup and has a zero; indeed, $e_s = x_0$ and $o_s = x_n$. Also, it is a regular semigroup and $\text{Mul}(S_V) \cong S_V$ because S_V has an identity.

Evidently, S_V is regular since each $s \in S_V$ is idempotent. The semigroup algebra $\ell^1(S_V)$ is a unital algebra because S_V has an identity. So by theorem 4.3 (ii) $\ell^1(S_V)$ is amenable.

Now, take $T = L_{x_k}$ for a fixed $x_k \in S$ where $k \geq 1$. By theorem 2.2, $(S_V)_T$ is commutative so is amenable. We show that T is neither injective and nor surjective.

Take $x_i \in S_V$, then $Tx_i = x_k x_i = x_{\text{max}\{k,i\}}$. So

$$T(S_V) = \{x_k, x_{k+1}, \dots, x_n\} \neq S_V.$$

Hence, T is not surjective.

Again, take distinct elements x_i, x_j in S_V for some $i, j < k$ such that $T(x_i) = T(x_j)$. Then we have $x_{\text{max}\{k,i\}} = x_{\text{max}\{k,j\}}$ but $x_i \neq x_j$. So T is not injective.

We prove that $(S_V)_T$ is not regular. If $(S_V)_T$ is regular, then for $x_{k-1} \in S_V$ there exists an element $x_j \in S_V$ such that

$$x_{k-1} = x_{k-1} \circ x_j \circ x_{k-1} = x_{\text{Max}\{k,j\}}.$$

That implies that $\max\{k, j\} = k - 1$; which is impossible. Consequently, by theorem 4.2 (ii) or 4.3 (ii), $\ell^1((\mathcal{S}_V)_T)$ is not amenable.

Also, the inequality $\mathcal{S}_V \circ \mathcal{S}_V = \{x_k, x_{k+1}, \dots, x_n\} \neq \mathcal{S}_V$ shows that $\ell^1((\mathcal{S}_V)_T)$ is not weakly amenable. In the next example we show that in the theorem 3.2 (iii) the condition "injectivity of " can not be omitted.

4.5 There are a semigroup \mathcal{S} and $T \in \text{Mul}_l(\mathcal{S})$ such that $T \in \text{Mul}_l(\mathcal{S})$ is not injective and the corresponding $\tilde{T} \in \text{Mul}_l(\ell^1(\mathcal{S}_T))$ is not an isometry.

Suppose that \mathcal{S}_V is a semigroup as in example 4.4 and $T = L_{x_k}$ for some fixed $1 < k < n$. If $f \in \ell^1(\mathcal{S}_V)$ then $f = \sum_{i=0}^n f(x_i) \delta_{x_i}$ and also $\tilde{T}(f) = \sum_{i=0}^n f(x_i) \delta_{T(x_i)}$. But

$$T(x_i) = \begin{cases} x_i & k < i \leq n \\ x_k & 0 \leq i \leq k \end{cases},$$

so

$$\tilde{T}(f) = \left(\sum_{i=0}^k f(x_i) \right) \delta_{x_k} + \sum_{i=k+1}^n f(x_i) \delta_{T(x_i)}.$$

Hence

$$\begin{aligned} \|\tilde{T}(f)\| &= \left| \sum_{i=0}^k f(x_i) \right| + \sum_{i=k+1}^n |f(x_i)| \\ &\leq \sum_{i=0}^k |f(x_i)| + \sum_{i=k+1}^n |f(x_i)| = \|f\|_1, \end{aligned}$$

It shows that \tilde{T} is not an isometry.

4.6. There are semigroups \mathcal{S} and $T \in \text{Mul}_l(\mathcal{S})$ such that $\ell^1(\mathcal{S})$ is semisimple. But $\ell^1(\mathcal{S}_T)$ is not semisimple. This example remind that, in theorem 3.1 the multiplier T must be injective.

Let \mathcal{S} be a set $\{x_0, x_1, \dots, x_n\}$ where $n \in \mathbb{N}$ and $n \geq 3$ is fixed. by operation given by $xy = x_{\min\{i, j\}}$, \mathcal{S} is a commutative semigroup. Since

$$\min\{i, \min\{j, k\}\} = \min\{\min\{i, j\}, k\} = \min\{i, j, k\} \quad (i, j, k \in \mathbb{N}).$$

We denote it briefly by \mathcal{S}_\wedge . For each $x, y \in \mathcal{S}$ the equality $x^2 = y^2 = xy$ implies $x = y$. So by Theorem 5.8 [8] $\ell^1(\mathcal{S}_\wedge)$ is semisimple.

Now, let $T = L_{x_k}$ for a fixed $1 \leq k < n - 1$. It is easy to see that $T(x_k) = T(x_n)$ but $x_k \neq x_n$. So the multiplier T is not injective.

We show that neither \mathcal{S}_\wedge nor $\ell^1(\mathcal{S}_\wedge)_T$ is semisimple.

Each ideal of \mathcal{S} is of the form

$$I_m = \{x_0, x_1, \dots, x_m\} \quad (m \leq n).$$

We claim that \mathcal{S}_T is not semisimple. Since for each $m \in \mathbb{N}$ we have

$$I_m \circ I_m = \begin{cases} I_m & m \leq k \\ I_k & m > k \end{cases}.$$

On the other hand, for each $x_i, x_j \in S$ where $i \neq j$ and $i, j > k$, we have $x_i \circ x_j = x_j \circ x_i = x_i \circ x_j = x_k$, while $x_i \neq x_j$. Thus, Theorem 5.8 [8] shows that $\ell^1(S_\wedge)_T$ is not semisimple.

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