Amenability and Weak Amenability of the Semigroup Algebra $\ell^1(S_T)$

*Mohammadi S.M.; Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran,
Laali J.; Department of Mathematics, Faculty of Mathematical Science and Computer, Kharazmi University

Received: 18 Nov 2013 Revised: 10 Nov 2014

Abstract

Let **S** be a semigroup with a left multiplier T on **S**. A new product on **S** is defined by T related to **S** and T such that **S** and the new semigroup $\mathbf{S_T}$ have the same underlying set as **S**. It is shown that if T is injective then $\ell^1(\mathbf{S_T}) \cong \ell^1(\mathbf{S})_{\tilde{T}}$ where, $\tilde{T}$ is the extension of T on $\ell^1(\mathbf{S})$. Also, we show that if T is bijective, then $\ell^1(\mathbf{S})$ is amenable if and only if $\ell^1(\mathbf{S_T})$ is so. Moreover, if **S** completely regular, then $\ell^1(\mathbf{S_T})$ is weakly amenable.

Mathematics Subject Classification: 43A20, 43A22, 43A07. 2010
Keywords: Semigroup,Semigroup algebra, Multiplier, Amenability, Weak amenability.

Introduction

Let S be a semigroup and T be a left multiplier on S. We present a general method of defining a new product on S which makes S a semigroup. Let S_T denote S with the new product. These two semigroups are sometims different and we try to find conditions on S and T such that the semigroups S and S_T have the same properties. This idea has started by Birtel in [1] for Banach algebras and continued by Larsen in [11]. Recently, this notion developed by some authors, for more details see [1], [10], [11], [12] and [15]. One of the best result in this work expresses that $L^1(G)_T$ is Arens regular if and only if G is a compact group [10]. We continue this direction on the regularity of S and S_T and the amenability of their semigroup algebras.

The term of semigroup will be a non-empty set S endowed with an associative binary operation on S , defined by $(s,t) \to st$. If S is also a Hausdorff topological space and the binary operation is jointly continuous, then S is called a topological semigroup.

*Corresponding author: s11mohamadi@iausr.ac.ir

Let $p \in S$. Then p is an idempotent if $p^2 = p$. The set of all idempotents of S is denoted by $E(S)$.

An element e is a left (right) identity if $es = s$ (resp. $se = s$) for all $s \in S$. An element $e\epsilon S$ is an identity if it is a left and a right identity. An element z is a left (resp. right) zero if $zs = z$ (resp. $sz = z$) for all $s \in S$. An element $z\epsilon S$ is a zero if it is a left and a right zero. We denote any zero of S by 0_S (or z_S). An element $p\epsilon S$ is a regular element of S if there exists $t\epsilon S$ such that $p = ptp$ and p is completely regular if it is regular and $pt = tp$. We say that $p \in S$ has an inverse if there exists $t\epsilon S$ such that $p = ptp$ and $t = tpt$. Note that the inverse of element $p \in S$ need not be unique. If $p\epsilon S$ has an inverse, then p is regular and vise versa. Since, if $p \in S$ is regular, there exists $s \in S$ such that $p = psp$. Let $t = sps$. Then

$p = psp = (psp)sp = p(sps)p = ptp$, $t = sps = s(psp)s = (sps)p(sps) = tpt$.

So p has an inverse. We say that S is a regular (resp. completely regular) semigroup if each $p\epsilon S$ is regular (resp. completely regular). Also S is an inverse semigroup if each $p \in S$ has a unique inverse. The map $T : S \to S$ is called a left (resp. right) multiplier if

$$T(st) = T(s)t \quad (\text{resp. } T(st) = sT(t)) \qquad (s, t\epsilon S).$$

The map $T : S \to S$ is a multiplier if it is a left and right multiplier.Let S be a topological semigroup. The net $(e_\alpha) \subseteq S$ is a left (resp. right) approximate identity if $\lim_\alpha e_\alpha t = t$. (resp. $\lim_\alpha t\, e_\alpha = t$) ($t\epsilon S$). The net $(e_\alpha) \subseteq S$ is an approximate identity if it is a left and a right approximate identity.

Let S be a discrete semigroup. We denote by $\ell^1(S)$ the Banach space of all complex function $f: S \to \mathbb{C}$ having the form

$$f = \sum_{s\epsilon S} f(s)\delta_s ,$$

such that $\sum_{s\epsilon S}|f(s)| = \|f\|_1$ is finite, where δ_s is the point mass at $\{s\}$.For $f, g\epsilon\ell^1(S)$ we define the convolution product on $\ell^1(S)$ as fallow:

$$f * g(s) = \sum_{t_1 t_2 = s} f(t_1)g(t_2) \qquad (s\epsilon S),$$

with this product $\ell^1(S)$ becomes a Banach algebra and is called the semigroup algebra on S.

Remark 1.1. If $f\epsilon\ell^1(S)$ then $f = 0$ on S except at most on a countable subset of S. In other words, the set $A = \{s\epsilon S : \ f(s) \neq 0\}$ is at most countable. Since, if $A_n = \left\{s\epsilon S : \ |f(s)| \geq \frac{1}{n}\right\}$, $A = \bigcup_{n\epsilon N} A_n$. Set $\|f\|_1 = M$ and $n\epsilon N$ is fixed. Then we have

$$M = \sum_{s\epsilon S}|f(s)| \geq \sum_{s\epsilon A_n}|f(s)| \geq \sum_{s\epsilon A_n}\frac{1}{n} = \frac{1}{n}|A_n| ,$$

where $|A_n|$ is the cardinality of A_n . So $|A_n| \leq nM$. Hence A_n is a finite subset of S and thus A is at most countable.

Semigroup S_T

Let $T \in Mul_l(\boldsymbol{S})$. Then we define a new binary operation "$\circ$" on $\boldsymbol{S}$ as follow :

$$s \circ t = s\,T(t) \quad (s, t \epsilon \boldsymbol{S}).$$

The set S equipt with the new operation "$\circ$" is denoted by S_T and sometimes called "induced semigroup of S" . Now we have the following results.

Theorem 2.1. Let **S** be a Semigroup. Then (i) if $T \in Mul_l(\boldsymbol{S})$ then $\boldsymbol{S}_T$ is a semigroup . The converse is true if **S** is left cancellative and T is surjective .

(ii) If $\boldsymbol{S}_T$ is left cancellative and T is surjective, then $T^{-1} \in Mul_l(\boldsymbol{S})$.

(iii) If **S** is a topological semigroup and $\boldsymbol{S}_T$ has a left approximate identity then $T^{-1} \in Mul_l(\boldsymbol{S})$.

Proof. i) Let $T \epsilon Mul_l(\boldsymbol{S})$ and take r,s,t ϵ**S**. Then

$$r \circ (s \circ t) = r\,T(s \circ t) = r\,T(\,s\,T(t)\,) = r\,T(s)T(t) = (\,r\,T(s)\,)\,T(\,t)$$
$$= (r \circ s) \circ t$$

So, $\boldsymbol{S}_T$ is a semigroup.

Conversely, suppose that **S** is left cancellative and take r,s,t ϵ**S**. Since T is surjective, there exists uϵ**S** such that $T(u) = t$. Then

$$rT(st) = rT(sT(u)) = r \circ (s \circ u) = (r \circ s) \circ u = (rT(s))T(u)$$
$$= r(T(s)t).$$

By the left cancellativity of **S**, we have $T(st) = T(s)t \quad (r, s\epsilon\boldsymbol{S})$. So, T is a left multiplier.

ii) We must prove that T is injective. To do this end, take r,s,uϵ**S** and let $T(r) = T(s)$. Then $u \circ r = uT(r) = uT(s) = u \circ s$. So $r = s$, since $\boldsymbol{S}_T$ is left cancellative. Hence T^{-1} exists.

Now, we show that $T^{-1} \in Mul_l(\boldsymbol{S})$. Take $r, s\epsilon\boldsymbol{S}$. Then

$$T^{-1}(rs) = T^{-1}[TT^{-1}(r)s] = T^{-1}[T(T^{-1}(r)s)]$$
$$= (T^{-1}T)[T^{-1}(r)s] = T^{-1}(r)s\,.$$

iii) It is enough to show that T is injective. Take $r, s\epsilon\boldsymbol{S}$ and suppose that $T(r) = T(s)$. Then

$$r = \lim_\alpha e_\alpha \circ r = \lim_\alpha e_\alpha T(r) = \lim_\alpha e_\alpha T(s) = \lim_\alpha e_\alpha \circ s = s.$$

There are many properties that induced from **S to** semigroup $\mathbf{S}_\mathrm{T}$. But sometimes they are different.

Theorem2.2. Let **S** be a Hausdorff topological semigroup and $\mathrm{T}\epsilon\mathrm{Mul}_\mathrm{l}(\boldsymbol{S})$. If **S** is commutative then so is $\mathbf{S}_\mathrm{T}$. The converse is true if $\overline{\mathrm{T}(\boldsymbol{S})} = \mathbf{S}$.

Proof. Suppose **S** is commutative and take $\mathrm{r}, \mathrm{s} \in \mathbf{S}$. Then

$$\mathrm{r} \circ \mathrm{s} = \mathrm{r}\,\mathrm{T}(\mathrm{s}) = \mathrm{T}(\mathrm{s})\mathrm{r} = \mathrm{T}(\mathrm{sr}) = \mathrm{T}(\mathrm{rs}) = \mathrm{T}(\mathrm{r})\,\mathrm{s} = \mathrm{s}\mathrm{T}(\mathrm{r}) = \mathrm{s} \circ \mathrm{r}\,.$$

So, $\mathbf{S}_\mathrm{T}$ is commutative.

Conversely, Let $\mathbf{S}_\mathbf{T}$ be commutative and take $\mathrm{r}, \mathrm{s}\, \epsilon \mathbf{S}$. Then there exist nets (r_α) and (s_β) in **S** such that $\lim_\alpha \mathrm{T}(\mathrm{r}_\alpha) = \mathrm{r}$ and $\lim_\beta \mathrm{T}(\mathrm{s}_\beta) = \mathrm{s}$.

So, we have

$$\mathrm{rs} = \lim_\alpha \lim_\beta \mathbf{T}(\mathrm{r}_{\alpha} \circ \mathrm{s}_\beta) = \lim_\propto \lim_\beta \mathbf{T}(\mathrm{s}_{\beta} \circ \mathrm{r}_\alpha) = \lim_\alpha \lim_\beta \mathbf{T}(\mathrm{s}_\beta)\, \mathbf{T}(\mathrm{r}_\alpha) = \mathrm{s\,r}.$$

Thus **S** is commutative .

In the sequel, we investigate some relations between two semigroup **S** and $\mathbf{S}_\mathbf{T}$ according to the role of the left multiplier T.

Theorem 2.3. Let **S** be a semigroup and $\mathrm{T}\epsilon\mathrm{Mul}_\mathrm{l}(\boldsymbol{S})$ .Then

(i) If T is surjective and $\mathbf{S}_\mathbf{T}$ is an inverse semigroup then **S** is an inverse semigroup and $\mathrm{T}(\mathrm{s}^{-1}) = \mathrm{T}(\mathrm{s})^{-1}$ for all $\mathrm{s}\epsilon\boldsymbol{S}$.

(ii) If $\mathbf{S}_\mathrm{T}$ is an inverse semigroup and T is injective then $\mathrm{T}(\mathbf{S})$ is an inverse subsemigroup of **S**.

(iii) If T is bijective then $\mathbf{S}_\mathrm{T}$ is an inverse semigroup if and only if **S** is an inverse semigroup.

Proof. i) Suppose that $\mathbf{S}_\mathrm{T}$ is an inverse semigroup and T is surjective. Define the map $\varphi: \mathbf{S}_\mathrm{T} \to \mathbf{S}$ by $\varphi(\mathrm{s}) = \mathrm{T}(\mathrm{s})$. Take $\mathrm{r}, \mathrm{s}\epsilon\mathbf{S}$, then

$$\varphi(r\circ s) = T(r\circ s) = T(r)T(s) = \varphi(r)\varphi(s).$$

So, φ is an epimorphism from $\boldsymbol{S}_T$ onto **S**, since T is surjective. By theorem 5.1.4[**7**], **S** is an inverse semigroup and $T(s^{-1}) = T(s)^{-1}$ for all $s\epsilon\boldsymbol{S}$.

ii) Suppose that T is injective and $\boldsymbol{S}_T$ is an inverse semigroup. Evidently, $T(\boldsymbol{S})$ is a subsemigroup of $\boldsymbol{S}$. We show that it is an inverse semigroup. Take $s\epsilon T(\boldsymbol{S})$. There exists $t\epsilon\boldsymbol{S}$ such that $s = T(t)$. Also, there exists a unique element $u\epsilon\boldsymbol{S}$ such that $t = t\,_0\, u\,_0\, t$, since $\boldsymbol{S}_T$ is an inverse semigroup. Therefore, $T(t) = T(t)T(u)T(t)$, or $s = s\,_0\, T(u)\,_0\, s$. Of course, $T(u)$ is unique because $u\epsilon\boldsymbol{S}$ is unique and T is injective. Hence $T(\boldsymbol{S})$ is an inverse subsemigroup of $\boldsymbol{S}$.

iii) Suppose that T is bijective and let $\boldsymbol{S}_T$ be an inverse semigroup. Since T is injective and surjective, by (i) and (ii), $\boldsymbol{S} = T(\boldsymbol{S})$ is an inverse semigroup.

Conversely, suppose that $\boldsymbol{S}$ is an inverse semigroup. Since T is bijective , by theorem 2.1(ii), $T^{-1} \epsilon Mul_l(\boldsymbol{S})$. So φ^{-1}: $\boldsymbol{S} \longrightarrow \boldsymbol{S}_T$ defined by $\varphi^{-1}(s) = T^{-1}(s)$ is an epimorphism. Hence by (i) $\boldsymbol{S}_T$ is an inverse semigroup

We say that $T \in Mul_l(\boldsymbol{S})$ is an inner left multiplier if it has the form $T = L_s$ for some $s \epsilon \boldsymbol{S}$ where $L_s(t) = s\,t \qquad (t \epsilon \boldsymbol{S})$.

If $T \epsilon Mul_l(\boldsymbol{S})$ is inner, then each ideal of $\boldsymbol{S}$ is permanent under T; that is $T(\boldsymbol{I}) \subseteq \boldsymbol{I}$ for all ideal **I** of $\boldsymbol{S}$. It is easily to see that if $\boldsymbol{S}$ has an identity, then each $T \in Mul_l(\boldsymbol{S})$ is inner.

Let **S** be a semigroup. Then **S** is called semisimple if $\boldsymbol{I}^2 = \boldsymbol{I}$ for all ideal **I** of **S** (see [**9**], page 95 for more details).

Theorem 2.4. Let $\boldsymbol{S}$ be a semigroup whit an identity and $T \in Mul_l(\boldsymbol{S})$. If $\boldsymbol{S}_T$ is semisimple, then $\boldsymbol{S}$ is so**.** The converse is true if $\boldsymbol{S}_T$ is left cancellative and T is surjective.

Proof. Since $\boldsymbol{S}$ is unital there exists $\mu \epsilon \boldsymbol{S}$ such that $\mathrm{T} = L_\mu$. Suppose that $\boldsymbol{S}_T$ is semisimple and $\boldsymbol{I}$ is an ideal of $\boldsymbol{S}$. Then

$$\boldsymbol{I} \circ \boldsymbol{S} = \boldsymbol{I}\, T(\boldsymbol{S}) \subseteq \boldsymbol{I}\,\boldsymbol{S} \subseteq \boldsymbol{I}\,.$$

Similarly, $\boldsymbol{S} \circ \boldsymbol{I} \subseteq \boldsymbol{I}$. It follows that $\boldsymbol{I}$ is an ideal of $\boldsymbol{S}_T$. By the hypothesis $(\boldsymbol{I}_T)^2 = \boldsymbol{I} \circ \boldsymbol{I} = \boldsymbol{I}$**.** Now, take $r \epsilon \boldsymbol{I}$ then there are $s, t \epsilon \boldsymbol{I}$ such that

$$r = s \circ t = sT(t) = s(\mu t) \in \boldsymbol{I}^2.$$

So we show that $\boldsymbol{I}^2 = \boldsymbol{I}$ and hence $\boldsymbol{S}$ is semisimple.

Conversely, assume that $\boldsymbol{S}_T$ is left cancellative and $T \in Mul_l(\boldsymbol{S})$ is surjective then by theorem 2.1(ii), $T^{-1} \epsilon Mul_l(\boldsymbol{S})$. So, there exists $b \epsilon \boldsymbol{S}$ such that $T^{-1} = L_b$. Suppose that $\breve{\boldsymbol{S}} = \boldsymbol{S}_{T^{-1}}$. Then we have .

$$\boldsymbol{S} = \boldsymbol{S}_{TT^{-1}} = (\boldsymbol{S}_{\boldsymbol{T}})_{T^{-1}} = \breve{\boldsymbol{S}}_{T^{-1}}.$$

By hypothesis and above the proof, $\breve{\boldsymbol{S}} = \boldsymbol{S}_{T^{-1}}$ is semisimple.

Semigroup Algebra $\ell^1(\boldsymbol{S}_{\boldsymbol{T}})$

We say that a discrete semigroup $\boldsymbol{S}$ is amenable if there exists a positive linear functional on $\ell^\infty(\boldsymbol{S})$ called a mean such that $m(\mathbf{1}) = 1$ and $m(l_s f) = m(f)$, $m(r_s f) = m(f)$ for each $s \epsilon \boldsymbol{S}$, where $l_s f(t) = f(st)$ and $r_s f(t) = f(ts)$ for all $t \epsilon \boldsymbol{S}$. The definition of amenable group is similar to semigroup case. Refer to [**12**] for more details.

Let $\mathfrak{A}$ be a Banach algebra and let X be a Banach $\mathfrak{A}$ –bimodule. A derivation from $\mathfrak{A}$ to X is a linear map $D: \mathfrak{A} \longrightarrow X$ such that

$$D(ab) = D(a) \cdot b + a \cdot D(b) \qquad (a, b \in \mathfrak{A}).$$

A derivation D is inner if there exists $x \epsilon X$ such that

$$D(a) = a \cdot x - x \cdot a \qquad (a \in \mathfrak{A}).$$

The Banch algebra $\mathfrak{A}$ is amenable if every bounded derivation $D: \mathfrak{A} \longrightarrow X^*$ is inner for all Banach $\mathfrak{A}$ –bimodule X. Where X^* is the dual space of X. We say that the Banch algebra $\mathfrak{A}$ is weakly amenable if any bounded derivation D from $\mathfrak{A}$ to $\mathfrak{A}^*$ is inner. Fore more details see [12] , [16] .

If $\boldsymbol{S}$ is a commutative semigroup, by theorem 5.8 of [**8**] $\ell^1(\boldsymbol{S})$ is called semisimple if and only if for all $x, y \epsilon \boldsymbol{S}$, $x^2 = y^2 = xy$ implies $x = y$.

Theorem 3.1. Let $\boldsymbol{S}$ be a commutative semigroup and let $T \in Mul_l(\boldsymbol{S})$ be injective . Then $\ell^1(\boldsymbol{S})$ is semisimple if and only if $\ell^1(\boldsymbol{S_T})$ is semisimple.

Proof. Take $r, s \in \boldsymbol{S}$. Then $r^2 = s^2 = rs$ if and only if $T(r^2) = T(s^2) = T(r)T(s)$ or equivalently $r\,{}_0\, r = s\,{}_0\, s = r\,{}_0\, s$, because T is injective**.** So, by theorem 5.8 [**8**], $\ell^1(\boldsymbol{S})$ is semisimple if and only if $\ell^1(\boldsymbol{S}_T)$ is semisimple.

Theorem 3.2. Let $\boldsymbol{S}$ be a discrete semigroup and $T \in Mul_l(\boldsymbol{S})$. Then

(i)The left multiplier T has an extension $\tilde{T} \epsilon Mul_l\left(\ell^1(\boldsymbol{S})\right)$ with the norm decreasing.

(ii) The left multiplier T is injective if and only if so is $\tilde{T}$.

(iii) If T is injective then $\tilde{T}$ is an isometry and also $\ell^1(\boldsymbol{S}_T)$ and $\left(\ell^1(\boldsymbol{S})\right)_T$ are isomorphic.

Proof. (i) An arbitrary element $f \epsilon\, \ell^1(\boldsymbol{S})$ is of the form $f: \boldsymbol{S} \to \mathbb{C}$ such that $f(x) = 0$ except at the most countable subset A of **S**. If A is a finite subset of **S** then $f = \sum_{k=1}^{n} f(x_k)\, \delta_{x_k}$for some fixed $n \epsilon \mathbb{N}$. So in general we have

$$f = \sum_{x \epsilon S} f(x) \delta_x = \sum_{x \epsilon A} f(x) \delta_x = \sum_{k=1}^{\infty} f(x_k)\, \delta_{x_k}.$$

Now, for each $n \epsilon \boldsymbol{N}$**,** let $f_n = \sum_{k=1}^{n} f(x_k) \delta_{x_k}$ and define $\tilde{T}: \ell^1(\boldsymbol{S}) \longrightarrow \ell^1(\boldsymbol{S})$ by

$$\tilde{T}(\delta_x) = \delta_{T(x)} \qquad (x \epsilon \boldsymbol{S}) ,$$

$$\tilde{T}(f_n) = \sum_{k=1}^{\mathrm{n}} f(x_k)\, \tilde{T}\left(\delta_{x_k}\right) = \breve{f}_n \ .$$

For each $m, n \epsilon \boldsymbol{N}$ where $n \geq m$, we have

$$\left\|\tilde{T}(f_n) - \tilde{T}(f_m)\right\|_1 = \left\|\breve{f}_n - \breve{f}_m\right\|_1 = \left\|\sum_{k=m}^{k=n} f(x_k)\ \tilde{T}\left(\delta_{x_k}\right)\right\| = \left\|\sum_{k=m}^{k=n} f(x_k)\, \delta_{T(x_k)} \right\|$$

$$\leq \sum_{k=m}^{k=n} |f(x_k)| = \|f_n - f_m\|_1 .$$

So $\{\tilde{T}(f_n)\}_n$ is a Cauchy sequence and it is convergent. Now, we define $\tilde{T}(f) = \lim_n \tilde{f_n}$. Then the definition is well defined. Hence

$$\tilde{T}(f) = \sum_{k=1}^{\infty} f(x_k)\,\tilde{T}\left(\delta_{x_k}\right) = \tilde{f}\ ,$$

also

$$\left\|\tilde{f}\right\|_1 \leq \sum_{x_k \epsilon A} |f(x_k)| = \|f\|_1 \ \text{ or } \ \left\|\tilde{T}(f)\right\|_1 \leq \|f\|_1.$$

It shows that $\tilde{T}$ is norm decreasing.

In the following, we extend $\tilde{T}$ by linearity. Let $f, g \epsilon\ \ell^1(\boldsymbol{S})$. Then there are two at most countable sub set A, B of **S** such that

$$f = \sum_{x\epsilon A} f(x)\delta_x\ ,\ \ g = \sum_{x\epsilon B} g(x)\delta_x\ .$$

Suppose that $D = A \cup B$. So we have $f + g = \sum_{x\epsilon D} (f(x) + g(x))\delta_x$.

Then, it follows that

$$\tilde{T}(f+g) = \widetilde{f+g} = \sum_{x\epsilon D}(f(x)+g(x))\tilde{T}(\delta_x) = \sum_{x\epsilon A} f(x)\,\tilde{T}(\delta_x) + \sum_{x\epsilon B} g(x)\,\tilde{T}(\delta_x)$$
$$= \tilde{f} + \tilde{g}\ .$$

Also, if $\alpha\epsilon\mathbb{C}$,we have

$$\tilde{T}(\alpha f) = \widetilde{\alpha f} = \sum_{x\epsilon A} \alpha f(x)\tilde{T}(\delta_x) = \alpha \sum_{x\epsilon A} f(s)\tilde{T}(\delta_x) = \alpha\tilde{T}(f).$$

Therefore, $\tilde{T}$ is a bounded linear isometry.

Now, we prove that $\tilde{T}\epsilon Mul_l\left(\ell^1(\boldsymbol{S})\right)$. Take $x, y\epsilon\boldsymbol{S}$. Then

$$\tilde{T}\left(\delta_x * \delta_y\right) = \tilde{T}\left(\delta_{xy}\right) = \delta_{T(xy)} = \delta_{T(x)y} = \delta_{T(x)} * \delta_y = \tilde{T}(\delta_x) * \delta_y.$$

Let $y\epsilon\boldsymbol{S}$ be fixed and $f, g\epsilon\ \ell^1(\boldsymbol{S})$. Then

$$\tilde{T}\left(f * \delta_y\right) = \tilde{T}\Big(\sum_{x\epsilon A} f(x)\,\delta_{xy}\Big) = \sum_{x\epsilon A} f(x)\tilde{T}\left(\delta_{xy}\right)$$
$$= \left(\sum_{x\epsilon A} \tilde{T}\ (\delta_x)\right) * \delta_y = \tilde{f} * \delta_y = \tilde{T}(f) * \delta_y.$$

In the general case, we have

$$\tilde{T}(f * g) = \tilde{T}\Big(\sum_{x\epsilon A} f(x)\left(\sum_{y\epsilon B} g(y)\right)\delta_{xy}\Big) = \sum_{x\epsilon A} f(x) \sum_{y\epsilon B}\ g(y)\tilde{T}(\delta_x) * \delta_y$$
$$= \sum_{x\epsilon A} f(x)\tilde{T}(\delta_x) * \sum_{y\epsilon B}\ g(y)\,\delta_y = \tilde{T}(f) * g\ .$$

This shows that $\tilde{T}$ is a multiplier on $\ell^1(\boldsymbol{S})$.

(ii) Let T be injective. Take $x, y\epsilon\boldsymbol{S}$ and suppose that $\tilde{T}(\delta_x) = \tilde{T}\left(\delta_y\right)$. Then $\delta_{T(x)} = \tilde{T}(\delta_x) = \tilde{T}\left(\delta_y\right) = \delta_{T(y)}$.

Therefore, $T(x) = T(y)$. Since T is injective, we have $x = y$. It follows that $\delta_x = \delta_y$, consequently $\tilde{T}$ is injective.

Conversely, the same argument shows that the converse holds.

(iii) Let T be injective and $f\epsilon\ell^1(\boldsymbol{S})$. Then there exists at most a countable subset $A \subseteq \boldsymbol{S}$ such that

$$f = \sum_{x\in A} f(x)\delta_x$$

Since A and $T(A)$ have the same cardinal number, $\|\tilde{T}(f)\| = \|\sum_{x\epsilon A} f(x)\,\delta_x\| = \sum_{x\epsilon A}|f(x)| = \|f\|_1$, so $\tilde{T}$ is an isometry.

Now, we can define a new multiplication " $\boxed{*}$ " on $\ell^1(\boldsymbol{S})$ as follow

$$f \boxed{*}\, g = f * \tilde{T}g \qquad (f, g\epsilon \ell^1(\boldsymbol{S}))\,.$$

By a similar argument in theorem1.31 [**10**], $\ell^1(\boldsymbol{S})$ with the new product is a Banach algebra that is denoted it by $\ell^1(\boldsymbol{S})_{\tilde{T}}$. We define the map $\Psi\colon \ell^1(\boldsymbol{S}_T) \longrightarrow \ell^1(\boldsymbol{S})_{\tilde{T}}$, by

$$\Psi(\delta_x) = \delta_x \qquad (\ x\epsilon\boldsymbol{S}\ ).$$

Take $x, y\ \epsilon\boldsymbol{S}$. Then

$$\begin{aligned}\Psi(\delta_x * \delta_y) &= \Psi(\delta_{x\circ y}) = \delta_{xT(y)} = \delta_x * \delta_{T(y)}\\ &= \delta_x * \tilde{T}(\delta_y) = \delta_x \boxed{*}\, \delta_y\\ &= \Psi(\delta_x) \boxed{*}\ \Psi(\delta_y).\end{aligned}$$

So, in general case, we have

$$\Psi(f * g) = \Psi\,(f) \boxed{*}\ \Psi(g) \qquad (f, g\epsilon\ell^1(\boldsymbol{S}))\,.$$

Thus, Ψ is an isomorphism. Therefore $\ell^1(\ \boldsymbol{S_T})$ and $\ell^1(\boldsymbol{S})_{\tilde{T}}$ are isomorphic

Theorem 3.3. Let **S** be a semigroup and $T \in Mul_l(\boldsymbol{S})$ be bijective. Then $\ell^1(\boldsymbol{S})$ is amenable if and only if $\ell^1(\ \boldsymbol{S}_T)$ is amenable.

Proof. By theorem 3.2, we have $\ell^1(\ \boldsymbol{S_T}) \cong \ell^1(\boldsymbol{S})_{\tilde{T}}$. Suppose that $\ell^1(\ \boldsymbol{S}_T)$ is amenable and define $\varphi\colon \ell^1(\boldsymbol{S})_{\tilde{T}} \longrightarrow \ell^1(\boldsymbol{S})$ by $\varphi(f) = \tilde{T}(f)$. Take $x, y\epsilon\boldsymbol{S}$. Then

$$\begin{aligned}\varphi(\delta_x \boxed{*}\, \delta_y) &= \tilde{T}(\delta_x \boxed{*}\, \delta_y) = \tilde{T}(\delta_{xT(y)}) = \tilde{T}(\delta_x * \delta_{T(y)}) = \tilde{T}(\delta_x) *\ \delta_{T(y)}\\ &=\ \tilde{T}(\delta_x) * \tilde{T}(\delta_y) = \varphi(\delta_x) * \varphi(\delta_y)\,.\end{aligned}$$

Now, by induction and continuity of $\tilde{T}$, we have

$$\varphi(f \boxed{*}\, g) = \varphi(f) * \varphi(g)\,.$$

If T is bijective, $\tilde{T}$ is bijective. Therefore φ is an epimorphism of $\ell^1(\ \boldsymbol{S_T})$ onto $\ell^1(\boldsymbol{S}\ )$.

Hence , by proposition 2.3.1 [**16**] $\ell^1(\boldsymbol{S}\ )$ is amenable.

Conversely, suppose that $\ell^1(\boldsymbol{S}\ )$ is amenable. Since T is bijective, $\tilde{T}$ is bijective. Therefore $\tilde{T}^{-1}$exists. Now define $\theta\colon \ell^1(\boldsymbol{S}) \longrightarrow \ell^1(\ \boldsymbol{S_T})\ [\cong \ell^1(\boldsymbol{S})_{\tilde{T}}]$ by $\theta(f) = \tilde{T}^{-1}\,(f)$.

Take $x, y\ \epsilon\boldsymbol{S}$ .Then

$$\begin{aligned}\theta(\delta_x * \delta_y) &= \tilde{T}^{-1}\,(\delta_{xy}) = \tilde{T}^{-1}\,(\delta_x)\tilde{T}\tilde{T}^{-1}\,(\delta_y) = \tilde{T}^{-1}\,(\delta_x) \boxed{*}\, \tilde{T}^{-1}\,(\delta_y)\\ &= \theta(\delta_x)\ \boxed{*}\, \theta(\delta_y)\,.\end{aligned}$$

Similarly θ is an epimorphism from $\ell^1(\boldsymbol{S})$ onto $\ell^1(\boldsymbol{S}_T)$. By proposition 2.3.1 [**16**] $\ell^1(\boldsymbol{S}_T)$ is amenable.

Note that, in general, it is not known when $\ell^1(\boldsymbol{S})$ is weakly amenable. For more detials see [**2**].

Theorem3.4. Let **S** be a semigroup and $T \in Mul_l(\boldsymbol{S})$ be bijective . Then, if **S** is completely regular then $\ell^1(\boldsymbol{S}_T)$ is weakly amenable.

Proof. It is enough to prove that $\boldsymbol{S}_T$ is completely regular, then by theorem 3.6 [**2**], $\ell^1(\boldsymbol{S_T})$ can be weakly amenable. Take $s\epsilon\boldsymbol{S}$. Then there exists $r\epsilon\boldsymbol{S}$ such that $T(s) = T(s)T(r)T(s)$, $T(r)T(s) = T(s)T(r)$, since T is bijective and **S** $= T(\boldsymbol{S})$ is completely regular. So we have $T(s) = T(s \circ r \circ s)$ and $T(r \circ s) = T(s \circ r)$. Hence $s = s \circ r \circ s$ and $r \circ s = s \circ r$ for some $r\epsilon\boldsymbol{S}$, since T is injective. Therefore $\boldsymbol{S}_T$ is completely regular.

Corollary.3.5. Suppose that **S** is a commutative completely regular semigroup and $T \in Mul_l(\boldsymbol{S})$ is injective. Then $\ell^1(T(\boldsymbol{S})_T)$ is weakly amenable.

Proof. [2, theorem 3.6] $\ell^1(\boldsymbol{S})$ is weakly amenable. Define $\varphi: \boldsymbol{S} \longrightarrow \ell^1(\boldsymbol{S})_T$ by

$$\varphi(s) = T^{-1}(s) \qquad (s\epsilon\boldsymbol{S}).$$

We show that φ is a homomorphism . Take $s\epsilon\boldsymbol{S}$, then we have

$$\varphi(rs) = T^{-1}(rs) = T^{-1}(r)s = T^{-1}(r) \circ (T^{-1}s) .$$

So φ is a homomorphism. Then by proposition 2.1[**7**], $\ell^1(T(\boldsymbol{S})_T)$ is weakly amenable. In the case that $\boldsymbol{S}$ is a group, it is easy to see that the amenability of $\boldsymbol{S}$ implies the amenability of $\ell^1(\boldsymbol{S}_T)$. Indeed, when $\boldsymbol{S}$ is a group, by theorem 2.1, $\boldsymbol{S}_T$ is a semigroup and one can easily prove that S_T is also a group. On the other hand, $Mul_l(\boldsymbol{S}) \cong \boldsymbol{S}$ because $\boldsymbol{S}$ is a unital semigroup, so each $T\epsilon Mul_l(\boldsymbol{S})$ is inner and of the form $T = L_s$ for some $s\epsilon\boldsymbol{S}$. Also $T^{-1} = L_{a^{-1}}$ exists, since $\boldsymbol{S}$ is a group. Then the map $\theta: \boldsymbol{S}_T \to \boldsymbol{S}$ defined by $\theta(s) = T(s)$ is an isomorphism; that is $\boldsymbol{S} \cong \boldsymbol{S}_T$. Thus we have the following result:

Corollary 3.6. Let $\boldsymbol{S}$ be a cancellative regular discrete semigroup. Then $\ell^1(\boldsymbol{S})$ is amenable if and only if $\ell^1(\boldsymbol{S}_T)$ is amenable.

Proof. By [**9**, Exercise 2.6.11] $\boldsymbol{S}$ is a group. So the assertion holds by [15, theorem 2.1.8]

Examples

In this section we present some examples which either comments on our results or indicate necessary condition in our theorems.

4.1. There are semigroups $\boldsymbol{S}$ and $T \in Mul_l(\boldsymbol{S})$ such that the background semigroups $\boldsymbol{S}$ are not commutative but their induced semigroups $\boldsymbol{S}_T$ are commutative.

This example shows that the condition $\overline{T(\boldsymbol{S})} = \boldsymbol{S}$, in theorem 2.2, can not be omitted.

Let $\boldsymbol{S}$ be the set {a, b, c, d, e} with operation table given by

.	a	b	c	d	e
a	a	a	a	d	d
b	a	b	c	d	d
c	a	c	b	d	d
d	d	d	d	a	a
e	d	e	e	a	a

Clearly$(\boldsymbol{S}, .)$ is a non-commutative semigroup. Now, put $T = L_a$ where $L_a(x) = ax$ for all $x \varepsilon \boldsymbol{S}$. One can get easily the operation table of $\boldsymbol{S}_T$ as fallow:

∘	a	b	c	d	e
a	a	a	a	d	d
b	a	a	a	d	d
c	a	a	a	d	d
d	d	d	d	a	a
e	d	d	d	a	a

The operation table shows that the induced semigroup $\boldsymbol{S}_T$ is commutative and $T(\boldsymbol{S}) \neq \boldsymbol{S}$. Also the other induced semigroup $\boldsymbol{S}_T$ is commutative for $T = L_d$ analogously.

Now we present some important theorems from[**14**] that we need in the following examples:

Theorem 4. 2. Let $\boldsymbol{S}$ be a semigroup. Suppose that $\ell^1(\boldsymbol{S})$ is amenable. Then

(i) $\boldsymbol{S}$ is amenable

(ii) $\boldsymbol{S}$ is regular.

(iii) $E(\boldsymbol{S})$ is finite.

(iv) $\ell^1(\boldsymbol{S})$has an identity.

Proof. (i) That is lemma 3 in [**5**].

(ii) and (iii) See theorem 2 in[**6**].

(iv) That is corollary 10.6 in[**4**].

Theorem 4.3. Let $\boldsymbol{S}$ be a finite semigroup. Then the following statements are equivalent:

(i) $\ell^1(\boldsymbol{S})$ is amenable.

(ii) $\boldsymbol{S}$ is regular and $\ell^1(\boldsymbol{S})$ is nuital.

(ii)) $\boldsymbol{S}$ is regular and $\ell^1(\boldsymbol{S})$ is semisimple.

Proof. Refer to [**3**].

4.4. There are semigroups $\boldsymbol{S}$ and $T \in Mul_l(\boldsymbol{S})$ such that $\boldsymbol{S}$ and $\ell^1(\boldsymbol{S})$ are amenable but $\boldsymbol{S}_T$ is not regular and also, $\ell^1(\boldsymbol{S}_T)$ is not amenable.

This example shows that two semigroup algebras $\ell^1(\boldsymbol{S})$ and $\ell^1(\boldsymbol{S}_T)$ can be different in some properties. Also, it notifies that the bijectivity of T in the theorem 3.3 is essential. Put $\mathbf{S} = \{x_0, x_1, x_2, \dots, x_n\}$ with the operation $x_i x_j = x_{Max\{i,j\}}$ $(0 \le i, j \le n,$ $n \ge 2$).

Then $\mathbf{S}$ is a semigroup. Since

$$Max\{i, Max\{j,k\}\} = Max\{Max\{i,j\},k\} = Max\{i,j,k\}.$$

We denote it by $\boldsymbol{S}_\vee$. This semigroup is commutative. So by (0.18) in [**12**], it is amenable. $\boldsymbol{S}_\vee$ is a unital semigroup and has a zero; indeed, $e_s = x_0$ and $o_s = x_n$. Also, it is a regular semigroup and $Mul(\boldsymbol{S}_\vee) \cong \boldsymbol{S}_\vee$ because $\boldsymbol{S}_\vee$ has an identity.

Evidently, $\boldsymbol{S}_\vee$ is regular since each $s \in \boldsymbol{S}_\vee$ is idempotent. The semigroup algebra $\ell^1(\boldsymbol{S}_\vee)$ is a unital algebra because $\boldsymbol{S}_\vee$ has an identity. So by theorem 4.3 (ii) $\ell^1(\boldsymbol{S}_\vee)$ is amenable.

Now, take $T = L_{x_k}$ for a fixed $x_k \epsilon \boldsymbol{S}$ where $k \ge 1$. By theorem 2.2, $(\boldsymbol{S}_\vee)_T$ is commutative so is amenable. We show that T is neither injective and nor surjective.

Take $x_i \epsilon \boldsymbol{S}_\vee$, then $Tx_i = x_k x_i = x_{max\{k,i\}}$. So

$$\mathrm{T}(\boldsymbol{S}_\vee) = \{x_k, x_{k+1}, \dots, x_n\} \ne \boldsymbol{S}_\vee.$$

Hence, T is not surjective.

Again, take distinct elements x_i, x_j in $\boldsymbol{S}_\vee$ for some $i, j < k$ such that $T(x_i) = T(x_j)$. Then we have $x_{max\{k,i\}} = x_{max\{k,j\}}$ but $x_i \ne x_j$. So T is not injective.

We prove that $(\boldsymbol{S}_\vee)_T$ is not regular. If $(\boldsymbol{S}_\vee)_T$ is regular, then for $x_{k-1} \epsilon \boldsymbol{S}_\vee$ there exists an element $x_j \epsilon \boldsymbol{S}_\vee$ such that

$$x_{k-1} = x_{k-1} \,{}_0\, x_j \,{}_0\, x_{k-1} = x_{Max\{k,j\}} .$$

That implies that $max\{k,j\} = k-1$; which is impossible. Consequently, by theorem 4.2 (ii) or 4.3 (ii), $\ell^1((\boldsymbol{S}_\vee)_T)$ is not amenable.

Also, the inequality $\boldsymbol{S}_\vee \circ \boldsymbol{S}_\vee = \{x_k, x_{k+1}, \dots, x_n\} \neq \boldsymbol{S}_\vee$ shows that $\ell^1((\boldsymbol{S}_\vee)_T)$ is not weakly amenable. In the next example we show that in the theorem 3.2 (iii) the condition "injectivity of " can not be omitted.

4.5 There are a semigroup $\boldsymbol{S}$ and $T \in Mul_l(\boldsymbol{S})$ such that $T \in Mul_l(\boldsymbol{S})$is not injecyive and the corresponding $\tilde{T} \in Mul_l\left(\ell^1(\boldsymbol{S}_T)\right)$ is not an isometry.

Suppose that $\boldsymbol{S}_\vee$ is a semigroup as in example 4.4 and $T = L_{x_k}$for some fixed $1 < k < n$. If $f \epsilon \ell^1(\boldsymbol{S}_\vee)$ then $f = \sum_{i=0}^n f(x_i)\delta_{x_i}$ and also $\tilde{T}(f) = \sum_{i=0}^n f(x_i)\delta_{T(x_i)}$. But

$$T(x_i) = \begin{cases} x_i & k < i \leq n \\ x_k & 0 \leq i \leq k \end{cases},$$

so

$$\tilde{T}(f) = \left(\sum_{i=0}^k f(x_i)\right)\delta_{x_k} + \sum_{i=k+1}^n f(x_i)\delta_{T(x_i)}.$$

Hence

$$\|\tilde{T}(f)\| = \left|\sum_{i=0}^k f(x_i)\right| + \sum_{i=k+1}^n |f(x_i)|$$

$$\leq \sum_{i=0}^k |f(x_i)| + \sum_{i=k+1}^n |f(x_i)| = \|f\|_1,$$

It shows that $\tilde{T}$ is not an isometry.

4.6. There are semigroups $\boldsymbol{S}$ and $T \in Mul_l(\boldsymbol{S})$ such that $\ell^1(\boldsymbol{S})$ is semisimple. But $\ell^1(\boldsymbol{S}_T)$ is not semisimple. This example remind that, in theorem 3.1 the multiplier T must be injective.

Let **S** be a set $\{x_0, x_1, \dots, x_n\}$ where $n \epsilon \boldsymbol{N}$ and $n \geq 3$ is fixed. by operation given by $xy = x_{min\{i,j\}}$, **S** is a commutative semigroup. Since

$$min\{i, min\{j,k\}\} = min\{min\{i,j\}, k\} = min\{i,j,k\} \qquad (i,j,k \epsilon \boldsymbol{N}).$$

We denote it briefly by $\boldsymbol{S}_\wedge$ For each $x, y \epsilon \boldsymbol{S}$ the equality $x^2 = y^2 = xy$ implies $x = y$. So by Theorem 5.8 [**8**] $\ell^1(\boldsymbol{S}_\wedge)$ is semisimple.

Now, let $T = L_{x_k}$ for a fixed $1 \leq k < n-1$. It is easy to see that $T(x_k) = T(x_n)$ but $x_k \neq x_n$. So the multiplier T is not injective.

We show that neither $\boldsymbol{S}_\wedge$ nor $\ell^1(\boldsymbol{S}_\wedge)_T$ is semisimple.

Each ideal of $\boldsymbol{S}$ is of the form

$$\boldsymbol{I}_m = \{x_0, x_1, \dots, x_m\} \qquad (m \leq n).$$

We claim that $\boldsymbol{S_T}$ is not semisimple. Since for each $m \epsilon \boldsymbol{N}$ we have

$$\boldsymbol{I}_m \circ \boldsymbol{I}_m = \begin{cases} \boldsymbol{I}_m & m \le k \\ \boldsymbol{I}_k & m > k \end{cases} .$$

On the other hand, for each $x_i, x_j \epsilon \boldsymbol{S}$ where $i \neq j$ and $i, j > k$, we have $x_i \circ x_i = x_j \circ x_j = x_i \circ x_j = x_k$, while $x_i \neq x_j$. Thus, Theorem 5.8 [**8**] shows that $\ell^1(\mathbf{S}_\wedge)_T$ is not semisimple .

Acknowledgment

The authors express their thanks to Professor A. R. Medghalchi for his valuable comments. Also we thank him for some corrections of this paper.

Reference

1. Birtel F. T., "Banach algebras of multipliers", Duke Math. J. 28 (1961) 203-211.
2. Blackmore T. D., "Weak amenability of discrete semigroup algebras", Semigroup Forum 55 (1997) 169-205.
3. Esslamzadeh G. H., "Ideal and representations of certain semigroup algebras", Semigroup Forum 69(2004) 51-62.
4. Dales H. G., Lau A. T.-M., Strauss D., "Banach algebras on semigroups and on their compactifications", Memoirs American Math. Soc. 205 (2010) 1-197.
5. Duncan J., Namioka I., "Amenability of inverse semigroup and their semigroup algebras", Proc, Royal. Edinburgh. Section A 80 (1978) 309-321.
6. Duncan J., Paterson A. L. T., "Amenability for discrete convolution semigroup algebras", Math. Scand. 66 (1990) 141-146.
7. Gronbaek N., "A characterization of weak amenability", Studia Math. 97 (1987) 149-162.
8. Hewitt E., Zuckerman H. S., "The ℓ^1 −algebra of a commutative semigroup", Trans. Amer. Math. Soc, 83 (1956) 70-97.
9. Howie J. M., "Fundamentals of Semigroup Theory", Claredon Press Oxford (2003).
10. Laali J., "The multipliers related products in banach alegebras", Quaestiones Mathematicae, 37 (2014) 1-17.
11. Larsen R., "An Introduction to the Theory of Multipliers", Springer-verlag, New York (1971).
12. Paterson A. L. T., "Amenability", American Mathematical Society (1988).

13. Medghalchi A. R., "Hypergroups, weighted hypergroups and modification by multipliers", Ph.D Thesis, University of Sheffild (1982).

14. Mewomo O. T., "Notions of amenability on semigroup algebras", J. Semigroup Theory Appl. 2013:8. ISSN 2051-2937 (2013) 1-18.

15. Mohammadi S. M., Laali J., "The Relationship between two involutive semigroups S and S_T is defined by a left multiplier T. Journal of Function Space", Article ID 851237 (2014).

16. Runde, "Lectures On Amenability", Springer-Verlag Berlin (2002).