

Exponential Stability of Linear Systems with Multiple Time Delays

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Abstract

In this paper, a class of linear systems with multiple time delays is studied. The problem of exponential stability of time-delay systems has been investigated by using Lyapunov functional method. We will convert the system of multiple time delays into a single time delay system and show that if the old system is stable then the new one is so. Then we investigate the stability of converted new system by using matrix decomposition and linear matrix inequality (LMI) technique. Some numerical examples are given to illustrate the efficiency of our method.

Keywords: Exponential stability, Multiple time-delays system, Matrix decomposition, Lyapunov-Krasovskii functional, LMI

1. Introduction

Time delays are often encountered due to measurement and computational delays, transmission and transport lags. Frequently, it is a source of the generation of oscillation and a source of instability in many biological and engineering systems. Therefore, stability testing and stabilization of time-delay systems is a problem of practical and theoretical interest. Over the past years, considerable efforts have been devoted to the analysis of the stability or stabilization for systems with time delays and many different methods have been proposed to deal with the stability or stabilization problem, see for example, [1,8].

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There are many different methods given to deal with the exponential stability problem. Among the well-known Lyapunov stability methods, the Lyapunov functional method is a powerful tool for studying system stability, even for linear systems. Numerous works deal with the stability of linear systems using the Lyapunov function method [1]. Time-delay stability conditions for time-invariant systems were formulated in both algebraic Riccati equation and (LMI) [1,3,4,5]. Some stability conditions are directly obtained from the eigenvalues or the robust stability conditions in terms of matrix measures and norm criteria [6], or in terms of the solution of some algebraic Riccati equations [7].

We provide an overview of two application problems, inverted pendulum systems and angiogenesis models with discrete delays.

Example 1: (Inverted pendulum) Consider the inverted pendulum on a cart such as in [12]. The physical structure is shown in Figure. 1 (which is taken from [12]). In this system, a pendulum is attached to the side of a cart by means of a pivot which allows the pendulum to show in the xy -plane. A force $F(t)$ is applied to the cart in the x direction, with the purpose of keeping the pendulum balanced upright. We assume that the pendulum may be modeled as a thin rod. Then, applying Newton's second law to the linear and angular displacement, we arrive at the equations of motion for the system

$$(M + m)\ddot{x} + \varepsilon \dot{x} + m\ddot{\theta}l \cos \theta - ml\dot{\theta}^2 \sin \theta = F(t), \quad (1)$$

$$ml\ddot{x} \cos \theta + \frac{4}{3}ml^2\ddot{\theta} - mgl \sin \theta = 0, \quad (2)$$

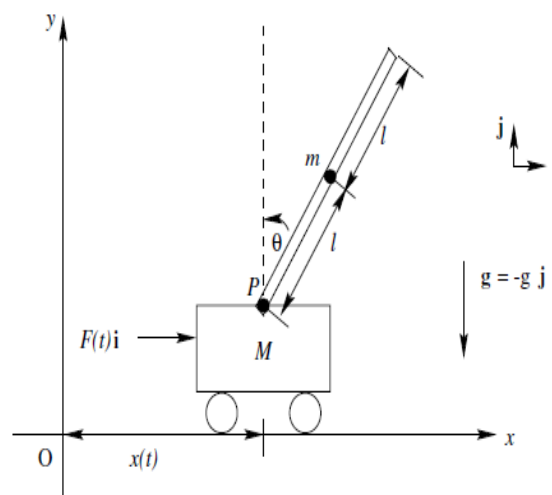


Figure1. A typical inverted pendulum fixed on a moving cart

A more convenient and form of the equations is found by solving for \ddot{x} and $\ddot{\theta}$ from

Table1. Notation of Example 1

$x(t)$	Displacement of the center of mass of the cart from point O
$\theta(t)$	Angle the pendulum makes with the top vertical
M	Mass of the cart
m	Mass of the pendulum
L	Length of the pendulum
l	Distance from pivot to the center of mass of the pendulum $l = \frac{L}{2}$
P	Pivot point of the pendulum
$F(t)$	Force applied to the cart

equations (1) and (2) introducing the variables

$$y = (y_1, y_2, y_3, y_4)^T = (x, \theta, \dot{x}, \dot{\theta})^T, \quad (3)$$

with the delay be h , then from [12] the system can be rewritten in the form of

$$\dot{y}(t) = Ay(t) + By(t-h), \quad (4)$$

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & c_1 & c_2 & 0 \\ 0 & c_3 & c_4 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ e_1 & e_2 & e_3 & e_4 \\ e_5 & e_6 & e_7 & e_8 \end{pmatrix},$$

where

$$c_1 = \frac{-3mg}{(m+4M)}, \quad c_2 = \frac{-3(\beta+\varepsilon)}{(m+4M)}, \quad c_3 = \frac{3(m+M)g}{l(m+4M)}, \quad c_4 = \frac{3(\beta+\varepsilon)}{l(m+4M)},$$

$$e_i = \frac{4d_i}{(m+4M)}, \quad i=1, \dots, 4, \quad e_i = \frac{-3d_i}{l(m+4M)}, \quad i=5, \dots, 8.$$

β, ε are constant numbers.

Example 2: (Angiogenesis models with discrete delays) consider a family of models with delays describing the process of angiogenesis that is a physiological process involving the growth of new blood vessels from pre-existing ones. This family includes the well known models of tumor angiogenesis proposed by Hahnfeldt et al. and Ergun et al. and is based on the Gompertz type of the tumor growth [13].

We consider the following system of differential equations with two discrete time delays describing the process of tumor angiogenesis

$$\frac{d}{dt} p(t) = -rp(t) \ln\left(\frac{p(t-h_1)}{q(t-h_1)}\right), \quad (5)$$

$$\frac{d}{dt} q(t) = q(t) \left[b \left(\frac{p(t-h_2)}{q(t-h_2)} \right)^\alpha - ap^{\frac{2}{3}}(t) - \mu \right]. \quad (6)$$

Where $p(t)$ and $q(t)$ represent tumor volume and the density of endothelial cells at time t , respectively. We follow the Hahnfeldt et al. [13] assumption that the tumor volume dynamics, described by Eq. (5), is governed by the Gompertz-type equation for which the carrying capacity of tumor cells is proportional to the density of endothelial cells. Eq (6) described the dynamics of endothelial cells which depends on the stimulation process initiated by poorly nourished tumor cells, vessels lost due to the accumulation of the inhibiting factors secreted by tumor cells within the tumor [13]. then from [13] the system can be rewritten in form of

$$\frac{d}{dt} x(t) = -rx(t-h_1) - y(t-h_1), \quad (7)$$

$$\frac{d}{dt} y(t) = be^{\alpha(x(t-h_2)-y(t-h_2))} - (b-\mu)e^{\frac{2}{3}x(t)} - \mu. \quad (8)$$

To close the system (7)-(8) we define an initial condition, that is a continuous function $\phi: [-h, 0] \rightarrow \mathbb{R}^2$ where $h = \max\{h_i : i = 1, 2\}$.

The linearization of Eq (7)-(8) then from [13] the system can be rewritten in form of

$$\dot{z}(t) = A_0 z(t) + \sum_{i=1}^2 A_i z(t-h_i), \quad (9)$$

where

$$z(t) = (x(t), y(t)).$$

In this paper we will convert a system of multiple time delays into a single time delay system and show that if the old system is stable the new one is so. Then we will consider the exponential stability problem for linear systems with single delay, based on the Lyapunov-Kradovskii functional approach and (LMI). Numerical examples are given to illustrate the efficiency of our method.

2. Preliminaries

In this section we state definitions and the required mathematical background, as well as the class of systems considered throughout the paper. First we introduce some

elementary definitions and lemmas which play an important role in the proof of the main results.

We consider the following linear systems with multiple time delays, described by

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^m A_i x(t - h_i), \quad t \in \mathbb{R}^+, \quad (10)$$

$$x(t) = \phi(t), \quad t \in [-h, 0],$$

where $h = \max\{h_i : i = 1, 2, \dots, m\}$, $A_i \in M^{n \times n}$, $i = 0, 1, 2, \dots, m$, are constant

and $\phi(t) \in C([-h, 0], \mathbb{R}^n)$, ($C([-h, 0], \mathbb{R}^n)$ denotes the Banach space of all piecewise-continuous vector functions mapping $[-h, 0]$ into \mathbb{R}^n).

Definition 1 [8] *The system (10) is said to be exponential stability with decay rate α , if there is a function $\eta: \mathbb{R}^n \rightarrow \mathbb{R}^n$*

such that for each $\phi(t) \in C([-h, 0], \mathbb{R}^n)$, the solution $x(t, \phi)$ of the system satisfies

$$\|x(t, \phi)\| \leq \eta(\|\phi\|) e^{-\alpha t}, \quad \text{for all } t \in \mathbb{R}^+.$$

Define an operator $D(x_t): C([-h, 0], \mathbb{R}^n) \rightarrow \mathbb{R}^n$ as

$$D(x_t) = x(t) + G_0 \int_{t-h}^t x(s) ds, \quad (11)$$

where $x_t(s) = x(t+s)$, and $G_0 \in \mathbb{R}^{n \times n}$ is a constant matrix.

We have the following fact about $D(x_t)$.

Lemma 1 [9] *The operator (11) is stable if there exist a scalar $0 < \delta < 1$ and positive symmetric M such that*

$$\begin{pmatrix} -\delta M & hG_0^T M \\ hM^T G_0 & -M \end{pmatrix} < 0.$$

Lemma 2 [10] *Assume that $S \in M^{n \times n}$ is symmetric positive definite matrix. Then for every $Q \in M^{n \times n}$ and $x, y \in \mathbb{R}^n$*

$$2\langle Qy, x \rangle - \langle Sy, y \rangle \leq \langle QS^{-1}Q^T x, x \rangle.$$

Lemma 3 [10] *Let $P, Q, S \in M^{n \times n}$ be given matrices such that $S > 0$, $S = S^T$. Then*

$$\begin{pmatrix} P & Q \\ Q^T & -S \end{pmatrix} < 0 \Leftrightarrow P + QS^{-1}Q < 0.$$

Lemma 4 [10] *Suppose U and V are real symmetric matrices and $U > 0$, $V \geq 0$.*

Then

$$U > V \Leftrightarrow \lambda_{\max}(VU^{-1}) < 1 \Leftrightarrow \lambda_{\max}(U^{-1/2}VU^{-1/2}) < 1.$$

Lemma 5 [10] *Suppose U and V are real symmetric matrices and $U > 0$, $V \geq 0$.*

Then there is a positive number σ such that $-U + \sigma V < 0$.

Lemma 6 [10] Suppose U and V are real symmetric matrices and $U > 0, V \geq 0$, a is a positive number. Then

$$aU > V \Leftrightarrow \lambda_{\max}(VU^{-1}) < a \Leftrightarrow \lambda_{\max}(U^{-\frac{1}{2}}VU^{-\frac{1}{2}}) < a.$$

Two problems are considered in the rest of this paper.

First, to show that if the system (10) is stable then the system (13) is so.

Theorem 1 [14] System (10) is uniformly asymptotically stable independent of delay if

$$\mu(A_0) + \sum_{i=1}^m \|A_i\| < 0. \quad (12)$$

Considering the time delay system described by (10) then the matrices $A_i, i=1,2,\dots,n$, are converted to diagonal form and subtract $\varepsilon > 0$ from each diagonal entry. Then we choose the matrix with maximum norm and denote it as G_ε . The rest of matrices denote as B_ε . Therefore from system (10) we have

$$\dot{x}(t) = A_0 x(t) + (G_\varepsilon + mB_\varepsilon)x(t-h), \quad t \in \mathbb{R}^+, \quad (13)$$

$$x(t) = \phi(t), \quad t \in [-h, 0],$$

where the matrix functions $A_0, (G_\varepsilon + mB_\varepsilon)$ are constant matrices and $B = (G_\varepsilon + mB_\varepsilon)$.

Remark 1: To show stability of the system (13), considering the time delay system described by (10) then the matrices $A_i, i=1,2,\dots,n$, are converted to diagonal form and subtract $\varepsilon > 0$ from each diagonal entry, also we assume that the system with multiple time delay is stable and satisfies in (12). We convert the system with multiple time delays to a stable system with single time-delay (13). Then, we know

$$\|G_\varepsilon + mB_\varepsilon\| \leq \|G_\varepsilon\| + m\|B_\varepsilon\| < \|G_{1\varepsilon}\| + \|G_{2\varepsilon}\| + \dots + \|G_{m\varepsilon}\| + m\|B_\varepsilon\|,$$

then by using theorem 1

$$\mu(A_0) + \|G_\varepsilon + mB_\varepsilon\| < \mu(A_0) + \sum_{i=1}^m \|A_i\| = \mu(A_0) + \sum_{i=1}^m \|G_{i\varepsilon} + mB_\varepsilon\| < 0.$$

Then system (13) is uniformly asymptotically stable independent of delay and we would get the result.

Second, stability of linear system with single delay will be presented.

3. Main results

In this section, exponential stability with decay rate α criterion for system (13) is derived by using Lyapunov functional and LMI approach.

Theorem 2 For delay time $h > 0$, system(13) is exponential stable with decay rate α , if there exists symmetric and positive-definite matrices $P > 0, Q > 0$ such that the following LMIs hold:

$$\begin{pmatrix} -I & he^{\alpha h} B^T \\ he^{\alpha h} B & -I \end{pmatrix} < 0, \quad (14)$$

$$\begin{pmatrix} \bar{A}^T P + P\bar{A} + hQ & he^{\alpha h} \bar{A}^T P B \\ he^{\alpha h} B^T P \bar{A} & -hQ \end{pmatrix} < 0, \quad (15)$$

where $\bar{A} = B_0 + Be^{\alpha h}$.

Proof. Consider the time delay systems (13) and use the following transformation:

$$y(t) = e^{\alpha t} x(t),$$

where $\alpha > 0$ is the stability degree, to transform (13) to

$$\dot{y}(t) = B_0 y(t) + e^{\alpha t} B y(t-h), \quad (16)$$

where $B_0 = A_0 + \alpha I$,

Consider the following Lyapunov- Krasovskii functional candidate :

$$V(y(t)) = D^T(y(t)) P D(y(t)) + \int_{-h}^0 \int_{t+s}^t y^T(\rho) Q y(\rho) d\rho ds, \quad (17)$$

where $D(y(t)) = y(t) + \int_{t-h}^t e^{\alpha s} B y(s) ds$.

Therefore we have

$$\begin{aligned} \dot{D}(y(t)) &= \dot{y}(t) + e^{\alpha h} B y(t) - e^{\alpha h} B y(t-h) \\ &= B_0 y(t) + e^{\alpha h} B y(t-h) + e^{\alpha h} B y(t) - e^{\alpha h} B y(t-h) = (B_0 + e^{\alpha h} B) y(t). \end{aligned}$$

Taking the time derivative of the Lyapunov functional along the trajectory of system (16), we have

$$\begin{aligned} \dot{V}(y(t)) &= \dot{D}^T P D + D^T P \dot{D} + y^T(t) h Q y(t) - \int_{t-h}^t y(s)^T Q y(s) ds = (\bar{A} y(t))^T P (y(t) + \\ &\int_{t-h}^t B e^{\alpha h} y(s) ds) + (y(t) + \int_{t-h}^t B e^{\alpha h} y(s) ds)^T P \bar{A} y(t) + y(t)^T h Q y(t) \\ &\quad - \int_{t-h}^t y(s)^T Q y(s) ds \\ &= y(t)^T (\bar{A}^T P + P\bar{A} + hQ) y(t) + 2y(t)^T \bar{A}^T P \int_{t-h}^t B e^{\alpha h} y(s) ds - \int_{t-h}^t y(s)^T Q y(s) ds. \quad (18) \end{aligned}$$

By assuming $2ab \leq a^T Q^{-1} a + b^T Q b$, we have

$$2y(t)^T \bar{A}^T P \int_{t-h}^t B e^{\alpha h} y(s) ds \leq y(t)^T h e^{2\alpha h} \bar{A}^T P B Q^{-1} B^T P \bar{A} y(t) + \int_{t-h}^t y(s)^T Q y(s) ds. \quad (19)$$

Hence, we obtain $\dot{V}(y(t)) \leq y(t)^T S_1 y(t)$, where

$$S_1 = \bar{A}^T P + P \bar{A} + h e^{2\alpha h} \bar{A}^T P B Q^{-1} B^T P \bar{A} + h Q. \quad (20)$$

Finally, using the Schur Complements, with some algebras, one can observe that condition (15) of Theorem 1 guaratees $\dot{V}(y(t)) < 0$.

We have $\dot{V}(x(t)) < 0$ if $S_1 < 0$, while $S_1 < 0$ if and only if condition (15) holds. Therefore, system (13) is exponential stable with decay rate α .

4. Illustrative examples

Example 3

Consider the linear delay system (10),

where

$$m = 2, \quad h_1 = 0.5, \quad h_2 = 1, \quad h = 1,$$

$$A_0 = \begin{pmatrix} -2 & 1 \\ -2 & -4 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0.05 & 0 \\ 0 & 0.05 \end{pmatrix},$$

with $\alpha = 0.8$ and $\varepsilon = 0.01$, then the matrices A_1, A_2 are converted to diagonal form and subtract $\varepsilon > 0$ from each diagonal entry. Then we choose the matrix with maximum norm and denote it as G_ε . The rest of matrices denote as B_ε , where

$$G_\varepsilon = \begin{pmatrix} 0.09 & 0 \\ 0 & 0.09 \end{pmatrix}, \quad B_\varepsilon = \begin{pmatrix} 0.01 & 0 \\ 0 & 0.01 \end{pmatrix}.$$

Therefore from system (10) we have

$$\dot{x}(t) = A_0 x(t) + (G_\varepsilon + m B_\varepsilon) x(t-h), \quad t \in \mathbb{R}^+,$$

$$x(t) = \varphi(t), \quad t \in [-h, 0],$$

where the matrix functions $A_0, (G_\varepsilon + m B_\varepsilon)$ are constant matrices. Let us set

$$B = (G_\varepsilon + m B_\varepsilon),$$

$$B_0 = A_0 + \alpha I,$$

$$\bar{A} = B_0 + B e^{\alpha h}.$$

Applying condition (15) Theorem 1, let $Q = I$ then we have a positive definite solution

$$P = \begin{pmatrix} 0.21 & 0.049 \\ 0.049 & 0.1075 \end{pmatrix}.$$

According to the Theorem 2, the system is 0.8-stable. This system is 0.5-stable in [10]. The stability of the system depend on ε . When the positive number ε decrease we obtain improvement of the result.

Example 4. Consider the linear delay system

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-1) + A_2 x(t-\sqrt{3}), \quad t \geq 0,$$

$$x(t) = \phi(t), \quad t \in [-\sqrt{3}, 0],$$

where

$$A_0 = \begin{pmatrix} -7 & -1 \\ 0.5 & -5.5 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 10.1 & 14.2 \\ -6.6 & -10.2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -5.2 & -6.2 \\ 3.6 & 5.1 \end{pmatrix},$$

with $\alpha = 0.7$ and $\varepsilon = 0.01$, then the matrices A_1, A_2 are converted in to diagonal form and subtract $\varepsilon > 0$ from each diagonal entry. Then we choose the matrix with maximum norm and denote it as G_ε . The rest of matrices denote as B_ε , we have

$$G_\varepsilon = \begin{pmatrix} -0.227 & 0 \\ 0 & 0.265 \end{pmatrix}, \quad B_\varepsilon = \begin{pmatrix} 0.01 & 0 \\ 0 & 0.01 \end{pmatrix}.$$

Therefore from system (10) we have

$$\dot{x}(t) = A_0 x(t) + (G_\varepsilon + mB_\varepsilon)x(t-h), \quad t \in \mathbb{R}^+,$$

$$x(t) = \varphi(t), \quad t \in [-h, 0],$$

where the matrix functions $A_0, (G_\varepsilon + mB_\varepsilon)$ are constant matrices. Let us set

$$B = (G_\varepsilon + mB_\varepsilon),$$

$$B_0 = A_0 + \alpha I,$$

$$\bar{A} = B_0 + Be^{ah}.$$

Applying condition (15) Theorem (1) with $Q = I$, we have

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 2.421 \end{pmatrix}.$$

According to the Theorem 2, the system is 0.7-stable. This system is 0.09-stable in [8].

5. Conclusion

In this brief we converted a system of multiple time delays into a single time delay system and show that if the old system is stable the new one is so. Then we consider the exponential stability problem for linear systems with single delay, based on the Lyapunov-Kradovskii functional approach and (LMI). Numerical examples show the effectiveness of theoretical results.

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